COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **12**, Number **6**, November **2013**

doi:10.3934/cpaa.2013.12.2543

pp. **2543–2564**

DYNAMICS OF VACUUM STATES FOR ONE-DIMENSIONAL FULL COMPRESSIBLE NAVIER-STOKES EQUATIONS

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(Communicated by Alain Miranville)

Abstract. In this paper, we consider the properties of the vacuum states for weak solutions to one-dimensional full compressible Navier-Stokes system with viscosity and heat conductivities for general equation of states. Under weak conditions on initial data, we prove that if there is no vacuum initially then the vacuum states do not occur in a finite time. In particular, the temperature variation has no immediate effects on the formation of the vacuum. There are no assumptions on density in large sets. Furthermore, we prove that two initially non interacting vacuum regions will never touch in the future.

1. **Introduction and main result.** The full compressible Navier-Stokes system in one-dimensional space expresses the conservation of mass, momentum

$$\rho_t + (\rho u)_x = 0, \tag{1}$$

$$(\rho u)_t + (\rho u^2 + P)_x = \mu u_{xx} + \rho f, \tag{2}$$

and the balance of internal energy

$$(\rho e)_t + (\rho e u)_x - \kappa \theta_{xx} = (\mu u_x - P)u_x, \tag{3}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$; $\rho(x,t) \geq 0,\ u(x,t),\ \theta(x,t) \geq 0,\ P(\rho,\theta)$ and $e(\rho,\theta)$ denote the density, velocity, temperature, pressure and internal energy respectively; $f = f(x,t) \in L^1([0,T]; L^\infty_{loc}(\mathbb{R}))$ is an external force; $\mu \geq 0$ is the constant viscosity coefficient depending on the local properties of isotropic fluid and is a measure of the internal friction opposing deformation of the fluid; $\kappa \geq 0$ is the constant heat conductivity coefficient; $P, e \in C^1([0,\infty)^2)$, $\frac{\partial e}{\partial \theta}(\rho,\theta) > 0$ on $[0,\infty)^2$, $P(\rho,\theta)$ is non-decreasing with respect to ρ for all $\theta \geq 0$ and

$$e(\rho, 0) = 0 \text{ for } \rho \geq 0; \quad P(0,\theta) = 0 \text{ for } \theta \geq 0. \tag{4}$$

The one-dimensional full compressible Navier-Stokes system has been investigated by a great many authors in a large variety of contexts. For the system away from vacuum, the local existence of a classical solution was proved by Nash in [27].

2010 *Mathematics Subject Classification.* 35Q30, 35Q35, 76N15.

Key words and phrases. Compressible Navier-Stokes equations, vacuum states, nonbarotropic, viscous fluid, weak solutions.

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Uniqueness had previously been obtained by Serrin in [28]. The global existence of classical solutions was obtained in [19, 20, 21]. For initial data with discontinuity, Hoff in [11] and [12] proved the local existence of solutions and global existence of solutions with small initial data, and Chen-Hoff-Trivisa [4] obtained global existence of solutions in bounded domain for large data. Besides, the reader may consult Amosov and Zlotnick [1], Fujita-Yashima, Padula and Novotny [10], and Jiang and Zhang [18] and the references contained therein. For the multidimensional system, the global classical solutions were first obtained by Matsumura-Nishida [26] for initial data close to a non-vacuum equilibrium.

In the present of vacuum (i.e. ρ may vanish), Cho and Kim [6] constructed a local strong solution, as long as a suitable compatibility condition is satisfied initially. It should be note that solutions of the Navier-Stokes equations show certain instabilities when vacuum states are allowed even for isentropic case (see Hoff-Serre [13]). Moreover, there is no a priori estimate on the temperature for vacuum states generally, and it is not clear how to set up the problem when concerning this physical dependence on temperature. By now, most results about global well-posedness of the full Navier-Stokes equations are limited to the existence of solutions with special pressure. More precisely, for specific pressure laws excluding the perfect gas, Feireisl in [9] got the existence of so-called "variational" solutions in dimension $N \geq 2$, where the temperature equation is satisfied only as an inequality. This work is the first attempt towards the existence of weak solutions to the full compressible Navier-Stokes system for large initial data with vacuum. Later on, for a very particular form of the viscosity coefficients depending on the density, Bresch and Desjardins [2] obtained global weak solutions in $\mathbb{T}^3$ or $\mathbb{R}^3$. Recently, Huang and Li [15] establish the global existence of classical and weak solutions to the 3D full compressible Navier-Stokes system with small energy but possibly large oscillations. For more related problems, please refer to [8, 16, 17, 22, 23, 24, 25, 29].

It is still open whether the global solutions with possible different states at $x = \pm\infty$ exist for Cauchy problem of 1D full compressible Navier-Stokes equations containing vacuum with general pressure. In this case, investigating the dynamics of vacuum states is one of the most important aspects in studying the weak solutions and the regularity of solutions. Hoff-Smollar [14] showed that if there is no vacuum initially, the weak solutions to one-dimensional compressible Navier-Stokes equations do no exhibit vacuum states for special "potential energy density". And for N(≥ 3) dimensional spherically symmetric compressible Navier-Stokes system, Xin-Yuan [30] obtained the same result, which holds in the region away from the origin. This is one of several important differences between the Navier-Stokes equations and the inviscid Euler equations, for which vacuum states may in fact occur for large initial data and for certain equations of state (see [3, 5]).

Our main purpose here is to investigate the dynamics of vacuum states for weak solutions to one dimensional full compressible Navier-Stokes system with viscosity and heat conductivity (1)-(3). We prove that either near or away from $x = \pm\infty$, the vacuum states cannot appear provided there is no initial vacuum. We remove the restriction on the quantity "potential energy density" in [14], and only assume that the mass and energy densities of the fluid is integrable. There are no other constrains on the pressure P except the second principle of thermodynamics and the results hold for the solutions possibly having different states at $x = \pm\infty$. Furthermore, we prove that two separate vacuum regions will never meet each other in the future.

First, let us define the weak solution of problem (1)-(3).

Definition 1.1. We say that (ρ, u, θ) is a weak solution of (1)-(3) on $\mathbb{R} \times [0,T]$ provided that, ρ, ρu and $\rho e(\cdot,\cdot)$ are in $C([0,T]; H^{-1}_{loc}(\mathbb{R}))$ with (ρ,θ) nonnegative; $\rho(\cdot,t)$, $\rho u(\cdot,t)$ and $\rho e(\cdot,t)$ are in $L^1_{loc}(\mathbb{R})$ for each $t \in [0,T]$; $P, \rho u^2, \rho u e(\cdot,\cdot)$, $\theta_x(\cdot,\cdot)$, u_x and $(P-\mu u_x)u_x$ are in $L^1([-L,L]\times[0,T])$ for every L. And for all test functions $\varphi \in C^1_0(\mathbb{R}\times\mathbb{R})$,

$$\int \rho\varphi\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int [\rho\varphi_t + \rho u\varphi_x]dxdt, \tag{5}$$

$$\int \rho u\varphi\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int [\rho u(\varphi_x + u\varphi_x) + (P-\mu u_x)\varphi_x + \rho f\varphi]dxdt, \tag{6}$$

and

$$\int \rho e\varphi\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int [\rho e(\varphi_x + u\varphi_x) - (P-\mu u_x)u_x\varphi - \kappa\theta_x\varphi_x]dxdt, \tag{7}$$

for all $t_1, t_2 \in [0,T]$.

