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DISCONJUGACY AND EXTREMAL SOLUTIONS OF NONLINEAR THIRD-ORDER EQUATIONS

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Abstract. In this paper, we make an exhaustive study of the third order linear operators $u'''+Mu$, $u'''+Mu'$ and $u'''+Mu''$ coupled with $(k,3-k)$-conjugate boundary conditions, where $k=1$, 2. We obtain the optimal intervals on which the Green's functions are of one sign. The main tool is the disconjugacy theory. As an application of our results, we develop a monotone iteration method to obtain positive solutions of the nonlinear problem $u'''+Mu''+f(t,u)=0$, $u(0)=u'(0)=u(1)=0$.

1. **Introduction.** Third-order differential equations arise in an important number of physical problems, such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves and gravity-driven flows [7]. The existence and multiplicity of solutions of boundary value problems of nonlinear third-order differential equations have been studied by many authors in the last three decades, see [1], [2], [5], [8]-[15] and the references therein.

Positivity of linear operators plays a very important role in the study of the corresponding nonlinear problems. It is well-known that the second order operator

$$u''+ru, \quad u\in\{v\in C^2[0,1]\,|\,v(0)=v(1)=0\} \tag{1}$$

is positive if and only if

$$r<\pi^2.$$

Recently, Cabada and Enguiça [3] showed that the fourth order operator

$$u''''+ru, \quad u\in\{v\in C^4[0,1]\,|\,v(0)=v(1)=v'(0)=v'(1)=0\} \tag{2}$$

is nonnegative if and only if $-\rho_1^4\le r\le\rho_0^4$, where

$$\rho_1\approx 4.73004, \quad \rho_0\approx 5.553.$$

However, relatively little is known about the third order linear operators $u'''+Mu$.

It is the purpose of this paper to make an exhaustive study of the several third order linear operators coupled with $(k,3-k)$-conjugate boundary conditions, where

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$k = 1, 2$. We will determine the optimal interval of M in which the sign of Green's functions are of one sign:

$$\begin{aligned}
&L_1u = u''' + Mu, \quad u \in \{v \in C^3[0,1] \,|\, v(0) = v'(0) = v(1) = 0\},\\
&L_2u = u''' + Mu', \quad u \in \{v \in C^3[0,1] \,|\, v(0) = v'(0) = v(1) = 0\},\\
&L_3u = u''' + Mu'', \quad u \in \{v \in C^3[0,1] \,|\, v(0) = v'(0) = v(1) = 0\},\\
&L_4u = u''' + Mu, \quad u \in \{v \in C^3[0,1] \,|\, v(0) = v(1) = v'(1) = 0\},\\
&L_5u = u''' + Mu', \quad u \in \{v \in C^3[0,1] \,|\, v(0) = v(1) = v'(1) = 0\},\\
&L_6u = u''' + Mu'', \quad u \in \{v \in C^3[0,1] \,|\, v(0) = v(1) = v'(1) = 0\}.
\end{aligned}$$

As an application of our results on linear problems, we develop a monotone iteration method to show the existence of positive solutions of the nonlinear problem

$$\begin{aligned}
&u''' + Mu'' + f(t,u) = 0, \quad t \in (0,1),\\
&u(0) = u'(0) = u(1) = 0,
\end{aligned} \tag{3}$$

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ satisfies some suitable conditions.

The rest of this paper is arranged as follows: In Section 2, we state some preliminary results about disconjugacy. In Section 3, we state and prove our results on the sign of Green's function of third-order linear operators via disconjugacy theory, and obtain the optimal intervals on which the Green's functions are of one sign. Finally in Section 4, we apply our results on linear problems to develop a monotone method for (3).

2. **Preliminaries.** Disconjugacy theory is crucial for study the positivity of linear differential operators. Since the required results are somewhat scattered in [4], [6], we give some details here.

Definition 2.1 ([4], Page 1)**.** Let $p_k \in C[a,b]$ for $k = 1, \cdots, n$. A linear differential equation of order n

$$Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \tag{4}$$

is said to be *disconjugate* on an interval $[a,b]$ if every nontrivial solution has less than n zeros on $[a,b]$, multiple zeros being accounted according to their multiplicity.

Definition 2.2 ([4])**.** The functions $y_1, \cdots, y_n \in C^n[a,b]$ are said to form a *Markov system* if the n Wronskians

$$W_k := W[y_1, \cdots, y_k] = \begin{vmatrix} y_1 & \cdots & y_k \\ \cdots & \cdots & \cdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix}, \qquad (k = 1, \cdots, n) \tag{5}$$

are positive throughout on $[a,b]$.

Lemma 2.3 ([4], Theorem 3 in Page 94)**.** *The equation* (4) *has a Markov fundamental system of solutions on* $[a,b]$ *if and only if it is disconjugate on* $[a,b]$.

