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Wanyong Wang & Jian Xu

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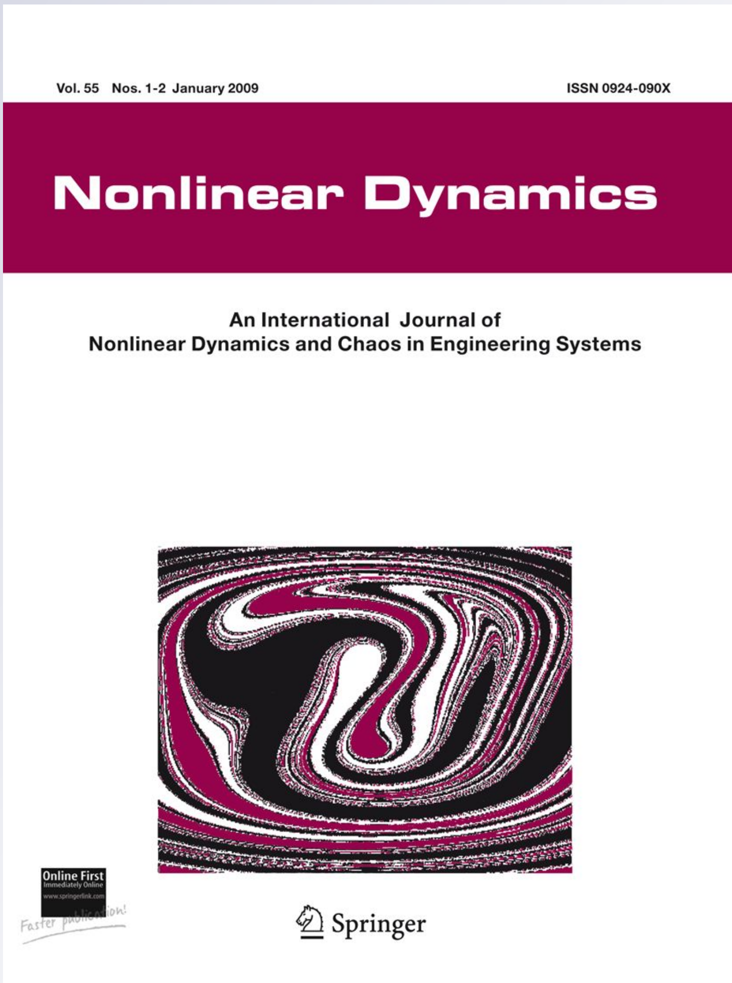

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ORIGINAL PAPER

Multiple scales analysis for double Hopf bifurcation with 1:3 resonance

Wanyong Wang · Jian Xu

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Abstract The dynamical behavior of a general n-dimensional delay differential equation (DDE) around a 1:3 resonant double Hopf bifurcation point is analyzed. The method of multiple scales is used to obtain complex bifurcation equations. By expressing complex amplitudes in a mixed polar-Cartesian representation, the complex bifurcation equations are again obtained in real form. As an illustration, a system of two coupled van der Pol oscillators is considered and a set of parameter values for which a 1:3 resonant double Hopf bifurcation occurs is established. The dynamical behavior around the resonant double Hopf bifurcation point is analyzed in terms of three control parameters. The validity of analytical results is shown by their consistency with numerical simulations.

Keywords Multiple scales · Resonance · Double Hopf bifurcation · Time delay

1 Introduction

The method of multiple scales (MMS) [1] is a powerful tool for the analysis of dynamical interaction phenomena occurring in weakly nonlinear systems under internal and/or external resonance conditions [2]. It has been used to reduce a multidimensional dynamical system into a lower-dimensional equivalent system which captures all the qualitative aspects of the original system behaviors [3]. Using the mixed (i.e., polar and Cartesian) form, the method leads to the standard normal form equations which are suitable to analyze the stability of incomplete solutions (i.e. solutions in which some amplitudes identically vanish) [4]. These standard normal form equations have been used to analyze nonresonant double Hopf bifurcation [5], codimension-three 1:2 and 1:3 resonant double Hopf bifurcation [6], 1:1 resonant double Hopf bifurcation [7], and multiple-Hopf bifurcations [8]. These works have successfully solved the problem of resonant double Hopf bifurcation without time delay. As an extension of the method introduced by Luongo et al. [6], we will investigate the dynamical behaviors of a general n-dimensional delay differential equation around a 1:3 resonant double Hopf bifurcation point.

As we know, time delay always exists and plays an important role in dynamical systems. Therefore, it is very essential to introduce time delay into dynamical systems. In fact, nonlinear differential equations with time delay have been studied in various scientific fields, and some publications on this topic are cited in [9, 10]. Some authors have begun to investigate the dynamical behavior of the systems with time delay near a resonant double Hopf bifurcation. Campbell and LeBlanc [11] used center manifold analysis to investigate a 1:2 resonant double Hopf bifurcation in

W. Wang · J. Xu (✉)
School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, P.R. China
e-mail: xujian@tongji.edu.cn

a DDE. Xu [12] developed an efficient method (PIS) for studying weak resonant double Hopf bifurcation in nonlinear systems with delayed feedbacks. Recently, the MMS has been directly applied to the systems with time delay [13–17] and used to study small perturbation of a harmonic oscillator by a small term with a large delay [18].

The outline of this paper is as follows. In Sect. 2, based on the MMS, a useful method is developed and used to analyze a 1:3 resonant double Hopf bifurcation of a dynamical system with time delay. Employing this method, complex bifurcation equations are obtained in real form. In Sect. 3, as an example, we consider two van der Pol oscillators with delay coupling and analyze the dynamical behaviors around a 1:3 resonant double Hopf bifurcation point. In Sect. 4, the discussion and conclusion are given.

2 Multiple scales analysis for 1:3 resonant double Hopf bifurcation

An n-dimensional, dynamical system with time delay is considered, governed by the following equation of motion:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{x}_\tau, \mu), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state variable depending on a set $\mu \in R^m$ of control parameter vector and $\mathbf{x}_\tau = \mathbf{x}(t-\tau)$ where τ is time delay. Without loss of generality, it is assumed that (1) admits the trivial equilibrium solution $\mathbf{x} = 0$ for any value of μ. According to the bifurcation theory, a 1:3 resonant double Hopf bifurcation satisfies three conditions among two eigenvalues, namely, $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0$, $\operatorname{Im}\lambda_2 = 3\operatorname{Im}\lambda_1$ [6]. Therefore, the critical values $\mu_\mathbf{c} = (\mu_{1c}, \mu_{2c}) \in R^2$ and $\tau = \tau_c \in R$ are taken.

It is assumed that $\mu_1 = \mu_{1c} + \mu_{1\varepsilon}$, $\mu_2 = \mu_{2c} + \mu_{2\varepsilon}$ and $\tau = \tau_c + \tau_\varepsilon$. The point $O = (\mathbf{x} = 0, \mu_\varepsilon = (\mu_{1\varepsilon}, \mu_{2\varepsilon}) = 0, \tau_\varepsilon = 0)$ is a 1:3 resonant double Hopf bifurcation point. Then the following conditions must be satisfied for the Jacobian matrix at O

$$\mathbf{F}_\mathbf{x}^\mathbf{0} := \left.\frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{x}_\tau, \mu_\varepsilon)}{\partial \mathbf{x}}\right|_{\mathbf{x}=0, \mathbf{x}_\tau=0, \mu_\varepsilon=0}, \qquad \mathbf{F}_{\mathbf{x}_\tau}^\mathbf{0} := \left.\frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{x}_\tau, \mu_\varepsilon)}{\partial \mathbf{x}_\tau}\right|_{\mathbf{x}=0, \mathbf{x}_\tau=0, \mu_\varepsilon=0}. \tag{2}$$

(C.1) The matrix $\mathbf{F}_\mathbf{x}^\mathbf{0} + \mathbf{F}_{\mathbf{x}_\tau}^\mathbf{0} e^{-\lambda\tau}$ has two pairs of purely imaginary eigenvalues $\lambda_{1,3} = \pm i\omega_1$, $\lambda_{2,4} = \pm i\omega_2$ and $\omega_2 = 3\omega_1$ and all the remaining eigenvalues λ_h $(h \geq 5)$ lie on the left side of the complex plane. Then the right $\mathbf{p}_j$ and left $\mathbf{q}_j$ $(j = 1, 2)$ eigenvectors of $\lambda_{1,3}$ and $\lambda_{2,4}$ are solutions of the following equations:

$$\begin{aligned} &\left(\mathbf{F}_\mathbf{x}^\mathbf{0} + \mathbf{F}_{\mathbf{x}_\tau}^\mathbf{0} e^{-i\omega_j\tau} - i\mathbf{E}\omega_j\right)\mathbf{p_j} = 0, \\ &\left(\left(\mathbf{F}_\mathbf{x}^\mathbf{0} + \mathbf{F}_{\mathbf{x}_\tau}^\mathbf{0} e^{i\omega_j\tau}\right)^\mathrm{T} + i\mathbf{E}\omega_j\right)\mathbf{q_j} = 0, \end{aligned} \tag{3}$$

where $\mathbf{E}$ is the identity matrix, $\mathbf{p_3} = \bar{\mathbf{p}}_\mathbf{1}$, $\mathbf{p_4} = \bar{\mathbf{p}}_\mathbf{2}$, $\mathbf{q_3} = \bar{\mathbf{q}}_\mathbf{1}$ and $\mathbf{q_4} = \bar{\mathbf{q}}_\mathbf{2}$. Right and left eigenvectors are orthonormal, i.e., $\mathbf{q_i}^\mathrm{H}\mathbf{p_i} = 1$, $(i = 1, 2)$, where H denotes the transpose conjugate.