We assume that the mass is locally finite for all t. That is, for any $L > 0$, there is a constant $C = C(L)$ such that

$$\int_{-L}^{L} \rho(\cdot,t)dx \le C(L), \quad \text{for all } t \in [0,T]. \tag{8}$$

Also, we assume that

$$u \in L^1([0,T]; L^1_{loc}(\mathbb{R})), \quad u_x \in L^1([0,T]; L^2_{loc}(\mathbb{R})).$$

In particular,

$$u(\cdot,t) \in L^1_{loc}(\mathbb{R}),\ u_x(\cdot,t) \in L^2_{loc}(\mathbb{R}) \quad \text{for almost all } t \in [0,T]. \tag{9}$$

Then we assume that, there is a constant $C = C(L)$ and a positive function $\gamma(t) \in L^1([0,T])$ such that, for all $L > 0$ and almost all $t \in [0,T]$,

$$\int_{-L}^{L} |u(x,t)|dx \le (1+L)\gamma(t), \tag{10}$$

$$\left(\int_{-L}^{L} u_x^2(x,t)dx\right)^{\frac{1}{2}} \le (1+L^{1/4})\gamma(t), \tag{11}$$

and

$$\left(\int_{-L}^{L} [\rho(u^2+e)]dx\right)^{\frac{1}{2}} \le \gamma(t)C(L). \tag{12}$$

Finally, we assume that the momentum is locally finite, that is, for every $L > 0$,

$$\int_{-L}^{L} (\rho|u|)(x,t)dx \le C(L), \quad \text{for almost all } t \in [0,T]. \tag{13}$$

Remark 1. If the total energy is strengthened slightly to be locally finite for all t, we can replace the right side of (12) by some constant $C = C(L) > 0$ for all t. Then the assumption (13) follows immediately from (8) and (12); that is, finite local mass and energy would imply finite local momentum. Moreover, if the states of the initial data $U_0 = (\rho_0, u_0, \theta_0)$ at $x = \pm\infty$ are defined by $U_\pm = (\rho_\pm, u_\pm, \theta_\pm)$, we fix a smooth function $\overline{U}(x) = \big(\overline{\rho}(x), \overline{u}(x), \overline{\theta}(x)\big)$ satisfying $\overline{U}(x) = U_\pm, (\pm x \ge 1)$ such that $\triangle U_0 = U_0 - \overline{U} \in L^2(\mathbb{R})$, as in [11, 12]. Then the assumptions (10) and (11) hold as long as $u - \bar{u}(x), u_x$ belong to $L^1([0,T]; L^2(\mathbb{R}))$, which are satisfied by the solutions constructed in Theorem 1.3 in [11], Theorem 1.2 in [15], and more general.

Now, we state our main results:

Theorem 1.2. *Let* (ρ, u, θ) *be a global weak solution of (1)-(4) satisfying assumptions (9)-(11). If* $\int_E \rho(x,0)dx > 0$ *for every open set* $E \subset \mathbb{R}$*, then for any* $t \in (0, +\infty)$*, it holds that*

$$\int_L^{2L} \rho(x,t)dx > 0 \quad \text{and} \quad \int_{-2L}^{-L} \rho(x,t)dx > 0, \tag{14}$$

for all $L \in (L^*(t), +\infty)$*, where*

$$L^*(t) = max\left\{1, C_1\left(\int_0^t \gamma(t)\right)^4\right\}, \quad \text{for some } C_1 > 0. \tag{15}$$

Remark 2. The above theory shows that if there is no vacuum initially, then the vacuum states can only possibly occur in a finite set in the future. We also note that in [14], Hoff and Smoller assumed that ρ cannot be close to zero on a too large set; i.e., they defined a "potential energy density" function G and assumed that

$$G(\rho, x, t) \geq C_0^{-1}, \quad \text{for some } C_0 > 0,$$

and there exist some constants $C_1 > 0$ and $\theta \in [0, 1)$, such that

$$\int_{x_0}^{x_0+L} G(\rho, x, t)dx \leq C_1 + \theta C_0^{-1}L, \quad \text{for all } x_0,\ L \in \mathbb{R}.$$

In this paper, we remove the restriction on "potential energy density" and Theorem 1.2 holds for general one-dimensional full compressible Navier-Stokes system.

Based on Theorem 1.2, we have

Theorem 1.3. *Let* (ρ, u, θ) *be a weak solution of (1)-(4) on* $\mathbb{R} \times [0, T]$ *satisfying assumptions (9)-(13). If*

$$\int_E \rho(x,0)dx > 0 \tag{16}$$

for every open set $E \subset \mathbb{R}$*, then*

$$\int_E \rho(x,t)dx > 0$$

for every open subset $E \subset \mathbb{R}$ *and for every* $t \in [0, T]$.

The next observation is about vacuum appears on two or more open intervals.

Theorem 1.4. *Let* (ρ, u, θ) *be a weak solution to (1)-(4) on* $\mathbb{R} \times [0, T]$ *with any* $T < +\infty$ *satisfying assumptions as in Theorem 1.3. If there exist constants* $a_i, b_i,\ i = 1, 2$ *such that*

$$-\infty < a_1 < b_1 < a_2 < b_2 < +\infty,$$

$$\int_{a_1}^{b_1} \rho(0,x)dx = \int_{a_2}^{b_2} \rho(0,x)dx = 0,$$

$$\int_{a_1-\varepsilon}^{b_1} \rho(0,x)dx > 0, \ \int_{a_2}^{b_2+\varepsilon} \rho(0,x)dx > 0, \quad \text{for all } \varepsilon \in (0, +\infty),$$

and

$$\int_{a_1}^{b_1+\varepsilon} \rho(0,x)dx > 0, \int_{a_2-\varepsilon}^{b_2} \rho(0,x)dx > 0, \quad \textit{for all } \varepsilon \in (0, a_2 - b_1).$$

Then there exist $y_i(t),\ z_i(t) \in C^1[0,T],\ i = 1,2,$ *such that*

$$y_i(0) = a_i,\ z_i(0) = b_i, \quad i = 1,2,$$
$$-\infty < y_1(t) < z_1(t) < y_2(t) < z_2(t) < +\infty, \quad \forall t \in [0,T],$$

with the following properties:

$$\int_{y_1(t)}^{z_1(t)} \rho(x,t)dx = \int_{y_2(t)}^{z_2(t)} \rho(x,t)dx = 0,$$
$$\int_{y_1(t)-\varepsilon}^{z_1(t)} \rho(x,t)dx > 0, \int_{y_2(t)}^{z_2(t)+\varepsilon} \rho(x,t)dx > 0, \quad \textit{for any } \varepsilon \in (0,+\infty),$$
$$\int_{y_1(t)}^{z_1(t)+\varepsilon} \rho(x,t)dx > 0, \int_{y_2(t)-\varepsilon}^{z_2(t)} \rho(x,t)dx > 0, \quad \textit{for any } \varepsilon \in (0, y_2(t) - z_1(t)),$$

for all $t \in [0,T]$.

In particular,

$$\int_{z_1(t)}^{y_2(t)} \rho(\cdot,t)dx = \int_{b_1}^{a_2} \rho(0,\cdot)dx, \quad \forall t \in [0,T]. \tag{17}$$

Remark 3. Our results also allow the possibility that U_0 have different states at $x = \pm\infty$. Also, the proof in the following sections shows that the temperature has no immediate effect on the dynamic of vacuum states.

The paper is organized as this. In section 2, we show that the flow velocity glows at most linearly and an estimate on the evolution of a vacuum interval is given. In section 3, we investigate the vacuum states at $x = \pm\infty$ and prove Theorem 1.2. In section 4, we focus on the properties of vacuum states and prove Theorem 1.3 by contradiction argument. In section 4, we investigate two vacuum states without interaction and prove Theorem 1.4.

2. **Preliminary Lemmas.**

In this section, we list two elementary facts which are useful for later analysis.