Suppose (4) is disconjugate on $[a,b]$. Let f be a continuous function on $[a,b]$. Let k be a positive integer. Then the two-point boundary value problem

$$Ly = f(t), \qquad t \in (a,b), \tag{6}$$

$$\begin{aligned}
&y^{(i)}(a) = 0, \ i = 0, 1, \cdots, k-1,\\
&y^{(j)}(b) = 0, \ j = 0, 1, \cdots, n-k-1
\end{aligned} \tag{7}$$

has a unique solution y. The solution can be represented in the form

$$y(t) = \int_a^b G(t,s)f(s)ds,$$

where the Green's function $G(t,s)$ is defined by the properties

(1) as a function of t, $G(t,s)$ is a solution of (4) on $[a,s)$ and on $(s,b]$ and satisfies the n boundary conditions (7);

(2) as a function of t, $G(t,s)$ and its first $n-2$ derivatives are continuous at $t=s$, while

$$G^{(n-1)}(s+0,s) - G^{(n-1)}(s-0,s) = 1.$$

Lemma 2.4. *Suppose (4) is disconjugate on* $[a,b]$. *Then*

$$(-1)^{n-k}G(t,s) > 0, \qquad a < s < b, \qquad a < t < b. \tag{8}$$

Proof. It is a immediate consequence of [[4], Theorem 11 in Page 106]. See also [[6], Theorem 0.13]. □

Definition 2.5 ([4], Page 99)**.** Suppose that the equation (4) is not disconjugate on $[a,b]$. Let $\eta(a)$ be the supremum of all $c > a$ such that (4) is discojugate on $[a,c]$. We call $\eta(a)$ the first *right conjugate* of a.

For $k \in \{1,\cdots,n\}$, let $y_k(t,a)$ be the unique solution of the initial value problem

$$Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \tag{9}$$

$$y_k^{(n-k)}(a) = 1, \qquad y_k^{(n-j)}(a) = 0 \quad (j = 1,\cdots n;\ j \neq k). \tag{10}$$

We denote by

$$W[y_1,\cdots,y_k](t,a) := \begin{vmatrix} y_1(a,t) & \cdots & y_k(a,t) \\ \cdots & \cdots & \cdots \\ y_1^{(k-1)}(a,t) & \cdots & y_k^{(k-1)}(a,t) \end{vmatrix} \tag{11}$$

the Wronskian of $y_1(a,t),\cdots,y_k(a,t)$.

Definition 2.6 ([4], Page 99)**.** Let $w(a)$ be the least $s > a$ in $[a,b]$, if one exists, at which one of the Wronskian $W_1(s,a),\cdots,W_{n-1}(s,a)$ vanishes.

Remark 1. It may be note that $W_n(s,a)$, $s \in [a,\, b]$ never vanish because the solutions $y_1,\cdots,y_n$ are linearly independent.

Lemma 2.7 ([4], Page 99)**.** $\eta(a) = w(a)$.

Remark 2. It is easy to check that if we replace $y_k(t,a)$, $k \in \{1,\cdots,n\}$ with the solutions $z_k(t,a)$ of the problem

$$Lz \equiv z^{(n)} + p_1(t)z^{(n-1)} + \cdots + p_n(t)z = 0, \tag{12}$$

$$z_k^{(n-k)}(a) = d_k > 0, \qquad z_k^{(n-j)}(a) = 0 \quad (j = 1,\cdots n;\ j \neq k), \tag{13}$$

where $d_k \in (0,\infty)$ are constants, $k \in \{1,\cdots,n\}$. Then the conclusion of Lemma 2.7 is still true. In fact, $z_k(t,a) = d_k y_k(t,a)$, so

$$W[y_1,\cdots,y_k](t,a) = cW[z_1,\cdots,z_k](t,a)$$

for some constant $c > 0$.

3. Sign of Green's function of third-order linear problems.

3.1. Disconjugacy for the equations $u''' + Mu = 0$.

From Definition 2.1, it is easy to verify that for $M \in \mathbb{R}$,

$$u''' + Mu = 0 \text{ is disconjugate on } [0,1] \iff u''' - Mu = 0 \text{ is disconjugate on } [0,1]. \tag{14}$$

In fact, $u(t)$ is the solution of the equation $u'''(t) + Mu(t) = 0, \quad t \in [0,1]$, if and only if $u(1-t)$ is the solution of $v'''(t) - Mv(t) = 0, \quad t \in [0,1]$. Moreover, $\tau \in [0,1]$ is a zero of $u(t)$ in $[0,1]$ if and only if $1-\tau \in [0,1]$ is a zero of $u(1-t)$ in $[0,1]$. So the number of zeros of the two functions $u(t)$ and $u(1-t)$ in $[0,1]$ must be same. Therefore, (14) is valid.

Let m_1 be the smallest positive solution of the equation

$$\frac{1}{3}e^{-m} - \frac{1}{3}e^{\frac{m}{2}}\cos\frac{\sqrt{3}m}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}}\sin\frac{\sqrt{3}m}{2} = 0. \tag{15}$$

Then

$$m_1 \approx 4.23321.$$

Theorem 3.1. *For every $M \in (-m_1^3, m_1^3)$, the equation*

$$u'''(t) + Mu(t) = 0 \tag{16}$$

is disconjugate on $[0,1]$. Moreover, the result is optimal.

Proof. By (14), we only need to deal with the case that $M = m^3 \geq 0$, i.e.

$$u'''(t) + m^3u(t) = 0. \tag{17}$$

We will divide two cases to prove the conclusion.

Case 1 $m > 0$.

From Lemma 2.7 and Remark 2, we consider the initial value problem(IVP)

$$\begin{aligned} &u''' + m^3u = 0,\\ &u^{(j)}(0) = 0, \quad j \neq 3-k,\\ &u^{(3-k)}(0) = m^{3-k}, \end{aligned}$$

where $k = 1, 2, 3$. Denote the unique solution of the above problem by u_k. Then

$$\begin{aligned} u_1(t) &= \frac{1}{3}e^{-mt} - \frac{1}{3}e^{\frac{m}{2}t}\cos\frac{\sqrt{3}mt}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}t}\sin\frac{\sqrt{3}mt}{2};\\ u_2(t) &= -\frac{1}{3}e^{-mt} + \frac{1}{3}e^{\frac{m}{2}t}\cos\frac{\sqrt{3}mt}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}t}\sin\frac{\sqrt{3}mt}{2};\\ u_3(t) &= \frac{1}{3}e^{-mt} + \frac{2}{3}e^{\frac{m}{2}t}\cos\frac{\sqrt{3}mt}{2}. \end{aligned}$$

Obviously, we have $W_1[u_1](0) = 0, W_2[u_1, u_2](0) = 0$. In order to construct a Markov system for (17) on $[0,1]$, we have to replace t with $t+\sigma$ in u_k, where $\sigma \in (0,1)$ is a small constant to be determined later. Define

$$y_1(t) = u_1(t+\sigma),\ y_2(t) = -u_2(t+\sigma),\ y_3(t) = u_3(t+\sigma), \tag{18}$$

we claim that $\{y_k(t)\}_{k=1}^3$ form a Markov system if $m \in (0, m_1)$ and σ is small enough.