(C.2) The critical eigenvalues $\lambda_{1,3} = \alpha_1(\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tau_\varepsilon) + i\omega_1(\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tau_\varepsilon)$ and $\lambda_{2,4} = \alpha_2(\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tau_\varepsilon) + i\omega_2(\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tau_\varepsilon)$ satisfy the transversality condition, $\alpha_j(0,0,0) = 0$ $(j = 1, 2)$ and $\omega_2(0,0,0) = 3\omega_1(0,0,0)$ at point O.

In the parameter space $(\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tau_\varepsilon)$, $\alpha_j(\mu_{1\varepsilon}, \mu_{2\varepsilon}, \tau_\varepsilon) = 0$ $(j = 1, 2)$ are the critical surfaces which bound the regions of linear stability of trivial solution. A double Hopf bifurcation occurs at the intersection of the two critical surfaces and a 1:3 resonant double Hopf bifurcation occurs at the point of the intersection of $\alpha_1 = 0$, $\alpha_2 = 0$ and $\omega_2 = 3\omega_1$.

In the following, the multiple scale method will be used to investigate the dynamical behavior around the double Hopf bifurcation point.

According to the MMS, a mono-parametric family of solution of the type is as follows:

$$\mathbf{x} = \mathbf{x}(\varepsilon, T_k, \ldots), \tag{4}$$

where $T_k = \varepsilon^k t$ $(k = 0, 1, 2, \ldots)$ and $\varepsilon \ll 1$.

1:3 internal resonances are associated with third-order effects, then the solutions do not dependent on the time scale T_1. Therefore, we assume a two scales expansion of the solution of (1)

$$\begin{aligned} \mathbf{x}(t) = {} & \varepsilon\mathbf{x}_1(T_0, T_2) + \varepsilon^2\mathbf{x}_2(T_0, T_2) \\ & + \varepsilon^3\mathbf{x}_3(T_0, T_2) + O\left(\varepsilon^4\right), \end{aligned} \tag{5}$$

while the vector parameters are ordered as

$$\mu_{1\varepsilon} = \varepsilon^2\hat{\mu}_{1\varepsilon}, \qquad \mu_{2\varepsilon} = \varepsilon^2\hat{\mu}_{2\varepsilon}, \qquad \tau_\varepsilon = \varepsilon^2\hat{\tau}_\varepsilon. \tag{6}$$

The delay term in (1) can be further expanded as

$$\begin{aligned}\mathbf{x}(t-\tau) &= \varepsilon\mathbf{x}_1\left(T_0-\tau_c-\varepsilon^2\hat{\tau}_\varepsilon, T_2-\varepsilon^2\tau_c-\varepsilon^4\hat{\tau}_\varepsilon\right)\\ &\quad+\varepsilon^2\mathbf{x}_2\left(T_0-\tau_c-\varepsilon^2\hat{\tau}_\varepsilon, T_2-\varepsilon^2\tau_c-\varepsilon^4\hat{\tau}_\varepsilon\right)\\ &\quad+\varepsilon^3\mathbf{x}_3\left(T_0-\tau_c-\varepsilon^2\hat{\tau}_\varepsilon, T_2-\varepsilon^2\tau_c-\varepsilon^4\hat{\tau}_\varepsilon\right)\\ &\quad+\cdots\\ &= \varepsilon\mathbf{x}_1(T_0-\tau_c, T_2)+\varepsilon^2\mathbf{x}_2(T_0-\tau_c, T_2)\\ &\quad+\varepsilon^3\left[-\hat{\tau}_\varepsilon D_0\mathbf{x}_1(T_0-\tau_c, T_2)\right.\\ &\quad-\tau_c D_2\mathbf{x}_1(T_0-\tau_c, T_2)\\ &\quad\left.+\mathbf{x}_3(T_0-\tau_c, T_2)\right]+O\left(\varepsilon^4\right), \end{aligned} \tag{7}$$

where $D_k = \partial/\partial T_k$.

By substituting (5), (6), and (7) into (1), expanding $\mathbf{F}$ as well and equating separately coefficients of like powers of ε, the following perturbative equations are obtained

$$D_0\mathbf{x}_1 - \mathbf{F}^0_{\mathbf{x}}\mathbf{x}_1 - \mathbf{F}^0_{\mathbf{x}_\tau}\mathbf{x}_{1\tau} = 0, \tag{8}$$

$$\begin{aligned}&D_0\mathbf{x}_2 - \mathbf{F}^0_{\mathbf{x}}\mathbf{x}_2 - \mathbf{F}^0_{\mathbf{x}_\tau}\mathbf{x}_{2\tau}\\ &\quad=\frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{x}_1^2+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{x}_1\mathbf{x}_{1\tau}+\frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{x}_{1\tau}^2,\end{aligned} \tag{9}$$

$$\begin{aligned}&D_0\mathbf{x}_3 - \mathbf{F}^0_{\mathbf{x}}\mathbf{x}_3 - \mathbf{F}^0_{\mathbf{x}_\tau}\mathbf{x}_{3\tau}\\ &\quad=\mathbf{F}^0_{\mathbf{x}\mu_\varepsilon}\mathbf{x}_1\hat{\mu}_\varepsilon+\frac{1}{6}\mathbf{F}^0_{\mathbf{xxx}}\mathbf{x}_1^3+\mathbf{F}^0_{\mathbf{xx}}\mathbf{x}_1\mathbf{x}_2+\mathbf{F}^0_{\mathbf{x}_\tau\mu_\varepsilon}\mathbf{x}_{1\tau}\hat{\mu}_\varepsilon\\ &\qquad+\frac{1}{2}\mathbf{F}^0_{\mathbf{xxx}_\tau}\mathbf{x}_1^2\mathbf{x}_{1\tau}+\frac{1}{2}\mathbf{F}^0_{\mathbf{xx}_\tau\mathbf{x}_\tau}\mathbf{x}_1\mathbf{x}_{1\tau}^2+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{x}_{1\tau}\mathbf{x}_2\\ &\qquad+\frac{1}{6}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{x}_{1\tau}^3+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{x}_1\mathbf{x}_{2\tau}+\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{x}_{1\tau}\mathbf{x}_{2\tau}\\ &\qquad-\tau_c\mathbf{F}^0_{\mathbf{x}_\tau}D_2\mathbf{x}_{1\tau}-\tau_\varepsilon\mathbf{F}^0_{\mathbf{x}_\tau}D_0\mathbf{x}_{1\tau}-D_2\mathbf{x}_1,\end{aligned} \tag{10}$$

where $\mathbf{F}^0_{\mathbf{x}\mu_\varepsilon} = \frac{\partial^2\mathbf{F}(\mathbf{0},\mathbf{0},0)}{\partial\mathbf{x}\partial\mu_\varepsilon}$, $\mathbf{F}^0_{\mathbf{x}_\tau\mu_\varepsilon} = \frac{\partial^2\mathbf{F}(\mathbf{0},\mathbf{0},0)}{\partial\mathbf{x}_\tau\partial\mu_\varepsilon}$ and similarly for higher-order derivatives. $\mathbf{x}_{j\tau} = \mathbf{x}_j(T_0-\tau_c, T_2)$, $j=1,2,3$.

Equation (8) has the following general solution:

$$\mathbf{x}_1 = A_1(T_2)\mathbf{p}_1e^{i\omega_1T_0}+A_2(T_2)\mathbf{p}_2e^{i\omega_2T_0}+c.c., \tag{11}$$

where A_j $(j=1,2)$ are complex constants, $\mathbf{p_j}$ are the right eigenvectors of $\mathbf{F}^0_{\mathbf{x}}+\mathbf{F}^0_{\mathbf{x}_\tau}e^{-i\omega_j\tau_c}$ associated with the eigenvalues $i\omega_j$ and $c.c.$ stands for the complex conjugate of the preceding terms.

Substituting (11) into (9), we obtain

$$\begin{aligned}&D_0\mathbf{x}_2 - \mathbf{F}^0_{\mathbf{x}}\mathbf{x}_2 - \mathbf{F}^0_{\mathbf{x}_\tau}\mathbf{x}_{2\tau}\\ &\quad=\left(\frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_1^2+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p}_1^2e^{-i\omega_1\tau_c}+\frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1^2e^{-2i\omega_1\tau_c}\right)\\ &\qquad\times A_1^2e^{2i\omega_1T_0}+\left(\frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_2^2+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p}_2^2e^{-i\omega_2\tau_c}\right.\\ &\qquad\left.+\frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_2^2e^{-2i\omega_2\tau_c}\right)A_2^2e^{2i\omega_1T_0}+\left(\frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_1\bar{\mathbf{p}}_1\right.\\ &\qquad\left.+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p}_1\bar{\mathbf{p}}_1e^{i\omega_1\tau_c}+\frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1\bar{\mathbf{p}}_1\right)A_1\bar{A}_1\\ &\qquad+\left(\frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_2\bar{\mathbf{p}}_2+\mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p}_2\bar{\mathbf{p}}_2e^{i\omega_2\tau_c}+\frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\right.\\ &\qquad\left.\times\mathbf{p}_2\bar{\mathbf{p}}_2\right)A_2\bar{A}_2+\left[\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_1\mathbf{p}_2+\mathbf{F}^0_{\mathbf{xx}_\tau}\left(\mathbf{p}_1\mathbf{p}_2e^{-i\omega_1\tau_c}\right.\right.\\ &\qquad\left.\left.+\mathbf{p}_1\mathbf{p}_2e^{-i\omega_2\tau_c}\right)+\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1\mathbf{p}_2e^{-i(\omega_1+\omega_2)\tau_c}\right]\\ &\qquad\times A_1A_2e^{i(\omega_1+\omega_2)T_0}\\ &\qquad+\left[\mathbf{F}^0_{\mathbf{xx}}\bar{\mathbf{p}}_1\mathbf{p}_2+\mathbf{F}^0_{\mathbf{xx}_\tau}\left(\bar{\mathbf{p}}_1\mathbf{p}_2e^{i\omega_1\tau_c}+\bar{\mathbf{p}}_1\mathbf{p}_2e^{-i\omega_2\tau_c}\right)\right.\\ &\qquad\left.+\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\bar{\mathbf{p}}_1\mathbf{p}_2e^{i(\omega_1-\omega_2)\tau_c}\right]\\ &\qquad\times\bar{A}_1A_2e^{i(\omega_2-\omega_1)T_0}+c.c.\end{aligned} \tag{12}$$

Solving (12), it yields

$$\begin{aligned}\mathbf{x}_2 &= A_1^2\mathbf{z}_{11}e^{2i\omega_1T_0}+A_2^2\mathbf{z}_{22}e^{2i\omega_2T_0}+A_1\bar{A}_1\mathbf{z}_{1\bar{1}}\\ &\quad+A_2\bar{A}_2\mathbf{z}_{2\bar{2}}+A_1A_2\mathbf{z}_{12}e^{-i(\omega_1+\omega_2)T_0}\\ &\quad+\bar{A}_1A_2\mathbf{z}_{\bar{1}2}e^{i(\omega_2-\omega_1)T_0}+c.c.,\end{aligned} \tag{13}$$

where the vectors $\mathbf{z_{rs}}$'s and $\mathbf{z_{r\bar{s}}}$'s $(r,s=1,2)\in C^n$ are obtained by solving (32) in Appendix A.