Lemma 2.1. *Let* (ρ, u, θ) *be a global weak solution of (1)-(3) satisfying assumptions (9)-(11). Then,* $u \in L^1([0,T]; L^\infty_{loc}(\mathbb{R}))$*; in fact, there is a constant* $C > 0$ *such that for any* $L > 0$,

$$\|u(\cdot,t)\|_{L^\infty(-L,L)} \le C\gamma(t)(L+1)$$

for almost all $t \in [0,T]$*, where* $\gamma(t)$ *is the same as in assumptions (10)-(12).*

Proof. For any $L > 0$, we choose $\xi_L(x) \in C_0^\infty(\mathbb{R})$ such that

$$\xi_L(x) = \begin{cases} 0 \le \xi_L \le 1, \\ |\xi_L'| \le C, \\ \xi_L = 1, \quad x \in [-L,L]; \\ \xi_L = 0, \quad |x| \ge L+1. \end{cases}$$

From hypothesis (9), $u(\cdot,t)\in H^1_{loc}(\mathbb{R})$ for almost all $t\in[0,T]$, pick such a t. Then, we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(-L,L)} =& ess\sup_{x\in[-L,L]}|u(x,t)\xi_L(x)|\\ =& ess\sup_{x\in[-L,L]}\left|\int_{-L-1}^{x}\partial_y(\xi_L(y)u(y,t))dy\right|\\ \le& \int_{-L-1}^{L}|\xi_L'(y)u(y,t)|dy+\int_{-L-1}^{L}|\xi_L(y)u_y(y,t)|dy\\ \le& C\int_{-L-1}^{-L}|u(y,t)|dy+\int_{-L-1}^{L}|u_y(y,t)|dy\\ \le& C\int_{-L-1}^{-L}|u(y,t)|dy+C(1+L^{\frac12})\left(\int_{-L-1}^{L}|u_y(y,t)|^2dy\right)^{\frac12}\\ \le& C(1+L)\gamma(t),\end{aligned}$$

where in the last inequality, we use the hypothesis (10) and (11). □

Based on Lemma 2.1, we can give an estimate on the evolution of an open interval of vacuum states.

Lemma 2.2. *Let (ρ,u,θ) be a global weak solution of (1)-(4) satisfying assumptions (9)-(11). For $t_1<T$ and suppose that $\rho(\cdot,t_1)=0$ a.e. on an open interval (a,b) with $-\infty<a<b<\infty$. Let*

$$t_0=\inf\left\{t\in[0,t_1]:\int_t^{t_1}\|u(\cdot,s)\|_{L^\infty(a,b)}<\frac{b-a}{2}\right\}$$

and

$$t_2=\sup\left\{t\in[t_1,T]:\int_{t_1}^{t}\|u(\cdot,s)\|_{L^\infty(a,b)}<\frac{b-a}{2}\right\}.$$

Then $t_0<t_1<t_2$, and for any $t\in(t_0,t_2)$, $\rho(\cdot,t)=0$ on the interval

$$\left(a+\left|\int_{t_1}^{t}\|u\|_{L^\infty(a,b)}ds\right|,b-\left|\int_{t_1}^{t}\|u\|_{L^\infty(a,b)}ds\right|\right).$$

Proof. Since the positive function $\gamma(t)$ is integrable, Lemma 2.1 and the definition of t_0, t_2 show that $t_0<t_1<t_2$.

Now suppose $t>t_1$; the proof for $t<t_1$ is similar, so it will be omitted. Let u^ε be the usual spatial regularization of u. Then for almost all $t\in[t_1,T]$,

$$\|u^\varepsilon\|_\infty:=\|u^\varepsilon\|_{L^\infty(a-\varepsilon,b+\varepsilon)}\le\|u\|_{L^\infty(a,b)}=:\|u\|_\infty.$$

Now for fixed $\delta\in(0,(b-a)/6)$, define the smooth monotone function $w^{\varepsilon\delta}$ by

$$w^{\varepsilon\delta}(x,t)=\begin{cases}\|u^\varepsilon\|_\infty, & x<\frac{b+a}{2}-\delta;\\ -\|u^\varepsilon\|_\infty, & x>\frac{b+a}{2}+\delta,\end{cases}$$

and the smooth function $\Psi^\delta(x)$ by

$$\Psi^\delta(x)=\begin{cases}0, & x>b-\delta;\\ 1, & a+2\delta<x<b-2\delta;\\ 0, & x<a+\delta,\end{cases}$$

where Ψ^δ is increasing in $(a+\delta, a+2\delta)$, and decreasing in $(b-2\delta, b-\delta)$. Moreover, let $\phi^{\varepsilon\delta}$ be the smooth solution of the problem

$$\begin{cases} \phi_t + w^{\varepsilon\delta}\phi_x = 0, \quad t > t_1; \\ \phi(\cdot, t_1) = \Psi^\delta. \end{cases} \tag{18}$$

Consider the curves $x_r^\varepsilon = x_r^\varepsilon(t)$, defined by

$$\begin{cases} \frac{dx}{dt} = w^{\varepsilon\delta}, \\ x|_{t_1} = r, \quad r \in \mathbb{R}. \end{cases} \tag{19}$$

Set

$$\begin{aligned} V_1 =& \{(x,t) : -\infty < x < x_{a+\delta}^\varepsilon(t),\ t \in [t_1, T]\}, \\ V_2 =& \{(x,t) : x_{a+\delta}^\varepsilon(t) < x < x_{a+2\delta}^\varepsilon(t),\ t \in [t_1, T]\}, \\ V_3 =& \{(x,t) : x_{a+2\delta}^\varepsilon(t) < x < x_{b-2\delta}^\varepsilon(t),\ t \in [t_1, T]\}, \\ V_4 =& \{(x,t) : x_{b-2\delta}^\varepsilon(t) < x < x_{b-\delta}^\varepsilon(t),\ t \in [t_1, T]\}, \\ V_5 =& \{(x,t) : x_{b-\delta}^\varepsilon(t) < x < +\infty,\ t \in [t_1, T]\}. \end{aligned}$$

It is easy to check that

$$\phi^{\varepsilon\delta}(x,t) = \left\{ \begin{array}{ll} 0, & (x,t) \in V_1 \cup V_5; \\ 1, & \quad (x,t) \in V_3, \end{array} \right. \tag{20}$$

and

$$\phi_x^{\varepsilon\delta} \left\{ \begin{array}{ll} > 0, & (x,t) \in V_2; \\ < 0, & (x,t) \in V_4; \\ = 0, & \text{otherwise}. \end{array} \right. \tag{21}$$

Thus $\phi^{\varepsilon\delta}$ is a test function for the weak formulation of solution of (1)-(3). In particular, from (5) we have

$$\begin{aligned} \int_{a+\delta}^{b-\delta} \rho\phi^{\varepsilon\delta}\Big|_{t_1}^{t} &= \int_{a+\delta}^{b-\delta}\int_{t_1}^{t} \rho(\phi_t^{\varepsilon\delta} + u\phi_x^{\varepsilon\delta})dxdt \\ &= \int_{a+\delta}^{b-\delta}\int_{t_1}^{t} \rho(u - w^{\varepsilon\delta})\phi_x^{\varepsilon\delta}dxdt. \end{aligned}$$

Since $\rho(x, t_1) = 0$ a.e. on (a, b), we have

$$\begin{aligned} \int_{a+\delta}^{b-\delta} (\rho\phi^{\varepsilon\delta})(x,t)dx = &\int_{a+\delta}^{b-\delta}\int_{t_1}^{t} \rho(u^\varepsilon - w^{\varepsilon\delta})\phi_x^{\varepsilon\delta}dxdt \\ &+ \int_{a+\delta}^{b-\delta}\int_{t_1}^{t} \rho(u - u^\varepsilon)\phi_x^{\varepsilon\delta}dxdt. \end{aligned} \tag{22}$$