In fact, for $t \in [0,1]$,

$$W_1[y_1](t) = \frac{1}{3}e^{-m(t+\sigma)} - \frac{1}{3}e^{\frac{m}{2}(t+\sigma)}\cos\frac{\sqrt{3}m(t+\sigma)}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}(t+\sigma)}\sin\frac{\sqrt{3}m(t+\sigma)}{2};$$

$$W_2[y_1,y_2](t) = m\Big[\frac{1}{3}e^{m(t+\sigma)} - \frac{1}{3}e^{-\frac{m}{2}(t+\sigma)}\cos\frac{\sqrt{3}m(t+\sigma)}{2} - \frac{\sqrt{3}}{3}e^{-\frac{m}{2}(t+\sigma)}\sin\frac{\sqrt{3}m(t+\sigma)}{2}\Big];$$

$$W_3[y_1,y_2,y_3](t) = m^3.$$

It follows from (15) that there exists $\sigma \in (0,1)$ such that

$$W_1[y_1](t) > 0, \quad W_2[y_1,y_2](t) > 0, \quad W_3[y_1,y_2,y_3](t) > 0, \qquad t \in [0,1].$$

Now, from the Definition 2.6 and Lemma 2.7, m_1 should be the smallest positive number of the zeros of $W_1(1) = 0$ and $W_2(1) = 0$. By computing, $W_2(1) > 0$, and consequently, m_1 is the smallest positive zero of $W_1(1) = 0$.

Case 2 $m = 0$

In this case, for each fixed $k \in \{1,2,3\}$, let v_k be the unique solution of the initial value problem

$$v''' = 0,$$
$$v^{(j)}(0) = 0, \qquad j \neq 3-k,$$
$$v^{(3-k)}(0) = 1.$$

Then

$$v_1(t) = \frac{1}{2}t^2; \quad v_2(t) = t; \quad v_3(t) = 1.$$

Applying the similar method to construct Markov system (18), let us define

$$z_1(t) = v_1(t+\sigma),\ z_2(t) = -v_2(t+\sigma),\ z_3(t) = v_3(t+\sigma),$$

where $\sigma \in (0,1)$ is a small constant. Then

$$W_1[z_1](t) = \frac{1}{2}(t+\sigma)^2 > 0; \quad W_2[z_1,z_2](t) = \frac{1}{2}(t+\sigma)^2 > 0; \quad W_3[z_1,z_2,z_3](t) = 1 \tag{19}$$

for any $t \in [0,1]$. From (19), it follows that the functions $\{z_1,\ z_2,\ z_3\}$ form a Markov fundamental system of solutions of (17) on $[0,1]$ if $m = 0$ and $\sigma \in (0,1)$ is small enough.

Hence, for every $M \in (-m_1^3, m_1^3)$, the equation (16) is disconjugate on $[0,1]$. Moreover, from Lemma 2.7, it follows that the result is optimal. □

3.2. **Disconjugacy for the equations $u''' + Mu' = 0$.** Let m_2 be the smallest positive solution of the equation

$$\cos m - 1 = 0.$$

Then

$$m_2 = 2\pi.$$

Theorem 3.2. *For every $m \in (0, m_2)$, the equation*

$$u'''(t) + m^2u'(t) = 0 \tag{20}$$

is disconjugate on $[0,1]$. Moreover, the result is optimal.

Proof. For each fixed $k \in \{1, 2, 3\}$, let u_k be the unique solution of the initial value problem

$$\begin{aligned} &u''' + m^2 u' = 0, \\ &u^{(j)}(0) = 0, \quad j \neq 3 - k, \\ &u^{(3-k)}(0) = m^{3-k}. \end{aligned}$$

Then

$$\begin{aligned} &u_1(t) = 1 - \cos(mt); \\ &u_2(t) = \sin(mt); \\ &u_3(t) = 1. \end{aligned}$$

Applying the similar method to construct Markov system (18), let us define

$$y_1(t) = u_1(t+\sigma), \ y_2(t) = -u_2(t+\sigma), \ y_3(t) = u_3(t+\sigma),$$

where $\sigma \in (0, 1)$ is a small constant. Then

$$\begin{aligned} &W_1[y_1](t) = 1 - \cos(m(t+\sigma)) > 0; \\ &W_2[y_1, y_2](t) = m[1 - \cos(m(t+\sigma))] > 0; \\ &W_3[y_1, y_2, y_3](t) = m^3 > 0 \end{aligned} \tag{21}$$

for any $t \in [0, 1]$. From (21), it follows that the functions $\{y_1, \ y_2, \ y_3\}$ form a Markov system of solutions of (20) on $[0, 1]$ if $0 < m < m_2$ and $\sigma \in (0, 1)$ is small enough.