Substituting (11) and (13) into (10) and eliminating the secular terms, we can obtain the equations including D_2A_1 and D_2A_2. Eliminating the coefficients of D_2A_1 and D_2A_2 by using the left eigenvectors and reabsorbing the parameter ε [6], the following bifurcation equations are determined:

$$\begin{aligned}\dot{A}_1 &= C_{1\mu}\mu_\varepsilon A_1+C_{11\bar{1}}A_1^2\bar{A}_1+C_{\bar{1}\bar{1}2}\bar{A}_1^2A_2\\ &\quad+C_{12\bar{2}}A_1A_2\bar{A}_2,\\ \dot{A}_2 &= C_{2\mu}\mu_\varepsilon A_2+C_{111}A_1^3+C_{1\bar{1}2}A_1\bar{A}_1A_2\\ &\quad+C_{22\bar{2}}A_2^2\bar{A}_2,\end{aligned} \tag{14}$$

where the expressions of the coefficients C_{ijk} and $C_{i\mu}\mu_\varepsilon$ are reported in Appendix B.

To express the bifurcation equations in real form, a mixed form representation for the complex amplitudes is introduced

$$A_1 = \frac{1}{2} a e^{i\theta}, \qquad A_2 = \frac{1}{2}(u + iv) e^{i3\theta}. \tag{15}$$

Substituting (15) into (14) and separating the real and imaginary parts in (14), the generalized amplitudes and phase modulation equations are drawn

$$\begin{aligned}
\dot{a} &= aR_1 + \frac{1}{4}R_{11\bar{1}}a^3 + \frac{1}{4}R_{\bar{1}\bar{1}2}a^2u - \frac{1}{4}I_{\bar{1}\bar{1}2}a^2v \\
&\quad + \frac{1}{4}R_{12\bar{2}}au^2 + \frac{1}{4}R_{12\bar{2}}av^2, \\
a\dot{\theta} &= aI_1 + \frac{1}{4}I_{11\bar{1}}a^3 + \frac{1}{4}R_{\bar{1}\bar{1}2}a^2v + \frac{1}{4}I_{\bar{1}\bar{1}2}a^2u \\
&\quad + \frac{1}{4}I_{12\bar{2}}av^2 + \frac{1}{4}I_{12\bar{2}}au^2, \\
\dot{u} &= 3v\dot{\theta} + uR_2 - vI_2 + \frac{1}{4}R_{111}a^3 + \frac{1}{4}R_{1\bar{1}2}a^2u \\
&\quad - \frac{1}{4}I_{1\bar{1}2}a^2v + \frac{1}{4}R_{22\bar{2}}u^3 + \frac{1}{4}R_{22\bar{2}}uv^2 \\
&\quad - \frac{1}{4}I_{22\bar{2}}u^2v - \frac{1}{4}I_{22\bar{2}}v^3, \\
\dot{v} &= -3u\dot{\theta} + vR_2 + uI_2 + \frac{1}{4}I_{111}a^3 + \frac{1}{4}R_{1\bar{1}2}a^2v \\
&\quad + \frac{1}{4}I_{1\bar{1}2}a^2u + \frac{1}{4}I_{22\bar{2}}u^3 + \frac{1}{4}I_{22\bar{2}}uv^2 \\
&\quad + \frac{1}{4}R_{22\bar{2}}u^2v + \frac{1}{4}R_{22\bar{2}}v^3,
\end{aligned} \tag{16}$$

where $R_i = \mathrm{Re}(C_{i\mu}\mu_\varepsilon)$, $I_i = \mathrm{Im}(C_{i\mu}\mu_\varepsilon)$, $R_{ijk} = \mathrm{Re}(C_{ijk})$ and $I_{ijk} = \mathrm{Im}(C_{ijk})$. If $a \neq 0$, from (16_2), $\dot{\theta}$ can be expressed as a function of a, u, v. Substituting $\dot{\theta}$ into (16_3) and (16_4), a set of three bifurcation equations in standard normal form is obtained.

$$\begin{aligned}
\dot{a} &= aR_1 + \frac{1}{4}R_{11\bar{1}}a^3 + \frac{1}{4}R_{\bar{1}\bar{1}2}a^2u - \frac{1}{4}I_{\bar{1}\bar{1}2}a^2v \\
&\quad + \frac{1}{4}R_{12\bar{2}}au^2 + \frac{1}{4}R_{12\bar{2}}av^2, \\
\dot{u} &= 3vI_1 + \frac{3}{4}I_{11\bar{1}}a^2v + \frac{3}{4}R_{\bar{1}\bar{1}2}av^2 + \frac{3}{4}I_{\bar{1}\bar{1}2}auv \\
&\quad + \frac{3}{4}I_{12\bar{2}}v^3 + \frac{3}{4}I_{12\bar{2}}u^2v + uR_2 - vI_2 \\
&\quad + \frac{1}{4}R_{111}a^3 + \frac{1}{4}R_{1\bar{1}2}a^2u - \frac{1}{4}I_{1\bar{1}2}a^2v \\
&\quad + \frac{1}{4}R_{22\bar{2}}u^3 + \frac{1}{4}R_{22\bar{2}}uv^2 - \frac{1}{4}I_{22\bar{2}}u^2v \\
&\quad - \frac{1}{4}I_{22\bar{2}}v^3, \\
\dot{v} &= -3uI_1 - \frac{3}{4}I_{11\bar{1}}a^2u - \frac{3}{4}R_{\bar{1}\bar{1}2}auv - \frac{3}{4}I_{\bar{1}\bar{1}2}au^2 \\
&\quad - \frac{3}{4}I_{12\bar{2}}uv^2 - \frac{3}{4}I_{12\bar{2}}u^3 + vR_2 + uI_2 \\
&\quad + \frac{1}{4}I_{111}a^3 + \frac{1}{4}R_{1\bar{1}2}a^2v + \frac{1}{4}I_{1\bar{1}2}a^2u + \frac{1}{4}I_{22\bar{2}}u^3 \\
&\quad + \frac{1}{4}I_{22\bar{2}}uv^2 + \frac{1}{4}R_{22\bar{2}}u^2v + \frac{1}{4}R_{22\bar{2}}v^3.
\end{aligned} \tag{17}$$

In order to obtain (17), we assume that $a \neq 0$, but by observing (17), we find that $a = 0$ still admits (17). If we assume that $a_1 = |A_1| = a$, $a_2 = |A_2| = \sqrt{u^2 + v^2}$, from (17), we obtain that

$$\begin{aligned}
\dot{a}_1 &= R_1a_1 + \frac{1}{4}R_{11\bar{1}}a_1^3 + \frac{1}{4}R_{\bar{1}\bar{1}2}a_1^2u - \frac{1}{4}I_{\bar{1}\bar{1}2}a_1^2v \\
&\quad + \frac{1}{4}R_{12\bar{2}}a_1a_2^2, \\
a_2\dot{a}_2 &= u\dot{u} + v\dot{v} \\
&= R_2a_2^2 + \frac{1}{4}R_{111}a_1^3u + \frac{1}{4}I_{111}a_1^3v \\
&\quad + \frac{1}{4}R_{1\bar{1}2}a_1^2a_2^2 + \frac{1}{4}R_{22\bar{2}}a_2^4.
\end{aligned} \tag{18}$$

Equation (18) admits the trivial solution $a_1 = a_2 = 0$. However, nontrivial steady-state solutions are possible causing monomodal or bimodal solutions. Monomodal solution occurs when one of the two model amplitudes vanishes. Thus, if $a_2 = 0$, namely, $u = v = 0$, (18_2) is identically satisfied, while (18_1) leads to

$$a_{10} = \sqrt{-\frac{4R_1}{R_{11\bar{1}}}}.$$

Similarly, if $a_1 = 0$, (18_1) is identically satisfied, while (18_2) leads to

$$a_{20} = \sqrt{-\frac{4R_2}{R_{22\bar{2}}}}.$$

Finally, if a_1 and a_2 are different from zero, a bimodal (quasiperiodic) solution exists. From (18_1) and (18_2), we have

$$\frac{1}{4}a_1u = \frac{1}{4a_1^2(R_{\bar{1}\bar{1}2}I_{111} + R_{111}I_{\bar{1}\bar{1}2})}$$

$$\times\left(-R_{11\bar{1}}I_{111}a_1^4-4R_1T_{111}a_1^2 - R_{12\bar{2}}I_{111}a_1^2a_2^2-R_{1\bar{1}2}I_{\bar{1}\bar{1}2}a_1^2a_2^2 - 4R_2I_{\bar{1}\bar{1}2}a_2^2-R_{22\bar{2}}I_{\bar{1}\bar{1}2}a_2^4\right),$$

$$\frac{1}{4}a_1v=\frac{1}{4a_1^2(R_{\bar{1}\bar{1}2}I_{111}+R_{111}I_{\bar{1}\bar{1}2})}\times\left(R_{111}R_{11\bar{1}}a_1^4+4R_1R_{111}a_1^2 + R_{111}R_{12\bar{2}}a_1^2a_2^2-R_{1\bar{1}2}R_{\bar{1}\bar{1}2}a_1^2a_2^2 - 4R_2R_{\bar{1}\bar{1}2}a_2^2-R_{22\bar{2}}R_{\bar{1}\bar{1}2}a_2^4\right).$$

By applying the relationship $u^2+v^2=a_2^2$, we get

$$\left(\frac{1}{4}a_1u\right)^2+\left(\frac{1}{4}a_1v\right)^2=\frac{1}{16}a_1^2a_2^2. \tag{19}$$

By instituting $\frac{1}{4}a_1u$ and $\frac{1}{4}a_1v$ into (19), we can obtain the relationship of a_1 and a_2, then we can get the region in which the bimodal (quasiperiodic) solution exists.