Now we define $T^{\varepsilon\delta}$ by

$$T^{\varepsilon\delta} = \sup\left\{ t \in [t_1, T] : x_{a+2\delta}^\varepsilon(s) < \frac{a+b}{2} - \delta,\ x_{b-2\delta}^\varepsilon(s) > \frac{a+b}{2} + \delta, \forall s \in [t_1, t] \right\}.$$

Without loss of generality, we may assume that $x^{\varepsilon}_{a+2\delta}(T^{\varepsilon\delta}) = \frac{a+b}{2} - \delta$, then we estimate $T^{\varepsilon\delta}$ from (19) that

$$\begin{aligned} \frac{b-a}{2} - 3\delta = (\frac{a+b}{2} - \delta) - (a+2\delta) &= \int_{t_1}^{T^{\varepsilon\delta}} w^{\varepsilon\delta} dt \\ &\leq \int_{t_1}^{T^{\varepsilon\delta}} \|u^{\varepsilon}\|_{\infty} dt \leq \int_{t_1}^{T^{\varepsilon\delta}} \|u\|_{\infty} dt. \end{aligned}$$

Now, we define T^{δ} by

$$T^{\delta} = \sup\left\{ t \in [t_1, T] : \int_{t_1}^{t} \|u\|_{\infty} < \frac{b-a}{2} - 3\delta \right\}, \tag{23}$$

then $T^{\varepsilon\delta} \geq T^{\delta}$.
Next, we will show

$$\rho(\cdot, t) = 0 \quad \text{a.e. on } I_{\delta}, \quad \text{for all } t \in [t_1, T^{\delta}], \tag{24}$$

where $I_{\delta}(t) = \left(a + 2\delta + \int_{t_1}^{t} \|u\|_{\infty},\ b - 2\delta - \int_{t_1}^{t} \|u\|_{\infty} \right)$.

To see this, first, if $t \in [t_1, T^{\delta}]$, then $t \in [t_1, T^{\varepsilon\delta}]$. So from (20),(21),(22) and the definition of $w^{\varepsilon\delta}$, we have

$$\int_{a+\delta}^{b-\delta} \int_{t_1}^{t} \rho(u^{\varepsilon} - w^{\varepsilon\delta}) \phi_x^{\varepsilon\delta} dx dt \leq 0. \tag{25}$$

Secondly, for the term $\int_{a+\delta}^{b-\delta} \int_{t_1}^{t} \rho(u - u^{\varepsilon}) \phi_x^{\varepsilon\delta} dx dt$, we differentiate (18) with respect to x and get

$$\begin{cases} \phi_{xt} + w^{\varepsilon\delta} \phi_{xx} = -w_x^{\varepsilon\delta} \phi_x, \quad t > t_1; \\ \phi_x(\cdot, t_1) = \Psi_x^{\delta}. \end{cases}$$

Then alone the characteristics, we have

$$\phi_x^{\varepsilon\delta}(x(t), t) = \Psi_x^{\delta}(x(t_1)) e^{-\int_{t_1}^{t} w_x^{\varepsilon\delta}(x(s), s) ds}. \tag{26}$$

Next, by the definition of $w^{\varepsilon\delta}$, we see that

$$|w_x^{\varepsilon\delta}(\cdot, s)| \leq C(\delta) \|u^{\varepsilon}\|_{\infty} \leq C(\delta) \|u\|_{\infty},$$

where $C(\delta) \to +\infty$ as $\delta \to 0$, and thus by (26),

$$\|\phi_x^{\varepsilon\delta}\|_{\infty} \leq C_1(\delta),$$

where $C_1(\delta)$ is a constant only depending on δ and $C_1(\delta) \to 0$ as $\delta \to 0$. Hence from (8) and note that for almost all $t \in [t_1, T]$, $u(\cdot, t) \in H^1_{loc}(\mathbb{R})$, we have

$$\begin{aligned} &\left| \int_{t_1}^{t} \int_{a+\delta}^{b-\delta} \rho(u - u^{\varepsilon}) \phi_x^{\varepsilon\delta} dx dt \right| \\ &\leq C_1(\delta) \int_{t_1}^{T} \|u(\cdot, t) - u^{\varepsilon}(\cdot, t)\|_{\infty} \|\rho(\cdot, t)\|_{L^1(a+\delta, b-\delta)} dt \\ &\leq C(a, b, T) C_1(\delta) \int_{t_1}^{T} \|u(\cdot, t) - u^{\varepsilon}(\cdot, t)\|_{\infty} dt \\ &\leq \varepsilon C(a, b, T) C_1(\delta) \int_{t_1}^{T} \|u(\cdot, t)\|_{H^1} dt \\ &\longrightarrow 0 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

That is

$$\lim_{\varepsilon\to 0}\int_{a+\delta}^{b-\delta}\int_{t_1}^{t}\rho(u-u^\varepsilon)\phi_x^{\varepsilon\delta}dxdt=0. \tag{27}$$

Now, from (20), (22) and (25)-(27), we get

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\int_{I_\delta}\rho(x,t)dx &= \overline{\lim_{\varepsilon\to 0}}\int_{I_\delta}(\rho\phi^{\varepsilon\delta})(x,t)dx\\ &\le \overline{\lim_{\varepsilon\to 0}}\int_{a+\delta}^{b-\delta}(\rho\phi^{\varepsilon\delta})(x,t)dx\le 0,\quad t\in[t_1,T^\delta],\end{aligned}$$

then (24) is proved.

If now $t\in(t_1,t_2)$, then $\int_{t_1}^t\|u\|_\infty<\frac{b-a}{2}$, and thus there exists a $\delta_0\in(0,\frac{b-a}{6})$ such that if $\delta\in(0,\delta_0)$, then

$$\int_{t_1}^t\|u\|_\infty<\frac{b-a}{2}-4\delta.$$

For such δ, (23) implies that $t\le T^\delta$ and then $\rho(\cdot,t)=0$ for almost all $x\in I_\delta(t)$. Letting $\delta\to 0$, we get that $\rho(\cdot,t)=0$ a.e. on

$$\left(a+\int_{t_1}^t\|u\|_\infty,\ b-\int_{t_1}^t\|u\|_\infty\right)$$

for all $t\in[t_1,t_2]$, thus the proof is completed. □

3. **Non formation of vacuum near $x=\pm\infty$.** In this section, by Lemma 2.1 and Lemma 2.2, we can show that if there is no vacuum initially, then there will be no formation of vacuum state near infinite.