Hence, for every $m \in (0, m_2)$, the equation (20) is disconjugate on $[0, 1]$. Moreover, from Lemma 2.7, it follows that the result is optimal. □

Theorem 3.3. *For every $m \in (0, \infty)$, the equation*

$$u'''(t) - m^2 u'(t) = 0 \tag{22}$$

is disconjugate on $[0, 1]$. Moreover, the result is optimal.

Proof. For each fixed $k \in \{1, 2, 3\}$, let v_k be the unique solution of the initial value problem

$$\begin{aligned} &v''' - m^2 v' = 0, \\ &v^{(j)}(0) = 0, \quad j \neq 3 - k, \\ &v^{(3-k)}(0) = m^{3-k}. \end{aligned}$$

Then

$$\begin{aligned} &v_1(t) = \cosh(mt) - 1; \\ &v_2(t) = \sinh(mt); \\ &v_3(t) = 1. \end{aligned}$$

Applying the similar method to construct Markov system (18), let us define

$$z_1(t) = v_1(t+\sigma), \ z_2(t) = -v_2(t+\sigma), \ z_3(t) = v_3(t+\sigma),$$

where $\sigma \in (0, 1)$ is a small constant. Then

$$\begin{aligned} &W_1[z_1](t) = \cosh(m(t+\sigma)) - 1 > 0; \\ &W_2[z_1, z_2](t) = m[\cosh(m(t+\sigma)) - 1] > 0; \\ &W_3[z_1, z_2, z_3](t) = m^3 > 0 \end{aligned} \tag{23}$$

for any $t \in [0,1]$. From (23), it follows that the functions $\{z_1,\ z_2,\ z_3\}$ form a Markov system of solutions of (22) on $[0,1]$ if $m \in (0,\infty)$ and $\sigma \in (0,1)$ is small enough.

Hence, for every $m \in (0,\infty)$, the equation (22) is disconjugate on $[0,1]$. Moreover, from Lemma 2.7, it follows that the result is optimal. □

Note that Theorem 3.2 and Theorem 3.3 remains valid for $m = 0$, it has been proved in Case 2 of Theorem 3.1, so for every $M \in (-\infty, 2\pi)$, the equation $u''' + Mu' = 0$ is disconjugate on $[0,1]$. Moreover, the interval of $M \in (-\infty, 2\pi)$ is optimal.

3.3. **Disconjugacy for the equations** $u'''+Mu''=0$**.** Applying the same method to get (14) with obvious changes, we may get that for $M \in \mathbb{R}$,

$$u'''+Mu''=0 \text{ is disconjugate on } [0,1] \iff u'''-Mu''=0 \text{ is disconjugate on } [0,1]. \tag{24}$$

Let m_3 be the smallest positive solution of the equation

$$me^{-m} + e^{-m} - 1 = 0.$$

Then

$$m_3 = \infty. \tag{25}$$

Theorem 3.4. *For every* $M \in (-\infty,\infty)$*, the equation*

$$u'''(t) + Mu''(t) = 0 \tag{26}$$

is disconjugate on $[0,1]$*. Moreover, the result is optimal.*

Proof. Obviously, in the case $M = 0$, the conclusion holds as in Case 2 of Theorem 3.1. By (24), we only need to deal with the case that $M = m > 0$, i.e.

$$u'''(t) + mu''(t) = 0. \tag{27}$$

For each fixed $k \in \{1,2,3\}$, let u_k be the unique solution of the initial value problem

$$\begin{aligned}
&u''' + mu'' = 0,\\
&u^{(j)}(0) = 0, \quad j \neq 3-k,\\
&u^{(3-k)}(0) = m^{3-k}.
\end{aligned}$$

Then

$$\begin{aligned}
&u_1(t) = e^{-mt} + mt - 1;\\
&u_2(t) = mt;\\
&u_3(t) = 1.
\end{aligned}$$

Applying the similar method to construct Markov system (18), let us define

$$y_1(t) = u_1(t+\sigma),\ y_2(t) = -u_2(t+\sigma),\ y_3(t) = u_3(t+\sigma),$$

where $\sigma \in (0,1)$ is a small constant. Then

$$\begin{aligned}
&W_1[y_1](t) = e^{-m(t+\sigma)} + m(t+\sigma) - 1 > 0;\\
&W_2[y_1,y_2](t) = -m[m(t+\sigma)e^{-m(t+\sigma)} + e^{-m(t+\sigma)} - 1] > 0;\\
&W_3[y_1,y_2,y_3](t) = m^3 > 0
\end{aligned} \tag{28}$$

for any $t \in [0,1]$. From (28), it follows that the functions $\{y_1,\ y_2,\ y_3\}$ form a Markov system of solutions of (27) on $[0,1]$ if $0 < m < \infty$ and $\sigma \in (0,1)$ is small enough.

Hence, for every $M \in (-\infty,\infty)$, the equation (26) is disconjugate on $[0,1]$. Moreover, from Lemma 2.7, it follows that the result is optimal. $\square$

3.4. **Sign of Green's functions.** Now, we are in the position to study the sign of Green's function G_j associated L_j, $j = 1, \cdots, 6$.

In the case $k = 2$ and $n = 3$, (7) reduces to

$$u(0) = u'(0) = u(1) = 0.$$

In the case $k = 1$ and $n = 3$, (7) reduces to

$$u(0) = u(1) = u'(1) = 0.$$

Now, from Theorem 3.1-3.4 and Lemma 2.4, we deduce that

Theorem 3.5. *(1) Let $M \in (-m_1^3, m_1^3)$. Then $G_1(t,s) < 0,\ G_4(t,s) > 0, \quad (t,s) \in (0,1)\times(0,1)$;*

(2) Let $M \in (-\infty, m_2^2)$. Then $G_2(t,s) < 0,\ G_5(t,s) > 0, \quad (t,s) \in (0,1)\times(0,1)$;

(3) Let $M \in (-\infty,\infty)$. Then $G_3(t,s) < 0,\ G_6(t,s) > 0, \quad (t,s) \in (0,1)\times(0,1)$.