3 An example

In this section, as a sample, we investigate the dynamics of two van der Pol oscillators with time delay coupling [19]. Our work is motivated by applications to laser dynamics and the coupling of microwave oscillators. The motion of this system is governed by the following delay differential equations:

$$\begin{aligned}&\ddot{y}_1+\omega_{10}^2y_1+\beta\dot{y}_1+\gamma y_1^2\dot{y}_1=\alpha\dot{y}_2(t-\tau),\\&\ddot{y}_2+\omega_{20}^2y_2+\beta\dot{y}_2+\gamma y_2^2\dot{y}_2=\alpha\dot{y}_1(t-\tau).\end{aligned}\tag{20}$$

Letting $x_1=y_1$, $x_2=\dot{y}_1$, $x_3=y_2$ and $x_4=\dot{y}_2$, (20) can be rewritten as

$$\begin{aligned}&\dot{x}_1=x_2,\\&\dot{x}_2=-\omega_{10}^2x_1-\beta x_2+\alpha x_4(t-\tau)-\gamma x_1^2x_2,\\&\dot{x}_3=x_4,\\&\dot{x}_4=-\omega_{20}^2x_3-\beta x_4+\alpha x_2(t-\tau)-\gamma x_3^2x_4.\end{aligned}\tag{21}$$

Linearizing (21) at $O(0,0,0,0)$, we obtain

$$\begin{aligned}&\dot{x}_1=x_2,\\&\dot{x}_2=-\omega_{10}^2x_1-\beta x_2+\alpha x_4(t-\tau),\\&\dot{x}_3=x_4,\\&\dot{x}_4=-\omega_{20}^2x_3-\beta x_4+\alpha x_2(t-\tau).\end{aligned}\tag{22}$$

The characteristic equation corresponding to (22) is

$$\left(\lambda^2+\beta\lambda+\omega_{10}^2\right)\left(\lambda^2+\beta\lambda+\omega_{20}^2\right)=\left(\alpha\lambda e^{-\lambda\tau}\right)^2. \tag{23}$$

Supposing that $\lambda_{1,3}=\pm i\omega$ ($\omega>0$) is one pair eigenvalues of (23), then $\lambda_{2,4}=\pm 3i\omega$ are also one pair eigenvalues of (23). Substituting $\lambda=i\omega$ into (23), we have

$$\begin{aligned}&\left(\omega_{10}^2-\omega^2\right)\left(\omega_{20}^2-\omega^2\right)-\beta^2\omega^2\\&\quad+i\beta\omega\left(\omega_{10}^2+\omega_{20}^2-2\omega^2\right)\\&\quad=-\alpha^2\omega^2\cos 2\omega\tau+\alpha^2\omega^2\sin 2\omega\tau.\end{aligned}$$

Separating the real and imaginary parts, we derive that

$$\begin{aligned}&\left(\omega_{10}^2-\omega^2\right)\left(\omega_{20}^2-\omega^2\right)-\beta^2\omega^2\\&\quad=-\alpha^2\omega^2\cos 2\omega\tau,\\&\beta\omega\left(\omega_{10}^2+\omega_{20}^2-2\omega^2\right)=\alpha^2\omega^2\sin 2\omega\tau.\end{aligned}\tag{24}$$

Similarly, substituting $\lambda=3i\omega$ into (23) and separating the real and imaginary parts, we have

$$\begin{aligned}&\left(\omega_{10}^2-9\omega^2\right)\left(\omega_{20}^2-9\omega^2\right)-9\beta^2\omega^2\\&\quad=-9\alpha^2\omega^2\cos 6\omega\tau,\\&3\beta\omega\left(\omega_{10}^2+\omega_{20}^2-18\omega^2\right)=9\alpha^2\omega^2\sin 6\omega\tau.\end{aligned}\tag{25}$$

From (24), we obtain that

$$\begin{aligned}&\left[\left(\omega_{10}^2-\omega^2\right)\left(\omega_{20}^2-\omega^2\right)-\beta^2\omega^2\right]^2\\&\quad+\left[\beta\omega\left(\omega_{10}^2+\omega_{20}^2-2\omega^2\right)\right]^2=\left(\alpha^2\omega^2\right)^2.\end{aligned}\tag{26}$$

Similarly, from (27), we get that

$$\begin{aligned}&\left[\left(\omega_{10}^2-9\omega^2\right)\left(\omega_{20}^2-9\omega^2\right)-9\beta^2\omega^2\right]^2\\&\quad+\left[3\beta\omega\left(\omega_{10}^2+\omega_{20}^2-18\omega^2\right)\right]^2=\left(9\alpha^2\omega^2\right)^2.\end{aligned}\tag{27}$$

Noticing that $\cos 3\omega\tau=(\cos\omega\tau)^3-3\cos\omega\tau(\sin\omega\tau)^2$, we have

$$\begin{aligned}&\frac{9\beta^2\omega^2-(\omega_{10}^2-9\omega^2)(\omega_{20}^2-9\omega^2)}{9\alpha^2\omega^2}\\&\quad=\left(\frac{\beta^2\omega^2-(\omega_{10}^2-\omega^2)(\omega_{20}^2-\omega^2)}{\alpha^2\omega^2}\right)^3\\&\qquad-3\left(\frac{\beta^2\omega^2-(\omega_{10}^2-\omega^2)(\omega_{20}^2-\omega^2)}{\alpha^2\omega^2}\right)\\&\qquad\times\left(\frac{\beta\omega(\omega_{10}^2+\omega_{20}^2-2\omega^2)}{\alpha^2\omega^2}\right)^2.\end{aligned}\tag{28}$$

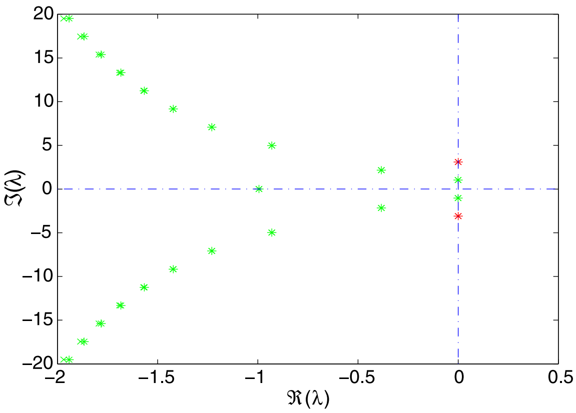


Fig. 1 The eigenvalues of system (21) at point $(\beta_c, \alpha_c, \tau_c) = (0, 1.00664, 1.5233)$ with $\omega_{10} = 1.1$ and $\omega_{10} = 2.9$

Fixing $\omega_{10} = 1.1$ and $\omega_{10} = 2.9$ and solving the (26), (27), and (28), we can obtain that $\omega = 1.03118$, $\beta = 0$, $\alpha = 1.00664$. Substituting this values into (26) and (27), we have

$$\begin{aligned} &\sin 2\omega\tau = 0, \qquad \cos 2\omega\tau = -1, \\ &\sin 6\omega\tau = 0, \qquad \cos 6\omega\tau = -1. \end{aligned} \tag{29}$$

It follows that

$$2\omega\tau = \pi + 2j\pi, \quad j = 0, 1, 2, \ldots, \tag{30}$$

then

$$\tau = \frac{\pi + 2j\pi}{2\omega}, \quad j = 0, 1, 2, \ldots. \tag{31}$$

Then a 1:3 resonant double Hopf bifurcation maybe occur at point $(\beta_c, \alpha_c, \tau_c) = (0, 1.00664, 1.5233)$ where $j = 0$. In order to verify the eigenvalue condition, we show all the eigenvalues of system (21) in Fig. 1. We can see that system (21) has two pairs of purely imaginary eigenvalues and all the remaining eigenvalues λ_h $(h \geq 5)$ have negative parts at bifurcation point. Then the point $(\beta_c, \alpha_c, \tau_c) = (0, 1.00664, 1.5233)$ is a 1:3 resonant double Hopf bifurcation point with the frequencies $\omega_1 = 1.03118$ and $\omega_2 = 3\omega_1$.