Proof of Theorem 1.2. We only prove the case that x is positive, the negative part is similar. If $L\ge 1$, as in Lemma 2.1, we can choose $\xi_L(x)\in C_0^\infty(\mathbb{R})$ such that

$$\xi_L(x)=\begin{cases}0\le\xi_L\le 1,\\ |\xi_L'|\le CL^{-1/2},\\ \xi_L=1,\quad x\in[-L,L];\\ \xi_L=0,\quad |x|\ge 3L/2.\end{cases}$$

Then, we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(-L,L)}&=ess\sup_{x\in[-L,L]}|u(x,t)\xi_L(x)|\\ &=ess\sup_{x\in[-L,L]}\left|\int_{-\frac{3L}{2}}^{x}\partial_y(\xi_L(y)u(y,t))dy\right|\\ &\le\int_{-\frac{3L}{2}}^{L}|\xi_L'(y)u(y,t)|dy+\int_{-\frac{3L}{2}}^{L}|\xi_L(y)u_y(y,t)|dy\\ &\le CL^{-1/2}\int_{-\frac{3L}{2}}^{L}|u(y,t)|dy+CL^{\frac12}\left(\int_{-\frac{3L}{2}}^{L}|u_y(y,t)|^2dy\right)^{\frac12}\\ &\le C\gamma(t)(1+L^{\frac34})\\ &\le CL^{\frac34}\gamma(t),\end{aligned}$$

which yields that

$$\int_0^t \|u\|_{L^\infty(L,2L)} \leq \int_0^t \|u\|_{L^\infty(-2L,2L)} \leq 2^{\frac{3}{4}} C L^{\frac{3}{4}} \int_0^t \gamma(s)ds, \tag{28}$$

for all $t \in (0, +\infty)$ and $L \in (1, +\infty)$. Let

$$L^*(t) = max\left\{1, 648C^4 \left(\int_0^t \gamma(s)ds\right)^4\right\}, \tag{29}$$

then

$$2^{\frac{3}{4}} C (L^*(t))^{\frac{3}{4}} \int_0^t \gamma(s)ds \leq L^*(t)/3 \tag{30}$$

for all $t \in (0, +\infty)$. We claim that for any $t \in (0, +\infty)$,

$$\int_L^{2L} \rho(x,t)dx > 0$$

for all $L \in (L^*(t), +\infty)$ with $L^*(t)$ defined by (29). In fact, if (14) is not true for some $t_0 \in (0, +\infty)$ and some $L_0 \in (L^*(t_0), +\infty)$, then $\rho(x, t_0) = 0$ a.e. on $(L_0, 2L_0)$. Then Lemma 2.2 shows that $\rho(\cdot, t) = 0$ a.e. on

$$\left(L_0 + \int_t^{t_0} \|u\|_{L^\infty(L_0,2L_0)},\ 2L_0 - \int_t^{t_0} \|u\|_{L^\infty(L_0,2L_0)}\right) \tag{31}$$

for all $t \in [t_*, t_0]$, where

$$t_* = \inf\left\{t \in [0, t_0] :\ \int_t^{t_0} \|u\|_{L^\infty(L_0,2L_0)} < \frac{L_0}{2}\right\}. \tag{32}$$

It follows from (28) and (30) that

$$\int_0^{t_0} \|u(\cdot,t)\|_{L^\infty(L_0,2L_0)} dt \leq 2^{\frac{3}{4}} C L_0^{\frac{3}{4}} \int_0^{t_0} \gamma(t)dt \leq \frac{L_0}{3} < \frac{L_0}{2} \tag{33}$$

for all $L_0 \in (L^*(t_0), +\infty)$. Combining (31)-(33) yields that $0 \in [t_*, t_0]$, that is

$$\rho_0(x) = \rho(x, 0) = 0$$

for almost all $x \in (L_0 + \frac{L_0}{3},\ 2L_0 - \frac{L_0}{3})$. This contradicts $\int_E \rho(x,0)dx > 0$ for any open set $E \subset \mathbb{R}$. We finish the proof of Theorem 1.2. ☐

4. Vacuum away from infinity.

In this section, we focus on the properties of vacuum states and prove Theorem 1.3 by contradiction argument. Let (ρ, u, e) be a weak solution of (1)-(3) on $\mathbb{R} \times [0, T]$ satisfying assumptions (9)-(13). We suppose that $\rho(\cdot, t_1) = 0$ a.e. on (a, b), where, Theorem 1.2 implies $-\infty < a < b + \infty$. The interval (a, b) and the time t_1 will be fixed in the rest of the argument.

Let t_0 be as in the statement of Lemma 2.2, and define for $t \in (t_0, t_1)$,

$$y(t) = \inf\left\{x :\ \rho(\cdot, t) = 0 \text{ a.e. on } (x, \frac{a+b}{2})\right\}, \tag{34}$$

$$z(t) = \sup\left\{x :\ \rho(\cdot, t) = 0 \text{ a.e. on } (\frac{a+b}{2}, x)\right\}, \tag{35}$$

and

$$y(t_1) = a, \quad z(t_1) = b. \tag{36}$$

We start with some elementary estimates and regularity properties for the curves $x = y(t)$ and $x = z(t)$.

Lemma 4.1. *There exists a constant $h_0 = h_0(a,b) > 0$ such that $y(t)$ and $z(t)$ are absolutely continuous functions on $[t_1 - h_0, t_1]$, and*

$$a_0 \le y(t) \le a_1 < b_1 \le z(t) \le b_0, \tag{37}$$

where $a_0 = a - \frac{b-a}{2}$, $a_1 = a + \frac{b-a}{4}$, $b_1 = b - \frac{b-a}{4}$ and $b_0 = b + \frac{b-a}{2}$.

Proof. First, from Lemma 2.2,

$$y(t) < a + \frac{b-a}{2} = \frac{a+b}{2} < b - \frac{b-a}{2} < z(t) \tag{38}$$

for all $t \in [t_0, t_1]$, where t_0 is as in Lemma 2.2. Let

$$\omega(t) = \max\{z(t), -y(t)\} \ge 0, \tag{39}$$

then $\omega(t_1) = \max\{b, -a\}$ and

$$\begin{aligned}\int_{t_1-h}^{t_1} \|u(\cdot,t)\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})} &\le \int_{t_1-h}^{t_1} \|u(\cdot,t)\|_{L^\infty(-2\omega(t_1), 2\omega(t_1))} \\ &\le C(2\omega(t_1)+1)\int_{t_1-h}^{t_1} \gamma(t)dt,\end{aligned}$$

where C is a constant as in Lemma 2.1. Choose $h_0 = h_0(a,b) > 0$ such that

$$C(2\omega(t_1)+1)\int_{t_1-h_0}^{t_1} \gamma(t)dt < \frac{b-a}{4},$$

so that

$$\int_{t_1-h_0}^{t_1} \|u(\cdot,t)\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})} < \frac{b-a}{4}. \tag{40}$$

Then we can prove that for all $t \in [t_1 - h_0, t_1]$,

$$z(t) \le b + \frac{b-a}{2}. \tag{41}$$

In fact, if (41) fails then $z(t) > b + \frac{b-a}{2}$ for some $\underline{t} \in [t_1 - h_0, t_1)$. Due to (38), one gets that $\rho(\cdot, \underline{t}) = 0$ a.e. on $(\frac{a+b}{2}, b + \frac{b-a}{2})$. Applying Lemma 2.2 we obtain $\rho(\cdot, t) = 0$ a.e. on

$$\left(\frac{a+b}{2} + \int_{\underline{t}}^{t} \|u\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})},\ b + \frac{b-a}{2} - \int_{\underline{t}}^{t} \|u\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})}\right)$$

for all $t \in (\underline{t}, \bar{t})$, where

$$\bar{t} = \sup\left\{t \in [\underline{t}, T] : \int_{\underline{t}}^{t} \|u\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})} < \frac{b-a}{2}\right\}.$$

Using (40) we have $t_1 \in (\underline{t}, \bar{t})$, then $\rho(\cdot, t_1) = 0$ a.e. on

$$\left(\frac{a+b}{2} + \int_{\underline{t}}^{t_1} \|u\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})},\ b + \frac{b-a}{2} - \int_{\underline{t}}^{t_1} \|u\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})}\right).$$

This contradicts (35) and (36) since

$$b + \frac{b-a}{2} - \int_{\underline{t}}^{t_1} \|u\|_{L^\infty(\frac{a+b}{2}, b+\frac{b-a}{2})} > b.$$

Similarly, we have $a - \frac{b-a}{2} \le y(t)$. Moreover,

$$\int_{t}^{t_1} \|u\|_{L^\infty(a,b)} \le \int_{t_1-h_0}^{t_1} \|u\|_{L^\infty(a,b)} \le \int_{t_1-h_0}^{t_1} \|u\|_{L^\infty(-2\omega(t_1), 2\omega(t_1))} < \frac{b-a}{4}$$

for all $t \in [t_1 - h_0, t_1]$, then (37) holds.