The Green's functions G_j can be explicitly given via the method introduced in [[4], 105-106].

Theorem 3.6. *(1) Let $m \in (-m_1, m_1)$. Then the Green's function of the problem*

$$\begin{aligned} &u^{(3)}(t) + m^3 u(t) = 0, \quad t \in (0,\ 1),\\ &u(0) = u'(0) = u(1) = 0 \end{aligned} \tag{29}$$

can be explicitly given by

$$G_1(t,s) = \begin{cases} K_1(t,s), & 0 \le t \le s \le 1,\\ K_2(t,s), & 0 \le s \le t \le 1,\end{cases}$$

where

$$\begin{aligned} K_1(t,s) = {} & \frac{[-\frac{1}{3}e^{-m(1-s)} + \frac{1}{3}e^{\frac{m}{2}(1-s)}\cos\frac{\sqrt{3}}{2}m(1-s)) - \frac{\sqrt{3}}{3}e^{\frac{m}{2}(1-s)}\sin\frac{\sqrt{3}}{2}m(1-s))]}{m^2[\frac{1}{3}e^{-m} - \frac{1}{3}e^{\frac{m}{2}}\cos\frac{\sqrt{3}m}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}}\sin\frac{\sqrt{3}m}{2}]}\\ & \times [\frac{1}{3}e^{-mt} - \frac{1}{3}e^{\frac{m}{2}t}\cos\frac{\sqrt{3}mt}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}t}\sin\frac{\sqrt{3}mt}{2}],\\ K_2(t,s) = {} & \frac{1}{m^2[\frac{1}{3}e^{-m} - \frac{1}{3}e^{\frac{m}{2}}\cos\frac{\sqrt{3}m}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}}\sin\frac{\sqrt{3}m}{2}]}\\ & \times \Big\{\frac{1}{9}e^{-mt}e^{\frac{m}{2}(1-s)}[\cos\frac{\sqrt{3}}{2}m(1-s) - \sqrt{3}\sin\frac{\sqrt{3}}{2}m(1-s)]\\ & + \frac{1}{9}e^{-m}e^{\frac{m}{2}(t-s)}[\sqrt{3}\sin\frac{\sqrt{3}}{2}m(t-s) - \cos\frac{\sqrt{3}}{2}m(t-s)]\\ & + \frac{1}{9}e^{-m(1-s)}e^{\frac{m}{2}t}[\cos\frac{\sqrt{3}}{2}mt - \sqrt{3}\sin\frac{\sqrt{3}}{2}mt]\\ & + \frac{1}{9}e^{\frac{m}{2}}e^{-m(t-s)}[-\cos\frac{\sqrt{3}}{2}m + \sqrt{3}\sin\frac{\sqrt{3}}{2}m]\\ & - \frac{4}{9}e^{\frac{m}{2}}e^{\frac{m}{2}(t-s)}\sin\frac{\sqrt{3}}{2}ms\sin\frac{\sqrt{3}}{2}m(1-t)]\Big\}. \end{aligned}$$

The Green's function of the problem

$$\begin{aligned} &u^{(3)}(t) + m^3 u(t) = 0, \quad t \in (0,\ 1),\\ &u(0) = u'(0) = u(1) = 0 \end{aligned}$$

can be explicitly given by

$$G_4(t,s)=\begin{cases} K_3(t,s), & 0\le t\le s\le 1,\\ K_4(t,s), & 0\le s\le t\le 1,\end{cases}$$

where

$$\begin{aligned}K_3(t,s)=&\frac{1}{m^2[\frac{1}{3}e^m-\frac{1}{3}e^{-\frac{m}{2}}\cos\frac{\sqrt{3}m}{2}-\frac{\sqrt{3}}{3}e^{-\frac{m}{2}}\sin\frac{\sqrt{3}m}{2}]}\\ &\times\Big\{-\frac{1}{9}e^{ms}e^{-\frac{m}{2}(1-t)}[\sqrt{3}\sin\frac{\sqrt{3}}{2}m(1-t)+\cos\frac{\sqrt{3}}{2}m(1-t)]\\ &+\frac{1}{9}e^{m}e^{-\frac{m}{2}(s-t)}[\sqrt{3}\sin\frac{\sqrt{3}}{2}m(s-t)+\cos\frac{\sqrt{3}}{2}m(s-t)]\\ &-\frac{1}{9}e^{m(1-t)}e^{-\frac{m}{2}s}[\cos\frac{\sqrt{3}}{2}ms+\sqrt{3}\sin\frac{\sqrt{3}}{2}ms]\\ &+\frac{1}{9}e^{-\frac{m}{2}}e^{m(s-t)}[\cos\frac{\sqrt{3}}{2}m+\sqrt{3}\sin\frac{\sqrt{3}}{2}m]\\ &+\frac{4}{9}e^{-\frac{m}{2}}e^{-\frac{m}{2}(s-t)}\sin\frac{\sqrt{3}}{2}m(1-s)\sin\frac{\sqrt{3}}{2}mt\Big\},\end{aligned}$$