In order to obtain the neighboring solutions derived from such double Hopf bifurcation point with 1:3 resonance, we let $\beta = \beta_c + \beta_\varepsilon$, $\alpha = \alpha_c + \alpha_\varepsilon$, $\tau = \tau_c + \tau_\varepsilon$, where β_ε, α_ε and τ_ε are very small. Then (20) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega_{10}^2 x_1 - \beta_\varepsilon x_2 + \alpha_c x_4(t-\tau) \\ &\quad + \alpha_\varepsilon x_4(t-\tau) - \gamma x_1^2 x_2, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -\omega_{20}^2 x_3 - \beta_\varepsilon x_4 + \alpha_c x_2(t-\tau) \\ &\quad + \alpha_\varepsilon x_2(t-\tau) - \gamma x_3^2 x_4. \end{aligned}$$

Then

$$\mathbf{F_x^0} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_{10}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_{20}^2 & 0 \end{pmatrix},$$

$$\mathbf{F_{x_\tau}^0} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_c \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_c & 0 & 0 \end{pmatrix}.$$

Using (3), we obtain that

$$\mathbf{p}_1 = \left(1, i\omega_1, \frac{\omega_{10}^2 - \omega_1^2}{\alpha_c\omega_1}, \frac{i(\omega_{10}^2 - \omega_1^2)}{\alpha_c}\right),$$

$$\mathbf{p}_2 = \left(1, i\omega_2, \frac{\omega_2^2 - \omega_{10}^2}{\alpha_c\omega_2}, \frac{i(\omega_2^2 - \omega_{10}^2)}{\alpha_c}\right),$$

$$\mathbf{q}_1 = e_1\left(-\frac{i\omega_{10}^2}{\omega_1}, 1, -\frac{i(\alpha_c^2 - \omega_1^2 + \omega_{10}^2)}{\alpha_c}, \frac{\omega_{10}^2 - \omega_1^2}{\alpha_c\omega_1}\right),$$

$$\mathbf{q}_2 = e_2\left(-\frac{i\omega_{10}^2}{\omega_2}, 1, \frac{i(\alpha_c^2 - \omega_2^2 + \omega_{10}^2)}{\alpha_c}, \frac{\omega_2^2 - \omega_{10}^2}{\alpha_c\omega_1}\right),$$

where

$$e_1 = \frac{i\alpha_c^2\omega_1}{2[\alpha_c^2\omega_{10}^2 + (\omega_{10}^2 - \omega_1^2)^2]},$$

$$e_2 = \frac{i\alpha_c^2\omega_2}{2[\alpha_c^2\omega_{10}^2 + (\omega_{10}^2 - \omega_2^2)^2]}.$$

Therefore, if we let $\beta_\varepsilon = \varepsilon^2\hat{\beta}_\varepsilon$, $\alpha_\varepsilon = \varepsilon^2\hat{\alpha}_\varepsilon$ and $\tau_\varepsilon = \varepsilon^2\hat{\tau}_\varepsilon$ and use the same procedure as in the previous section, we can get $C_{j\mu\mu_\varepsilon}$ and C_{ijk}. Substituting $C_{j\mu\mu_\varepsilon}$ and C_{ijk} into (17), three bifurcation equations in real form are obtained.

In the following, our analysis is performed around the bifurcation point $(0, 1.00664, 1.5233)$. For (20), one is usually interested in the effects of variations of either the delay τ or the feedback α. Therefore, we assume that $\hat{\beta}_\varepsilon$ is fixed at 0, then we analyze the effects of variations of $\hat{\tau}_\varepsilon$ and $\hat{\alpha}_\varepsilon$ in the $(\hat{\tau}_\varepsilon, \hat{\alpha}_\varepsilon)$-plane. As

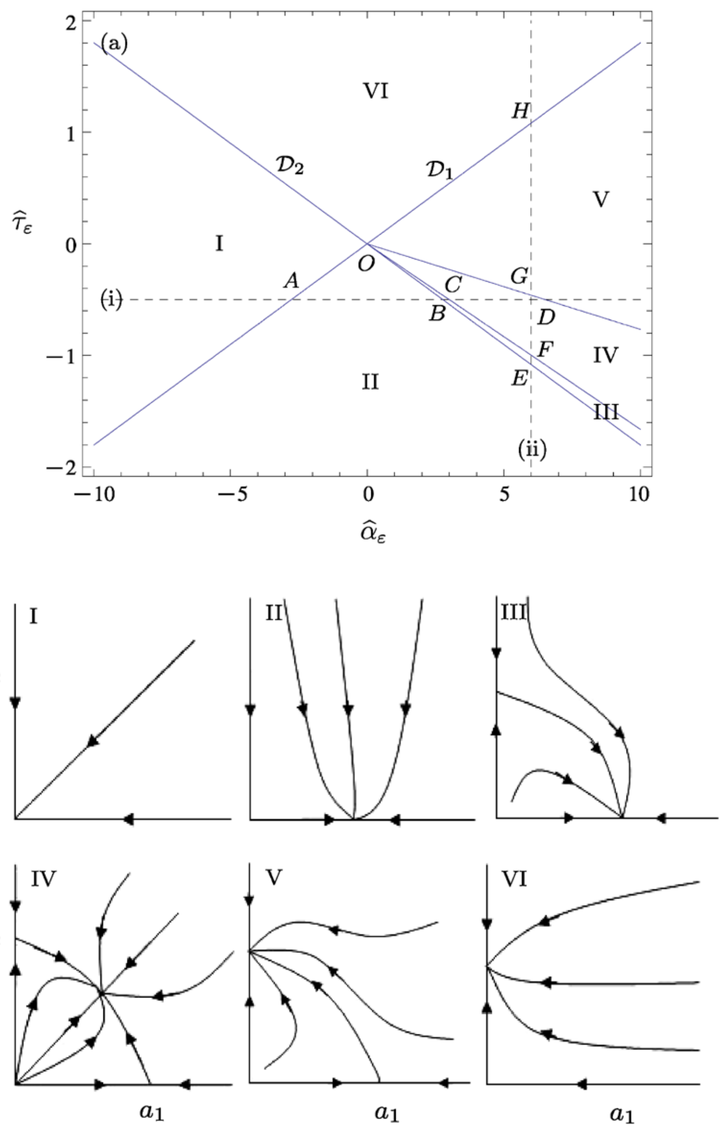
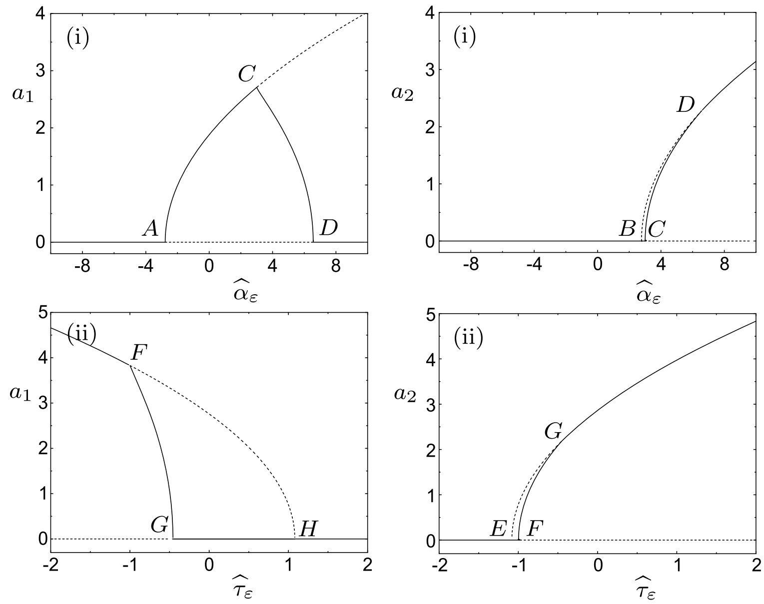


Fig. 2 The cross-section and phase portraits for 1:3 resonant double Hopf bifurcation: (**a**) the cross-section at $\hat{\beta}_\varepsilon = 0$; Fig. (I) to (VI): phase portraits in regions (I) to (VI); (i) and (ii): two selected paths along which the bifurcation diagrams are drawn

shown in Fig. 2, the classification and phase-portraits for 1:3 resonant double Hopf bifurcation are given. Hopf bifurcations for the state variables **x** appear as divergence bifurcations for the amplitude variables (a_1, a_2). The critical boundaries labeled as $\mathcal{D}_1$ and $\mathcal{D}_2$ have been represented in Fig. 2(a). The $(\hat{\tau}_\varepsilon, \hat{\alpha}_\varepsilon)$-plane is divided into six regions and the number and the stability of analytical solutions in regions (I)–(VI) are displayed by a_1 versus a_2. As shown in Fig. 2, there is a stable trivial solution $E_0(0,0)$ in region (I) which is an amplitude death region. With $(\hat{\tau}_\varepsilon, \hat{\alpha}_\varepsilon)$ changing into region (II), the trivial solution loses its stability and a stable solution $E_1(a_{10}, 0)$ appears. When $(\hat{\tau}_\varepsilon, \hat{\alpha}_\varepsilon)$ enters into region (III), the solutions $(0,0)$ and $(a_{10}, 0)$ still exist, and a unstable solution $E_2(0, a_{20})$ appears. In region (IV), a stable nontrivial solution $E_3(a_{12}, a_{22})$ appears together with two unstable solutions $(a_{10}, 0)$ and $(0, a_{20})$. In region (V), there are three solutions $(0,0)$, $(a_{10}, 0)$ and $(0, a_{20})$. In contrast with the situation in region (III), the solution $(0, a_{20})$ is stable and $(a_{10}, 0)$ is unstable. In region (VI), the solution $(a_{10}, 0)$ disappears and the solution $(0, a_{20})$ become stable solution.

In order to have a clearer picture of the analytical predictions, in Fig. 2(a), two straight paths are considered, along which the bifurcation diagrams of the amplitudes a_1 and a_2 are built up. Along path (i) ($\hat{\tau}_\varepsilon = -0.5$), a bifurcation takes place at point A and a

Fig. 3 Bifurcation diagrams along paths (i) and (ii) for the 1:3 resonance



stable branch E_1 arises, then another unstable branch E_2 bifurcates from $(0, 0)$ at point B. At point C, E_1 loses its stability and an stable branch E_3 occurs. At point D, E_3 collapses with E_2, then E_3 disappears and E_2 becomes stable (Fig. 3(i)). Along path (ii) ($\hat{\alpha}_\varepsilon = 6$), in contrast to the former path, the parameter $\hat{\alpha}_\varepsilon$ is fixed and the parameter $\hat{\tau}_\varepsilon$ is varied. The relevant bifurcation diagrams are displayed in Fig. 3(ii). The equilibrium paths E_1 and E_2 bifurcate from point H and E, respectively, are both unstable. At point F, E_2 bifurcates and its stability changes, then an stable branch E_3 occurs at point F and disappears at point G. The results are in accord with those of previous analytical predictions.