Next, we can prove that $y(t)$, $z(t)$ are absolutely continuous by definition and follow the ideas in [14]. $\square$

Let S be defined as the set of all $t > 0$ such that there exist extensions of $y(t)$ and $z(t)$ to $[t, t_1]$ with the following three properties:

1. $y(t)$ and $z(t)$ are absolutely continuous on $[t, t_1]$;
2. $y(t) < z(t)$ on $[t, t_1]$;
3. $\int_{y(s)-\varepsilon}^{z(s)} \rho(x,s)dx$ and $\int_{y(s)}^{z(s)+\varepsilon} \rho(x,s)dx$ are both positive for all $\varepsilon > 0$ and $s \in [t, t_1]$, and $\int_{y(s)}^{z(s)} \rho(x,s)dx = 0$.

It follows from Lemma 4.1 that S is not a empty set and thus let

$$\tau = \inf S. \tag{42}$$

Concerning τ, according to Lemma 2.1, Lemma 2.2 and Lemma 4.1, we have the following result.

Lemma 4.2. *$y(t)$ and $z(t)$ have absolutely continuous extensions to time τ, $y(\tau) = z(\tau)$, and there exists an $L = C'(\max\{-a_0, b_0\} + 1) > 0$ for some constant $C' > 0$, such that for all $t \in [\tau, t_1]$, $-L \le y(t) < z(t) \le L$, where a_0, b_0 as in (37).*

Proof. The proof is similar to Lemma 2.4 in [14]. For any $\widetilde{t} \in (\tau, t_1]$, $y(t)$ and $z(t)$ are absolutely continuous on $[\widetilde{t}, t_1]$, and $y(t) < z(t)$ for all $t \in [\widetilde{t}, t_1]$. Therefore for any $t \in [\widetilde{t}, t_1 - \frac{h_0}{2}]$, Lemma 2.2 shows that there is an $h = h(t) > 0$ such that

$$C \int_{t-h}^{t+h} \gamma(s)ds \le \frac{1}{4}, \tag{43}$$

and if $|s - t| \le h$, then $\rho(\cdot, s) = 0$ a.e. on the interval

$$\left(y(t) + \left| \int_s^t \|u\|_{L^\infty(-2\omega(t), 2\omega(t))} \right|, \ z(t) - \left| \int_s^t \|u\|_{L^\infty(-2\omega(t), 2\omega(t))} \right| \right),$$

where C is as in Lemma 2.1, $h_0 = h_0(a, b)$ is as in Lemma 4.1 and $\omega(t)$ is defined by (39). Thus

$$z(s) \ge z(t) - \left| \int_s^t \|u\|_{L^\infty(-2\omega(t), 2\omega(t))} \right|$$

and

$$y(s) \le y(t) + \left| \int_s^t \|u\|_{L^\infty(-2\omega(t), 2\omega(t))} \right|,$$

then we get

$$\begin{aligned}
\omega(s) &\ge \omega(t) - \left| \int_s^t \|u\|_{L^\infty(-2\omega(t), 2\omega(t))} \right| \\
&\ge \omega(t) - C \left| \int_s^t \gamma \right| (2\omega(t) + 1) \\
&\ge \left(1 - 2C \left| \int_s^t \gamma \right| \right) \omega(t) - C \left| \int_s^t \gamma \right|.
\end{aligned}$$

Thus for $|t - s| \le h(t)$, we have

$$0 < \omega(t) \le \left(1 + 2C \left| \int_s^t \gamma \right| \right) \left[\omega(s) + C \left| \int_s^t \gamma \right| \right] \tag{44}$$

for some positive constant C. On the other hand, let

$$d(\widetilde{t}) = \inf_{t\in[\widetilde{t},t_1]} |y(t) - z(t)| > 0. \tag{45}$$

Choose constants A, B depending on t and $\widetilde{t}$, such that

$$-2\omega(t) < y(t) < A < B < z(t) < 2\omega(t)$$

and

$$(A - y(t)) + (z(t) - B) < \frac{d}{4}.$$

If $h(t)$ is further reduced, it follows from the continuity of $y(t)$ and $z(t)$ that if $|s - t| \le h$, then

$$|y(s) - y(t)| < |A - y(t)| \quad \text{and} \quad |z(s) - z(t)| < |B - z(t)|,$$

which means

$$-2\omega(t) < y(s) < A < B < z(s) < 2\omega(t).$$

For such s, using Lemma 2.2, we find that there exists a $\sigma = \sigma(t)$ depending on $\frac{B-A}{2}$, such that if $s \le \widetilde{s} \le s + \sigma$, then $\rho(\cdot, \widetilde{s}) = 0$ a.e. on

$$\left(y(s) + \int_s^{\widetilde{s}} \|u\|_{L^\infty(-2\omega(t),2\omega(t))},\ z(s) - \int_s^{\widetilde{s}} \|u\|_{L^\infty(-2\omega(t),2\omega(t))} \right).$$

We can further reduce $h(t)$ so that $h(t) \le \sigma(t)$. Thus if $t - h(t) \le s \le t$, then $s \le t \le s + h(t) \le s + \sigma(t)$, and we may take $\widetilde{s} = t$ to obtain

$$\omega(t) \ge \omega(s) - \int_s^t \|u\|_{L^\infty(-2\omega(t),2\omega(t))} dt$$

and then

$$\omega(s) \le \omega(t) + C(2\omega(t) + 1) \left(\int_s^t \gamma \right). \tag{46}$$

We now cover the interval $[\widetilde{t}, t_1 - \frac{h_0}{2}]$ by finitely many intervals $B_{h_j}(s_j)$, where $s_1 > s_2 > \cdots > s_p$ and $h_j = h_j(s_j)$ as above with $h_j < \frac{h_0}{2}$, that is

$$\begin{cases} [\widetilde{t}, t_1 - \dfrac{h_0}{2}] \subset \cup_{j=1}^p (s_j - h_j, s_j + h_j), \\ \widetilde{t} \in (s_p - h_p, s_p + h_p), \\ t_1 - \dfrac{h_0}{2} \in (s_1 - h_1, s_1 + h_1). \end{cases}$$

If $\tau_j \in B_{h_{j+1}}(s_{j+1}) \cap B_{h_j}(s_j)$, then by (46),

$$\omega(s_{j+1}) \le \omega(\tau_j) + C(2\omega(\tau_j) + 1) \left(\int_{s_{j+1}}^{\tau_j} \gamma \right),$$

also from (44),

$$0 < \omega(\tau_j) < \left(1 + 2C \left| \int_{s_j}^{\tau_j} \gamma \right| \right) \left[\omega(s_j) + C \left| \int_{s_j}^{\tau_j} \gamma \right| \right],$$

then

$$\begin{aligned}\omega(s_{j+1}) \leq &\left(1+2C\int_{s_{j+1}}^{\tau_j}\gamma\right)\left(1+2C\int_{\tau_j}^{s_j}\gamma\right)\left[\omega_1 + C\int_s^t \gamma\right]\\ &+\left(1+2C\int_{s_{j+1}}^{\tau_j}\gamma\right)\int_s^t\gamma,\end{aligned} \tag{47}$$

which gives

$$\begin{aligned}\omega(s_p) \leq \omega(s_{j+1}) \leq &\prod_{j=0}^{p-1}\left(1+2C\int_{s_{j+1}}^{\tau_j}\gamma\right)\left(1+2C\int_{\tau_j}^{s_j}\gamma\right)\left[\omega_1 + C\int_s^t \gamma\right]\\ &+\sum_{k=0}^{p-1}\prod_{j=k}^{p-1}\left(1+2C\int_{s_{j+1}}^{\tau_j}\gamma\right)\left(1+2C\int_{\tau_j}^{s_j}\gamma\right)\int_s^t\gamma.\end{aligned}$$