and

$$\begin{aligned}K_4(t,s)=&\frac{[\frac{1}{3}e^{ms}-\frac{1}{3}e^{-\frac{m}{2}s}\cos\frac{\sqrt{3}}{2}ms-\frac{\sqrt{3}}{3}e^{-\frac{m}{2}s}\sin\frac{\sqrt{3}}{2}ms]}{m^2[\frac{1}{3}e^m-\frac{1}{3}e^{-\frac{m}{2}}\cos\frac{\sqrt{3}m}{2}-\frac{\sqrt{3}}{3}e^{-\frac{m}{2}}\sin\frac{\sqrt{3}m}{2}]}\\ &\times\Big[\frac{1}{3}e^{m(1-t)}-\frac{1}{3}e^{-\frac{m}{2}(1-t)}\cos\frac{\sqrt{3}m(1-t)}{2}-\frac{\sqrt{3}}{3}e^{-\frac{m}{2}(1-t)}\sin\frac{\sqrt{3}m(1-t)}{2}\Big].\end{aligned}$$

(2) Let $M\in(-\infty,m_2^2)$. If $M=m^2\in(0,m_2^2)$, then the Green's function of the problem

$$\begin{aligned}&u^{(3)}(t)+m^2u'(t)=0,\quad t\in(0,\ 1),\\ &u(0)=u'(0)=u(1)=0\end{aligned}$$

can be explicitly given by

$$G_2(t,s)=\begin{cases}\frac{\sin(ms)[\sin m-\sin(mt)-\sin m(1-t)]+(\cos(mt)-\cos m)(1-\cos(ms))}{m^2(1-\cos m)}, & 0\le s\le t\le 1,\\ \frac{[\cos m(1-s)-1](1-\cos(mt))}{m^2(1-\cos m)}, & 0\le t\le s\le 1.\end{cases}$$

The Green's function of the problem

$$\begin{aligned}&u^{(3)}(t)+m^2u'(t)=0,\quad t\in(0,\ 1),\\ &u(0)=u(1)=u'(1)=0\end{aligned}$$

can be explicitly given by

$$G_5(t,s)=\begin{cases}\frac{[1-\cos m(1-t)](1-\cos(ms))}{m^2(1-\cos m)}, & 0\le s\le t\le 1,\\ \frac{\sin(mt)[\sin(ms)+\sin m(1-s)-\sin m]+(\cos m-\cos(ms))(1-\cos(mt))}{m^2(1-\cos m)}, & 0\le t\le s\le 1.\end{cases}$$

If $M=-m^2<0$, then the Green's function of the problem

$$\begin{aligned}&u^{(3)}(t)-m^2u'(t)=0,\quad t\in(0,\ 1),\\ &u(0)=u'(0)=u(1)=0\end{aligned}$$

can be explicitly given by

$$G_2(t,s)=\begin{cases}\frac{\sinh(ms)[\sinh(mt)+\sinh m(1-t)-\sinh m]+(\cosh m-\cosh(mt))(\cosh(ms)-1)}{m^2(\cosh m-1)}, & s\le t,\\ \frac{[1-\cosh m(1-s)](\cosh(mt)-1)}{m^2(\cosh m-1)}, & t\le s.\end{cases}$$

The Green's function of the problem

$$u^{(3)}(t)-m^2u'(t)=0,\quad t\in(0,\ 1),$$
$$u(0)=u(1)=u'(1)=0$$

can be explicitly given by

$$G_5(t,s)=\begin{cases}\frac{[\cosh m(1-t)-1](\cosh(ms)-1)}{m^2(\cosh m-1)}, & s\le t,\\ \frac{\sinh(mt)[\sinh m-\sinh(ms)-\sinh m(1-s)]+(\cosh(ms)-\cosh m)(\cosh(mt)-1)}{m^2(\cosh m-1)}, & t\le s.\end{cases}$$

(3) Let $M\in(-\infty,\infty)$. Then the Green's function of the problem

$$u^{(3)}(t)+mu''(t)=0,\quad t\in(0,\ 1),$$
$$u(0)=u'(0)=u(1)=0$$

can be explicitly given by

$$G_3(t,s)=\begin{cases}K_5(t,s) & 0\le s\le t\le 1,\\ K_6(t,s) & 0\le t\le s\le 1,\end{cases}$$

where

$$\begin{aligned}K_5(t,s)=&\frac{1}{e^{-m}+m-1}\big[(m^2s-m^2+m)e^{-m(t+s)}+(mt-1)[m+m^2(s-1)]e^{-ms}\\&-me^{-m(1+t)}+m(1-mt)e^{-m}\big],\\ K_6(t,s)=&\frac{1}{e^{-m}+m-1}\big[(m^2s-m^2+m)e^{-m(t+s)}+m^3s(t-1)e^{-ms}\\&+(m^2-m)e^{-mt}+[m^2(t-s)-m]e^{-m(1+s)}+(m-m^2t)e^{-m}\big].\end{aligned}$$

The Green's function of the problem

$$u^{(3)}(t)+mu''(t)=0,\quad t\in(0,\ 1),$$
$$u(0)=u(1)=u'(1)=0$$

is

$$G_6(t,s)=\begin{cases}K_7(t,s) & 0\le s\le t\le 1,\\ K_8(t,s) & 0\le t\le s\le 1,\end{cases}$$

where

$$\begin{aligned}K_7(t,s)=&\frac{1}{e^{-m}(1+m)-1}\big[(m^2s+m)e^{-m(t+s)}+[m^2(t-s)-m]e^{-ms}\\&-(m^2+m)e^{-m(1+t)}+m^3t(s-1)e^{-m(1+s)}+[m+m^2(1-t)]e^{-m}\big],\\ K_8(t,s)=&\frac{1}{e^{-m}(1+m)-1}\big[(m^2s+m)e^{-m(t+s)}-me^{-mt}+(m^2s+m)(mt-m-1)\\&\times e^{-m(1+s)}+[m^2(1-t)+m]e^{-m}\big].\end{aligned}$$