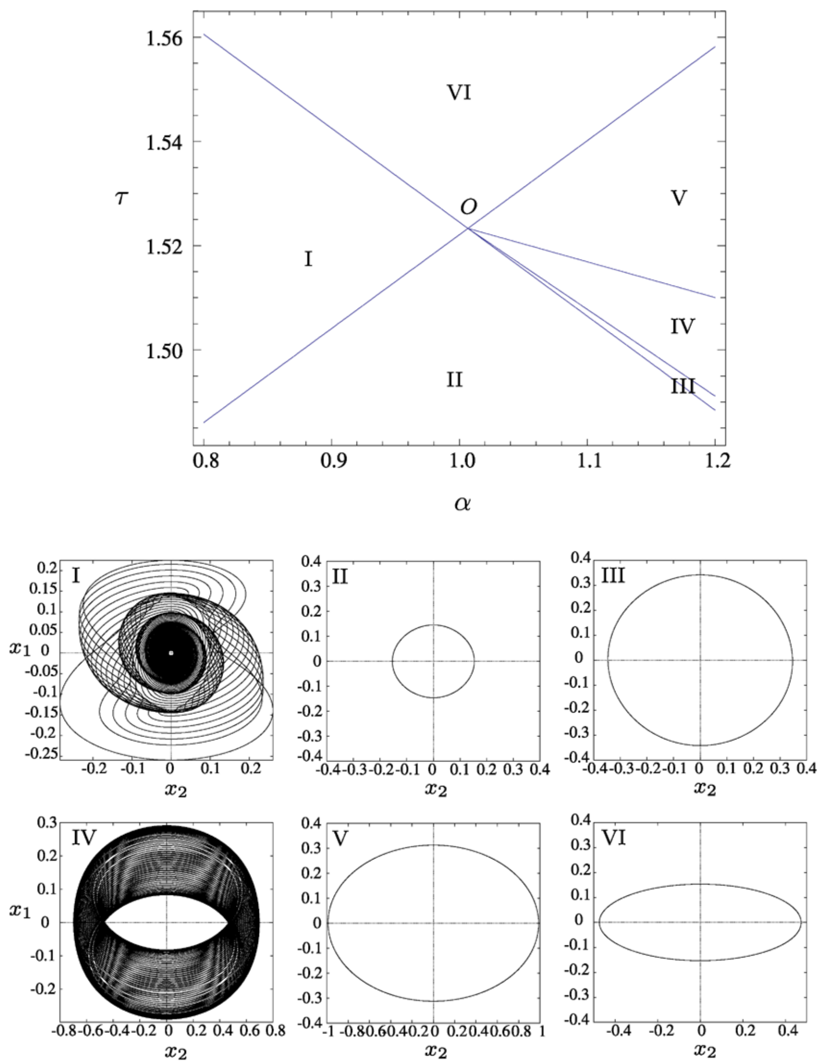
In Fig. 4, numerical simulations for the original system in the plane of $\alpha - \tau$ are given. Six regions responding to the six regions in Fig. 2 are divided. In regions I, the origin is stable. In regions II and III, there is a stable periodic solution with a frequency ω_1 responding to the monomodal solution $E_1(a_{10}, 0)$ in the region II and III of Fig. 2(a). In regions V and VI, a stable periodic solution with frequency ω_2 responding to the monomodal solution $E_2(0, a_{20})$ in the region V and VI of Fig. 2(a) exists. In region IV, a stable quasiperiodic solution exists. Obviously, the results of numerical simulations are in accord with those of the analysis in Fig. 2.

In this paper, there are three control parameters α, β and τ. In the following, the effects of the third control parameter on (21) will be shown. By fixing $\hat{\alpha}_\varepsilon = 0$, the dynamical behavior around the bifurcation point are classified in the plane of $\hat{\beta}_\varepsilon - \hat{\tau}_\varepsilon$ (Fig. 5). In Fig. 5, we can find that there are different dynamical behavior when the parameter $\hat{\beta}_\varepsilon$ changes into different region. Therefore, $\hat{\beta}_\varepsilon$ is a bifurcation parameter. In addition, some numerical simulation results in the plane of $\beta - \tau$ of original system are given in Fig. 6. The results of numerical simulations are in accord with those of analytical predications. Some interesting phenomena such as amplitude death, periodic solution, and period three solution occur. In Fig. 7, the Poincaré map in region IV in Fig. 6 is given. The Poincaré map has three points and it means that the system has a period three solution.

4 Conclusion

We have proposed an analytical method to investigate a general n-dimensional delay differential equations undergoing a 1:3 resonant double Hopf bifurcation. By using MMS and truncating the analysis at the ε^3-order, first-order bifurcation equations are obtained. In order to obtain a set of three bifurcation equations in standard normal form equations, a mixed polar-Cartesian representation for the amplitudes is used. Then the relevant complex bifurcation equations are recast in real form. By analyzing the bifurcation equations, the dynamical behaviors of the original system are obtained.

Fig. 4 The cross-section and phase portraits for (21) when $\beta = 0$, Fig. (I) to (VI): the phase portraits in regions (I) to (VI)



As an example, a pair of coupled van der Pol oscillators are studied. For a suitable choice of three control parameters, a 1:3 resonant double Hopf bifurcation can occur. By varying the set of control parameters, the dynamical behaviors of the system in the neighborhood of the bifurcation point are investigated. By employing the software package AUTO [20], the bifurcation diagrams are obtained by fixing two of the three control parameters. Then the dynamical behaviors arising from the bifurcation are classified qualitatively in a two-parameter cross-section. The results show the rich dynamical behaviors including the amplitude death, periodic solution, quasiperiodic solution and period three solution. In order to verify the theoretical results, the Runge–Kutta scheme is adopt to produce the numerical results. The comparison between analytical predictions and numerical results reveals a qualitatively excellent agreement.

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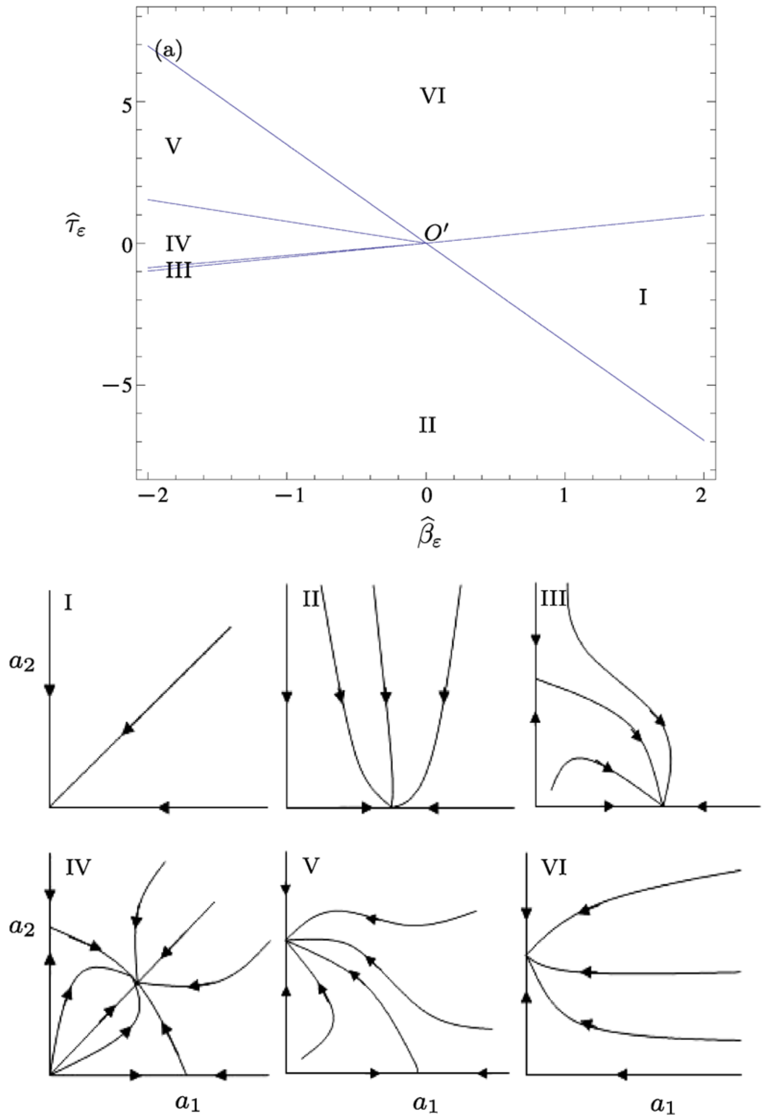
Appendix A

The vectors $\mathbf{z_{rs}}$'s and $\mathbf{z_{r\bar{s}}}$'s $(r, s = 1, 2) \in C^n$ appearing in (13) are obtained by solving the following equa-