Now if $\varepsilon_1 + \cdots + \varepsilon_q = \varepsilon$, and each $\varepsilon_i > 0$, then

$$\prod(1+\varepsilon_i) \leq \left(1+\frac{\varepsilon}{q}\right)^q \leq e^{\varepsilon}.$$

Therefore,

$$\begin{aligned}\omega(s_p) &\leq e^{2\varepsilon}\left[\omega(s_1) + C\int_0^T\gamma\right] + e^{\varepsilon}p\int_0^T\gamma\\ &\leq e^{2\int_0^T\gamma}\left[\omega(s_1) + C\int_0^T\gamma\right] + e^{\int_0^T\gamma}p\int_0^T\gamma\\ &\leq C'(\omega(s_1)+1),\end{aligned} \tag{48}$$

for some constant C'. As $s_1 \in [t_1 - h_0, t_1]$, by Lemma 4.1, we can bound $\omega(s_1)$ independent of t, and so (48) bound $\omega(t)$ on $[\widetilde{t}, t_1]$ for all $\widetilde{t} \in (\tau, t_1]$, independent of t. Thus there exists

$$L = C'\left(\max\{-a_0, b_0\} + 1\right) > 0,$$

such that

$$-L \leq y(t) < z(t) \leq L, \quad \text{for all } t \in (\tau, t_1],$$

where a_0, b_0 as in (37).

Next, we will show that $y(t)$ and $z(t)$ are uniformly continuous on the interval $(\tau, t_1]$. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that if $0 \leq s < t \leq T$ and $|s-t| < \delta$, then

$$\int_s^t \|u\|_{L^\infty(-L,L)} < \frac{\varepsilon}{5}.$$

Now, if $t \in (\tau, t_1]$, we can find $h(t) > 0$ such that if $|t-s| < h(t)$, then

$$|z(s) - z(t)| \leq \left|\int_s^t \|u\|_{L^\infty(-L,L)}\right|.$$

Now fix $s < t$ with $|s-t| < \delta$ and $s, t \in (\tau, t_1]$, then $[s,t]$ is covered by $\cup_{k=1}^N B_{\frac{h_k}{2}}(s_k)$, $s_1 < s_2 < \cdots < s_N$, where $s_j + \frac{h_j}{2} > s_{j+1} - \frac{h_{j+1}}{2}$ and $h_j < \delta$ for each j. Then $|s_{j+1} - s_j| \leq \frac{h_j + h_{j+1}}{2} \leq \max\{h_j, h_{j+1}\} < \delta$. Thus

$$|z(s_j) - z(s_{j+1})| \leq \left|\int_{s_j}^{s_{j+1}} \|u\|_{L^\infty(-L,L)}\right|.$$

Now if for some j and k, $s \in B_{\frac{h_j}{2}}(s_j),\ t \in B_{\frac{h_k}{2}}(s_k)$, then

$$\begin{aligned}|z(s)-z(t)| &\le |z(s)-z(s_j)| + |z(s_{j+1})-z(s_j)| + \cdots + |z(s_k)-z(t)| \\ &\le \int_{s_j}^{s_{j+1}} + \cdots + \int_{s_{k-1}}^{s_k} + \left|\int_s^{s_j} \|u\|_{L^\infty(-L,L)}\right| + \left|\int_{s_k}^{t} \|u\|_{L^\infty(-L,L)}\right| \\ &\le \int_s^t \|u\|_{L^\infty(-L,L)} + 2\left|\int_s^{s_j} \|u\|_{L^\infty(-L,L)}\right| + 2\left|\int_{s_k}^{t} \|u\|_{L^\infty(-L,L)}\right| \\ &< \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} + \frac{2\varepsilon}{5} = \varepsilon.\end{aligned}$$

It follows that $y(t)$ and $z(t)$ have absolutely continuous extension to time τ. To complete the proof, we have to show that $y(\tau) = z(\tau)$. Otherwise, $y(\tau) < z(\tau)$, and if $\tau = 0$, then $\int_{y(\tau)}^{z(\tau)} \rho(x,0)dx = 0$ which contradicts the assumption (16); or if $\tau > 0$, then τ would not be the minimal, since we can do the absolutely continuous extensions of $y(t)$ and $z(t)$ to time $\tau - h(\tau)$ for some $h(\tau) > 0$. $\square$

Now, we define the vacuum region V by

$$V = \{(x,t) :\ y(t) < x < z(t), \tau < t \le t_1\}. \tag{49}$$

The function u in V has following representation:

Lemma 4.3. *Let (ρ, u, θ) be a global weak solution of (1)-(4), then*

$$u(x,t) = \alpha(t)x + \beta(t)$$

for all $x \in (y(t), z(t))$ and almost all $t \in (\tau, t_1]$, where $\alpha(t) \in L^1_{loc}((\tau, t_1])$ and $\beta(t) \in L^1_{loc}((\tau, t_1])$.

Proof. This can be seen formally from the equation (2) and we omit the details. $\square$

Based on Lemma 4.2 and Lemma 4.3, we can estimate the growth of $x = y(t)$ and $x = z(t)$ more precisely.

Lemma 4.4. *It holds that*

$$\frac{dz}{dt} \le \alpha z + \beta \quad \textit{and} \quad \frac{dy}{dt} \ge \alpha y + \beta \tag{50}$$

for almost all $t \in (\tau, t_1]$.

Proof. The proof can follow the idea of Hoff-Smoller [14] and Duan-Zhao [7]. $\square$

Remark 4. Lemma 4.4 shows that integral curves of u which start in V must remain in V on $(\tau, t_1]$. In fact, fix $w_1 \in (a,b)$ and for $\tau < t \le t_1$, $w(t)$ is defined by

$$\begin{cases} \dfrac{dw}{dt} = \alpha w + \beta, \\ w(t_1) = w_1 < b = z(t_1). \end{cases} \tag{51}$$

Then by Lemma 4.4, we have

$$\begin{cases} \dfrac{d(z-w)}{dt} \le \alpha(z-w) \quad \text{a.e.}, \\ (z-w)(t_1) = b - w_1 > 0. \end{cases}$$

Thus

$$0 < z(t_1) - w_1 \le c^{\int_s^{t_1} \alpha}(z(t) - w(t)),$$

so that $w(t) < z(t)$; similarly, we have $w(t) > y(t)$. That is

$$y(t) < w(t) < z(t) \quad \text{for all } t \in (\tau, t_1]. \tag{52}$$

Corollary 1. *It holds that*

$$\lim_{t\to\tau}\int_t^{t_1}\alpha(s)ds = +\infty.$$

Proof. For $a < w_1 < w_2 < b$ and the corresponding function $w_i(t)(i = 1, 2)$ defined by (51), we have

$$\begin{cases} \dfrac{d(w_1(t) - w_2(t))}{dt} = \alpha(w_1(t) - w_2(t)) + \beta, \\ w_1(t_1) - w_2(t_1) = w_1 - w_2, \end{cases}$$

then

$$w_1(t) - w_2(t) = (w_1 - w_2)e^{-\int_t^{t_1}\alpha(s)ds}. \tag{53}$$

From (52), $\lim_{t\to\tau}(w_1(t) - w_2(t)) = 0$ and (53) gives that $\lim_{t\to\tau}\int_t^{t_1}\alpha(s)ds = +\infty$. □