Finally, it is worth remarking that the intervals in Theorem 3.5 are optimal. This can be deduced from the following

Theorem 3.7. *(1) Let $m^* \in (m_1, \infty)$ be such that $u''' + {m^*}^3 u = 0$ has no right conjugate point in $(\frac{m_1}{m^*}, 1)$, i.e. $\eta(\frac{m_1}{m^*}) \geq 1$. Then for $m \in (m_1, m^*)$ and $s \in (\frac{m_1}{m}, 1)$, we have*

$$G_i\big(\frac{m}{m_1}, s\big) = 0, \qquad \frac{d}{dt} G_i\big(\frac{m}{m_1}, s\big) \neq 0, \qquad i = 1, 4.$$

(2) Let $\hat{m} \in (m_2, \infty)$ be such that $u''' + \hat{m}^2 u' = 0$ has no right conjugate point in $(\frac{m_2}{\hat{m}}, 1)$, i.e. $\eta(\frac{m_2}{\hat{m}}) \geq 1$. Then for $m \in (m_2, \hat{m})$ and $s \in (\frac{m_2}{m}, 1)$,

$$G_j\big(\frac{m_2}{m}, s\big) = 0, \qquad \frac{d}{dt} G_j\big(\frac{m_2}{m}, s\big) \neq 0, \qquad j = 2, 5.$$

Proof. We only show that the results are valid for the Green's function G_1 of (29). The other cases can be treated by the same method.

To show G_1 changes its sign in $[0,1] \times [0,1]$, it is enough to show that $K_1(t,s)$ changes its sign on $\{(t,s) \in [0,1] \times [0,1] \,|\, t \leq s\}$. Since $m \in (m_1, m^*)$ and $s \in (\frac{m_1}{m}, 1)$, it follows from (15) that

$$\begin{aligned} G_1(\frac{m_1}{m}, s) = &\frac{[-\frac{1}{3}e^{-m(1-s)} + \frac{1}{3}e^{\frac{m}{2}(1-s)} \cos \frac{\sqrt{3}}{2}m(1-s)) - \frac{\sqrt{3}}{3}e^{\frac{m}{2}(1-s)} \sin \frac{\sqrt{3}}{2}m(1-s))]}{m^2[\frac{1}{3}e^{-m} - \frac{1}{3}e^{\frac{m}{2}} \cos \frac{\sqrt{3}m}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}} \sin \frac{\sqrt{3}m}{2}]} \\ &\times [\frac{1}{3}e^{-m_1} - \frac{1}{3}e^{\frac{m_1}{2}} \cos \frac{\sqrt{3}m_1}{2} + \frac{\sqrt{3}}{3}e^{\frac{m_1}{2}} \sin \frac{\sqrt{3}m_1}{2}] = 0. \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial G_1}{\partial t}(\frac{m_1}{m}, s) = &\frac{[-\frac{1}{3}e^{-m(1-s)} + \frac{1}{3}e^{\frac{m}{2}(1-s)} \cos \frac{\sqrt{3}}{2}m(1-s)) - \frac{\sqrt{3}}{3}e^{\frac{m}{2}(1-s)} \sin \frac{\sqrt{3}}{2}m(1-s))]}{m[\frac{1}{3}e^{-m} - \frac{1}{3}e^{\frac{m}{2}} \cos \frac{\sqrt{3}m}{2} + \frac{\sqrt{3}}{3}e^{\frac{m}{2}} \sin \frac{\sqrt{3}m}{2}]} \\ &\times [-\frac{1}{3}e^{-m_1} + \frac{1}{3}e^{\frac{m_1}{2}} \cos \frac{\sqrt{3}m_1}{2} + \frac{\sqrt{3}}{3}e^{\frac{m_1}{2}} \sin \frac{\sqrt{3}m_1}{2}] \neq 0, \end{aligned}$$

it follows that $G_1(t,s)$ must change its sign in any small neighborhood of $\big(\frac{m_1}{m}, s\big)$. $\square$

4. **Some applications.**

As applications of the results in previous sections, let us consider the existence of positive solutions of the nonlinear third-order boundary value problem

$$\begin{aligned} &u'''(t) + Mu''(t) + f(t, u(t)) = 0, \ 0 < t < 1, \\ &u(0) = u'(0) = u(1) = 0. \end{aligned} \tag{30}$$

Definition 4.1. We say that $\alpha \in C^3[0,1]$ is a lower solution of (30) if α satisfies

$$\begin{aligned} &\alpha'''(t) + M\alpha''(t) + f(t, \alpha(t)) \geq 0, \ 0 < t < 1, \\ &\alpha(0) \leq 0, \ \alpha'(0) \leq 0, \ \alpha(1) \leq 0. \end{aligned} \tag{31}$$

We say that $\beta \in C^3[0,1]$ is an upper solution of (30) if β the reversed inequalities of the definition of lower solution.

Lemma 4.2. *Let $h \in C([0,1], [0,\infty))$, r_1, r_2, $r_3 \geq 0$ are constants. For every fixed $M \in (-\infty, \infty)$, if $u \in C^3[0,1]$ and satisfies*

$$\begin{aligned} &u'''(t) + Mu''(t) + h(t) = 0, \quad t \in (0,1), \\ &u(0) = r_1, \ u'(0) = r_2, \ u(1) = r_3. \end{aligned} \tag{32}$$

Then $u \geq 0$ on $[0,1]$.