Fig. 5 The cross-section and phase portraits for 1:3 resonant double Hopf bifurcation: (**a**) the cross-section at $\hat{\alpha}_\varepsilon = 0$; Fig. (I) to (VI): phase portraits in regions (I) to (VI)



tions:

$$\left(2i\omega_1\mathbf{E} - \mathbf{F}^0_{\mathbf{x}} - \mathbf{F}^0_{\mathbf{x}_\tau} e^{-2i\omega_1\tau_c}\right)\mathbf{z_{11}} = \frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_1^2 + \mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p}_1^2 e^{-i\omega_1\tau_c} + \frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1^2 e^{-2i\omega_1\tau_c},$$

$$\left(2i\omega_2\mathbf{E} - \mathbf{F}^0_{\mathbf{x}} - \mathbf{F}^0_{\mathbf{x}_\tau} e^{-2i\omega_2\tau_c}\right)\mathbf{z_{22}} = \frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p}_2^2 + \mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p}_2^2 e^{-i\omega_2\tau_c} + \frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_2^2 e^{-2i\omega_2\tau_c},$$

$$\left(-\mathbf{F}^0_{\mathbf{x}} - \mathbf{F}^0_{\mathbf{x}_\tau}\right)\mathbf{z_{1\bar{1}}} = \frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p_1}\bar{\mathbf{p}}_1 + \mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p_1}\bar{\mathbf{p}}_1 e^{i\omega_1\tau_c} + \frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p_1}\bar{\mathbf{p}}_1,$$

$$\left(-\mathbf{F}^0_{\mathbf{x}} - \mathbf{F}^0_{\mathbf{x}_\tau}\right)\mathbf{z_{2\bar{2}}} = \frac{1}{2}\mathbf{F}^0_{\mathbf{xx}}\mathbf{p_2}\bar{\mathbf{p}}_2 + \mathbf{F}^0_{\mathbf{xx}_\tau}\mathbf{p_2}\bar{\mathbf{p}}_2 e^{i\omega_1\tau_c} + \frac{1}{2}\mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p_2}\bar{\mathbf{p}}_2, \tag{32}$$

$$\left(i\omega_1\mathbf{E} + i\omega_2\mathbf{E} - \mathbf{F}^0_{\mathbf{x}} - \mathbf{F}^0_{\mathbf{x}_\tau} e^{-i(\omega_1+\omega_2)\tau_c}\right)\mathbf{z_{12}} = \mathbf{F}^0_{\mathbf{xx}}\mathbf{p_1}\mathbf{p_2} + \mathbf{F}^0_{\mathbf{xx}_\tau}\left(\mathbf{p_1}\mathbf{p_2}e^{-i\omega_1\tau_c} + \mathbf{p_1}\mathbf{p_2}e^{-i\omega_2\tau_c}\right) + \mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p_1}\mathbf{p_2}e^{-i(\omega_1+\omega_2)\tau_c},$$

$$\left(-i\omega_1\mathbf{E} + i\omega_2\mathbf{E} - \mathbf{F}^0_{\mathbf{x}} - \mathbf{F}^0_{\mathbf{x}_\tau} e^{i(\omega_1-\omega_2)\tau_c}\right)\mathbf{z_{\bar{1}2}} = \mathbf{F}^0_{\mathbf{xx}}\bar{\mathbf{p}}_1\mathbf{p_2} + \mathbf{F}^0_{\mathbf{xx}_\tau}\left(\bar{\mathbf{p}}_1\mathbf{p_2}e^{i\omega_1\tau_c} + \bar{\mathbf{p}}_1\mathbf{p_2}e^{-i\omega_2\tau_c}\right) + \mathbf{F}^0_{\mathbf{x}_\tau\mathbf{x}_\tau}\bar{\mathbf{p}}_1\mathbf{p_2}e^{i(\omega_1-\omega_2)\tau_c}.$$

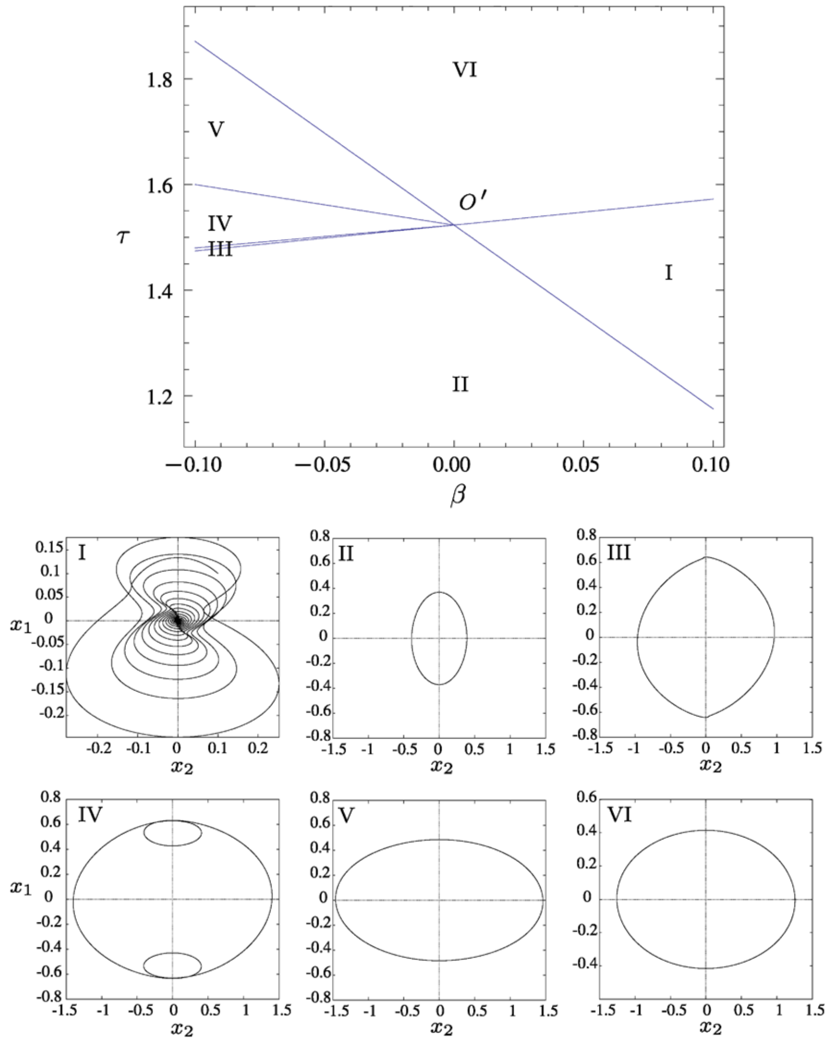


Fig. 6 The cross-section and phase portraits for (21) when $\alpha = 1.00664$, Fig. (I) to (VI): the phase portraits in regions (I) to (VI)

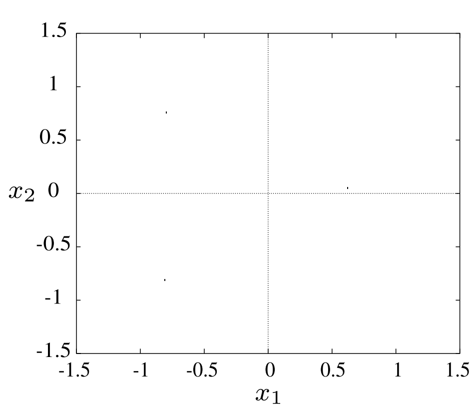


Fig. 7 The Poincaré map in region IV in Fig. 6

Appendix B

The coefficients C_{ijk} and $C_{i\mu\mu_\varepsilon}$ appearing in (14) are

$$
\begin{aligned}
C_{1\mu\mu_\varepsilon} = {} & \frac{1}{M_1}\mathbf{q}_1^{\mathbf{H}}\big(\mathbf{F}^{\mathbf{0}}_{\mathbf{x}\mu}\hat{\mu}_\varepsilon \mathbf{p}_1 + \mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau \mu_\varepsilon}\hat{\mu}_\varepsilon \mathbf{p}_1 e^{-i\omega_1\tau_c} \\
& - i\omega_1\hat{\tau}_\varepsilon \mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau}\mathbf{p}_1 e^{-i\omega_1\tau_c}\big), \\
C_{11\bar{1}} = {} & \frac{1}{M_1}\mathbf{q}_1^{\mathbf{H}}\Big[\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}}\mathbf{p}_1^2\bar{\mathbf{p}}_1 + \mathbf{F}^{\mathbf{0}}_{\mathbf{xx}}(\mathbf{p}_1\bar{\mathbf{z}}_{1\bar{1}} + \mathbf{z}_{11}\bar{\mathbf{p}}_1 \\
& + \mathbf{z}_{1\bar{1}}\mathbf{p}_1) + \frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}_\tau}\big(2\mathbf{p}_1^2\bar{\mathbf{p}}_1 e^{-i\omega_1\tau_c} \\
& + \mathbf{p}_1^2\bar{\mathbf{p}}_1 e^{i\omega_1\tau_c}\big) + \mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\mathbf{p}_1\bar{\mathbf{z}}_{1\bar{1}} e^{-i\omega_1\tau_c} \\
& + \mathbf{z}_{11}\bar{\mathbf{p}}_1 e^{i\omega_1\tau_c} + \mathbf{z}_{1\bar{1}}\mathbf{p}_1 e^{-i\omega_1\tau_c}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau\mathbf{x}_\tau}\left(2\mathbf{p}_1^2\bar{\mathbf{p}}_1+\mathbf{p}_1^2\bar{\mathbf{p}}_1e^{-2i\omega_1\tau_c}\right)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1^2\bar{\mathbf{p}}_1e^{-i\omega_1\tau_c}+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\mathbf{p}_1\bar{\mathbf{z}}_{1\bar{1}}\\
&+\mathbf{z}_{11}\bar{\mathbf{p}}_1e^{-2i\omega_1\tau_c}+\mathbf{z}_{1\bar{1}}\mathbf{p}_1\big)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau}\big(\mathbf{p}_1\bar{\mathbf{z}}_{1\bar{1}}e^{-i\omega_1\tau_c}+\mathbf{z}_{11}\bar{\mathbf{p}}_1e^{-i\omega_1\tau_c}\\
&+\mathbf{z}_{1\bar{1}}\mathbf{p}_1e^{-i\omega_1\tau_c}\big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{\bar{1}\bar{1}2}=\frac{1}{M_1}\mathbf{q}_1^{\mathbf{H}}\Big[&\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}}\mathbf{p}_2\bar{\mathbf{p}}_1^2+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}}(\bar{\mathbf{z}}_{11}\mathbf{p}_2+\mathbf{z}_{\bar{1}2}\bar{\mathbf{p}}_1)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}_\tau}\left(2\mathbf{p}_2\bar{\mathbf{p}}_1^2e^{i\omega_1\tau_c}+\mathbf{p}_2\bar{\mathbf{p}}_1^2e^{-i\omega_2\tau_c}\right)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\left(\bar{\mathbf{z}}_{11}\mathbf{p}_2e^{-3i\omega_1\tau_c}+\mathbf{z}_{\bar{1}2}\bar{\mathbf{p}}_1e^{i\omega_1\tau_c}\right)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau\mathbf{x}_\tau}\left(2\mathbf{p}_2\bar{\mathbf{p}}_1^2e^{-2i\omega_1\tau_c}+\mathbf{p}_2\bar{\mathbf{p}}_1^2e^{2i\omega_1\tau_c}\right)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_2\bar{\mathbf{p}}_1^2e^{-i\omega_1\tau_c}\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\left(\mathbf{p}_2\bar{\mathbf{z}}_{11}e^{2i\omega_1\tau_c}+\mathbf{z}_{\bar{1}2}\bar{\mathbf{p}}_1e^{-2i\omega_1\tau_c}\right)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau}\left(\mathbf{p}_2\bar{\mathbf{z}}_{11}e^{-i\omega_1\tau_c}+\mathbf{z}_{\bar{1}2}\bar{\mathbf{p}}_1e^{-i\omega_1\tau_c}\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{12\bar{2}}=\frac{1}{M_1}\mathbf{q}_1^{\mathbf{H}}\Big[&\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}}\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}}(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_1+\bar{\mathbf{z}}_{\bar{1}2}\mathbf{p}_2\\
&+\mathbf{z}_{12}\bar{\mathbf{p}}_2+\mathbf{z}_{2\bar{2}}\mathbf{p}_1)+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}_\tau}\big(2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2e^{-i\omega_1\tau_c}\\
&+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2e^{i\omega_2\tau_c}+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2e^{-i\omega_2\tau_c}\big)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_1e^{-i\omega_1\tau_c}+\bar{\mathbf{z}}_{\bar{1}2}\mathbf{p}_2e^{-3i\omega_1\tau_c}\\
&+\mathbf{z}_{12}\bar{\mathbf{p}}_2e^{3i\omega_1\tau_c}+\mathbf{z}_{2\bar{2}}\mathbf{p}_1e^{-i\omega_1\tau_c}\big)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau\mathbf{x}_\tau}\big(2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2e^{2i\omega_1\tau_c}\\
&+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2e^{-4i\omega_1\tau_c}\big)+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_2e^{-i\omega_1\tau_c}\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_1+\bar{\mathbf{z}}_{\bar{1}2}\mathbf{p}_2e^{2i\omega_1\tau_c}+\mathbf{z}_{12}\bar{\mathbf{p}}_2e^{-4i\omega_1\tau_c}\\
&+\mathbf{z}_{2\bar{2}}\mathbf{p}_1\big)+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau}\big(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_1e^{-i\omega_1\tau_c}+\bar{\mathbf{z}}_{\bar{1}2}\mathbf{p}_2e^{-i\omega_1\tau_c}\\
&+\mathbf{z}_{12}\bar{\mathbf{p}}_2e^{-i\omega_1\tau_c}+\mathbf{z}_{2\bar{2}}\mathbf{p}_1e^{-i\omega_1\tau_c}\big)\Big], \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
C_{2\mu\mu_\varepsilon}=\frac{1}{M_2}\mathbf{q}_2^{\mathbf{H}}\big(&\mathbf{F}^{\mathbf{0}}_{\mathbf{x}\mu_\varepsilon}\hat{\mu}_\varepsilon\mathbf{p}_2+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mu_\varepsilon}\hat{\mu}_\varepsilon\mathbf{p}_2e^{-i\omega_2\tau_c}\\
&-i\omega_2\hat{\tau}_\varepsilon\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau}\mathbf{p}_2e^{-i\omega_2\tau_c}\big),
\end{aligned}
$$

$$
\begin{aligned}
C_{111}=\frac{1}{M_2}\mathbf{q}_2^{\mathbf{H}}\Big[&\frac{1}{6}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}}\mathbf{p}_1^3+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}}\mathbf{p}_1\mathbf{z}_{11}+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}_\tau}\\
&\times\mathbf{p}_1^3e^{-i\omega_1\tau_c}+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\mathbf{p}_1\mathbf{z}_{11}\left(e^{-i\omega_1\tau_c}+e^{-2i\omega_1\tau_c}\right)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau\mathbf{x}_\tau}\mathbf{p}_1^3e^{-2i\omega_1\tau_c}+\frac{1}{6}\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1^3e^{-3i\omega_1\tau_c}\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1\mathbf{z}_{11}e^{-3i\omega_1\tau_c}\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{1\bar{1}2}=\frac{1}{M_2}\mathbf{q}_2^{\mathbf{H}}\Big[&\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}}\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}}(\bar{\mathbf{z}}_{1\bar{1}}\mathbf{p}_2+\mathbf{z}_{12}\bar{\mathbf{p}}_1\\
&+\mathbf{z}_{1\bar{1}}\mathbf{p}_2+\mathbf{z}_{\bar{1}2}\mathbf{p}_1)+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}_\tau}\big(2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1e^{-i\omega_2\tau_c}\\
&+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1e^{i\omega_1\tau_c}+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1e^{-i\omega_1\tau_c}\big)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\bar{\mathbf{z}}_{1\bar{1}}\mathbf{p}_2e^{-3i\omega_1\tau_c}+\mathbf{z}_{12}\bar{\mathbf{p}}_1e^{i\omega_1\tau_c}\\
&+\mathbf{z}_{1\bar{1}}\mathbf{p}_2e^{-3i\omega_1\tau_c}+\mathbf{z}_{\bar{1}2}\mathbf{p}_1e^{-i\omega_1\tau_c}\big)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau\mathbf{x}_\tau}\big(2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1e^{-2i\omega_1\tau_c}\\
&+2\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1e^{-4i\omega_1\tau_c}\big)+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_1\mathbf{p}_2\bar{\mathbf{p}}_1e^{-3i\omega_1\tau_c}\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\bar{\mathbf{z}}_{1\bar{1}}\mathbf{p}_2+\mathbf{z}_{12}\bar{\mathbf{p}}_1e^{-4i\omega_1\tau_c}+\mathbf{z}_{1\bar{1}}\mathbf{p}_2\\
&+\mathbf{z}_{\bar{1}2}\mathbf{p}_1e^{-2i\omega_1\tau_c}\big)+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau}\big(\bar{\mathbf{z}}_{1\bar{1}}\mathbf{p}_2e^{-3i\omega_1\tau_c}\\
&+\mathbf{z}_{12}\bar{\mathbf{p}}_1e^{-i\omega_1\tau_c}+\mathbf{z}_{1\bar{1}}\mathbf{p}_2e^{-3i\omega_1\tau_c}\\
&+\mathbf{z}_{\bar{1}2}\mathbf{p}_1e^{-3i\omega_1\tau_c}\big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_{22\bar{2}}=\frac{1}{M_2}\mathbf{q}_2^{\mathbf{H}}\Big[&\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}}\mathbf{p}_2^2\bar{\mathbf{p}}_2+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}}(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_2+\mathbf{z}_{22}\bar{\mathbf{p}}_2\\
&+\mathbf{z}_{2\bar{2}}\mathbf{p}_2)+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{xxx}_\tau}\big(2\mathbf{p}_2^2\bar{\mathbf{p}}_2e^{-i\omega_2\tau_c}\\
&+\mathbf{p}_2^2\bar{\mathbf{p}}_2e^{i\omega_2\tau_c}\big)+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_2e^{-3i\omega_1\tau_c}\\
&+\mathbf{z}_{22}\bar{\mathbf{p}}_2e^{3i\omega_1\tau_c}+\mathbf{z}_{2\bar{2}}\mathbf{p}_2e^{-3i\omega_1\tau_c}\big)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau\mathbf{x}_\tau}\big(\mathbf{p}_2^2\bar{\mathbf{p}}_2e^{-6i\omega_1\tau_c}+\mathbf{p}_2^2\bar{\mathbf{p}}_2\big)\\
&+\frac{1}{2}\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau\mathbf{x}_\tau}\mathbf{p}_2^2\bar{\mathbf{p}}_2e^{-3i\omega_1\tau_c}+\mathbf{F}^{\mathbf{0}}_{\mathbf{xx}_\tau}\big(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_2\\
&+\mathbf{z}_{22}\bar{\mathbf{p}}_2e^{-6i\omega_1\tau_c}+\mathbf{z}_{2\bar{2}}\mathbf{p}_2\big)\\
&+\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau\mathbf{x}_\tau}\big(\bar{\mathbf{z}}_{2\bar{2}}\mathbf{p}_2e^{-3i\omega_1\tau_c}+\mathbf{z}_{22}\bar{\mathbf{p}}_2e^{-3i\omega_1\tau_c}\\
&+\mathbf{z}_{2\bar{2}}\mathbf{p}_2e^{-3i\omega_1\tau_c}\big)\Big],
\end{aligned}
$$

$$
\mathbf{q}_1^{\mathrm{H}}\left(\tau_c\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau}e^{-i\omega_1\tau_c}+\mathbf{E}\right)\mathbf{p}_1=M_1,
$$

$$
\mathbf{q}_2^{\mathrm{H}}\left(\tau_c\mathbf{F}^{\mathbf{0}}_{\mathbf{x}_\tau}e^{-i\omega_2\tau_c}+\mathbf{E}\right)\mathbf{p}_2=M_2.
$$

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