Proof. It's easy to verify that such a u can be given by the expression

$$u(t) = \int_0^1 (-1)G_3(t,s)h(s)ds + R(t), \qquad t \in [0,1],$$

where

$$R(t) = \begin{cases} r_1 + r_2 t + \dfrac{r_3 - r_1 - r_2}{e^{-M} + M - 1}[e^{-Mt} + Mt - 1], & M \neq 0, \quad t \in [0,1], \\ r_1 + r_2 t + (r_3 - r_2 - r_1)t^2, & M = 0, \quad t \in [0,1], \end{cases}$$

and $G_3(t,s)$ is the Green's function of L_3.

From Theorem 3.5, $G_3(t,s) \leq 0,\ (t,s) \in [0,1] \times [0,1]$. Now we only need to prove

$$R(t) \geq 0, \ t \in [0,1].$$

In fact, for $M \in (-\infty, \infty)$ and $M \neq 0$, it's clear to see that

$$R(t) = r_1\Big(1 - \frac{e^{-Mt} + Mt - 1}{e^{-M} + M - 1}\Big) + r_2\Big(t - \frac{e^{-Mt} + Mt - 1}{e^{-M} + M - 1}\Big) + r_3 \frac{e^{-Mt} + Mt - 1}{e^{-M} + M - 1},$$

$0 \leq \frac{e^{-Mt}+Mt-1}{e^{-M}+M-1} \leq 1$, and $t - \frac{e^{-Mt}+Mt-1}{e^{-M}+M-1} \geq 0$, for $t \in [0,1]$. It follows that $R(t) \geq 0$ for $t \in [0,1]$.

If $M = 0$, we have

$$R(t) = r_1(1 - t^2) + r^2 t(1-t) + r^3 t^2 \geq 0 \ \text{ for } t \in [0,1].$$

Thus, $u \geq 0,\ t \in [0,1]$. □

Theorem 4.3. *For every fixed $M \in (-\infty, \infty)$, suppose that $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and α, β are respectively a lower and upper solutions of (30), which satisfy $\alpha \leq \beta$ and*

$$f(t,u_1) \leq f(t,u_2), \quad \alpha(t) \leq u_1(t) \leq u_2(t) \leq \beta(t), \qquad t \in [0,1]. \tag{33}$$

Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, nondecreasing and non-increasing, respectively, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions of the problem (30) in $[\alpha, \beta]$.

Proof. Let $T : C[0,1] \to C[0,1]$ as follows

$$Tu(t) = \int_0^1 (-1)G_3(t,s)f(s,u(s))ds, \quad t \in [0,1].$$

Then T is a completely continuous operator such that Tu is the unique solution of the problem (30). Now, we divide the proof into three steps.

Step 1. We show

$$T(K) \subset K, \tag{34}$$

where $K = \{u \in C[0,1] \,|\, \alpha \leq u \leq \beta\}$ is a nonempty bounded closed subset in $C[0,1]$.

In fact, for $u \in K$, set $w = Tu(t)$. From the definitions of α, β and K, we have that

$$\begin{aligned} &(\beta - w)'''(t) + M(\beta - w)''(t) + [f(t,\beta(t)) - f(t,u(t))] \leq 0, \\ &(\beta - w)(0) \geq 0, \ (\beta - w)'(0) \geq 0, \ (\beta - w)(1) \geq 0. \end{aligned} \tag{35}$$

Using Lemma 4.2, we get that $\beta \geq w$. Analogously we can prove that $w \geq \alpha$. Thus, (34) holds.

Step 2. Let $u_1 = T\eta_1$, $u_2 = T\eta_2$, where $\eta_1,\ \eta_2 \in K$ satisfy $\alpha \leq \eta_1 \leq \eta_2 \leq \beta$. We show

$$u_1 \leq u_2.$$

In fact, let $v := u_2 - u_1$, it from Theorem 3.5 and (33) follows that

$$v(t) = T\eta_2 - T\eta_1 = \int_0^1 G_3(t,s)[f(t,\eta_1) - f(t,\eta_2)] \geq 0.$$

Step 3. The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence: $\alpha_0 = \alpha,\ \beta_0 = \beta,\ \alpha_n = T\alpha_{n-1},\ \beta_n = T\beta_{n-1},\ n = 1, 2, \cdots$. From the results of Step 1 and Step 2, we have

$$\alpha = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_0 = \beta. \tag{36}$$

Moreover, from the definition of T, we have

$$\begin{aligned} &\alpha_n''' + M\alpha_n'' + f(t,\alpha_{n-1}) = 0, \qquad t \in (0,1), \\ &\alpha_n(0) = \alpha_n'(0) = \alpha_n(1) = 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} &\beta_n''' + M\beta_n'' + f(t,\beta_{n-1}) = 0, \qquad t \in (0,1), \\ &\beta_n(0) = \beta_n'(0) = \beta_n(1) = 0. \end{aligned} \tag{38}$$

By combining (36) and (37), we can easily get that there is C_α depending only on α but not on n or t, such that $|\alpha_n| \leq C_\alpha$, so we know that $\{\alpha_n\}$ is bounded in $C[0,1]$. Similarly, $\{\beta_n\}$ is bounded in $C[0,1]$. Now, by using the fact that $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded in $C[0,1]$, we can conclude that $\{\alpha_n\}$ and $\{\beta_n\}$ converge uniformly to the extremal solutions in $[0,1]$ of the problem (30). □

Remark 3. Using the same argument, with obvious changes, we may develop the monotone method for nonlinear problems associated with L_j for $j = 1, 2, 4, 5, 6$.

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