We now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $c(t) \equiv w_1(t) < w_2(t) \equiv d(t)$ be two curves defined by (51). Then $0 \le d(t) - c(t) \to 0$ as $t \to \tau$. Let $\omega^\varepsilon = (\alpha^\varepsilon(t)x + \beta^\varepsilon(t))\chi(x)$, where α^ε, β^ε are regularizations of α and β and $\chi(x)$ is a smooth function defined as following:

$$\begin{cases} spt\chi(x) \subset (a, b), \\ \{x : \chi(x) = 1\} = (a_1, b_1), \quad \text{where } (a_1, b_1) \subset (c(t_1), b) \text{ and } a_1 < d(t_1), \\ 0 \le \chi(x) \le 1. \end{cases}$$

Define the smooth function $\psi(x)$ by

$$\begin{cases} spt\psi(x) \subset (a_1, \widetilde{b}), & \text{where } \widetilde{b} > b, \\ \{x : \psi(x) = 1\} = (a_2, b_2), & \text{where } a_1 < a_2 < d(t_1) \text{ and } b < b_2 < \widetilde{b}, \\ 0 \le \psi(x) \le 1. \end{cases}$$

Now let ϕ^ε be the solution of the initial value problem

$$\begin{cases} \phi_t + \omega^\varepsilon\phi_x = 0, \\ \phi^\varepsilon(t_1) = \psi(x), \end{cases}$$

then ϕ^ε is a smooth compactly supported function. Consider the curves $x_r^\varepsilon = x_r^\varepsilon(t)$ defined by

$$\begin{cases} \dfrac{dx}{dt} = \omega^\varepsilon, & t \in (\tau, t_1], \\ x(t_1) = r, & r \in [0, +\infty), \end{cases}$$

Set

$$V_1 = \left\{(x, t) : \ x_{a_1}^\varepsilon(t) < x < x_{a_2}^\varepsilon(t), t \in (\tau, t_1]\right\},$$
$$V_2 = \left\{(x, t) : \ x_{b_2}^\varepsilon(t) < x < x_{\widetilde{b}}^\varepsilon(t), t \in (\tau, t_1]\right\}.$$

Then $spt\phi_x^\varepsilon$ is contained in V_1 and V_2 with the corresponding characteristics $\dot{x} = \alpha^\varepsilon x + \beta^\varepsilon$ and $\dot{x} = 0$ respectively. Thus from (6), we have for $\tau < t < t_1$,

$$\begin{aligned}\int \rho u\phi^\varepsilon \Big|_t^{t_1} dx &= \iint \left[\rho u(\phi_t^\varepsilon + u\phi_x^\varepsilon) + (P - \mu u_x)\phi_x^\varepsilon + \rho f\phi^\varepsilon\right] dxdt \\ &= \iint \left[\rho u(u - \omega^\varepsilon)\phi_x^\varepsilon + (P - \mu u_x)\phi_x^\varepsilon + \rho f\phi^\varepsilon\right] dxdt.\end{aligned} \tag{54}$$

Now, first $\left|\int \rho u\phi^\varepsilon\big|_t^{t_1} dx\right| \le C$ by (13), where C is independent of t. Then the term $\left|\iint \rho f\phi^\varepsilon\right|$ is bounded because of the regularity of f and (8). Also, since $\rho = 0$ a.e. on V_1,

$$\int_t^{t_1}\int_{V_1} \rho u(u - \omega^\varepsilon)\phi_x^\varepsilon = 0.$$

In V_2, $\omega^\varepsilon = 0$ and $\phi_x^\varepsilon = \psi_x^\varepsilon$, then in view of (12),

$$\left|\int_t^{t_1}\int_{V_2} \rho u(u - \omega^\varepsilon)\phi_x^\varepsilon\right| = \left|\int_t^{t_1}\int_{V_2} \rho u^2\psi_x\right| \le C(t_1, \tau).$$

Next, by the assumptions (4), (9)-(12) and Definition 1.1, we have

$$\left|\iint_{V_2} (P - \mu u_x))\phi_x^\varepsilon\right| \le \iint_{V_2} (|P| + \mu|u_x|)|\psi_x| \le C.$$

Finally, since $P(0, \theta) = 0$, we have

$$\begin{aligned}\iint_{V_1} (P - \mu u_x)\phi_x^\varepsilon &= -\iint_{V_1} \mu u_x\phi_x^\varepsilon \\ &= -\int_t^{t_1}\int_{c(t)}^{d(t)} \mu\alpha(s)\phi_x^\varepsilon \\ &= -\int_t^{t_1} \mu\alpha(s)(\phi^\varepsilon(d(s), s) - \phi^\varepsilon(c(s), s))ds \\ &= -\int_t^{t_1} \mu\alpha(s),\end{aligned}$$

because $c(t_1) < a_1 < a_2 < d(t_1)$ and thus $\phi^\varepsilon(d(s), s) = 1,\ \phi^\varepsilon(c(s), s) = 0$. Then by (54)

$$\begin{aligned}\left|\mu\int_t^{t_1} \alpha(s)\right| &= \left|\iint_{V_1} (P - \mu u_x)\phi_x^\varepsilon\right| \\ &\le \left|\int \rho u\phi^\varepsilon\Big|_t^{t_1} dx\right| + \left|\iint \rho u(u - \omega^\varepsilon)\phi_x^\varepsilon + \rho f\phi_x^\varepsilon\right| \\ &\quad + \left|\iint_{V_2} (P - \mu u_x))\phi_x^\varepsilon\right| \\ &\le C\end{aligned} \tag{55}$$

where C is independent of t. Letting $t \downarrow \tau$, (55) contradicts Corollary 1. We finish the proof of Theorem 1.3. $\square$

Remark 5. It is clear that if the vacuum states appear on an open interval initially, then the interval of vacuum state will persist in time. In fact, if $\rho(x, 0) = 0$ a.e.

on $(a,b) \subset \mathbb{R}$, we can define $y(t),\ z(t)$ and set S as before for $t \in [0,T]$ with any $T < +\infty$, and let

$$\overline{\tau} = \sup S. \tag{56}$$

Then from the proof of Theorem 1.3, we get

$$-L < y(t) < z(t) < L, \quad \text{for all } t \in [0,\overline{\tau}) \text{ and some } L > 0;$$

$y(t),\ z(t)$ have absolutely continuous extensions to time $\overline{\tau}$ and are uniform continuous on $[0,\overline{\tau})$. Now if $y(\overline{\tau}) = z(\overline{\tau})$, we obtain Corollary 1 which contradicts (55), which means the momentum is locally finite. Thus $y(\overline{\tau}) < z(\overline{\tau})$ and $\overline{\tau} = T$, since otherwise $\overline{\tau} < T$, then $\overline{\tau}$ would not be the maximal.

5. **Vacuum states on two intervals.** In this section we show that two initial non interacting vacuum regions will never meet each other in the future.

Proof of Theorem 1.4. According to Remark 5, we see that the curves $y_i(t),\ z_i(t)$, $i = 1,2$ are absolutely and uniformly continuous functions. Also, by Lemma 4.1, there exists $h_0 = h_0(a_i, b_i) > 0$ such that for any $t \in [0, h_0]$, we have

$$\begin{aligned} -\infty < y_1(t) < a_1 + \frac{b_1 - a_1}{4} < b_1 - \frac{b_1 - a_1}{4} < z_1(t) \\ < y_2(t) < a_2 + \frac{b_2 - a_2}{4} < b_2 - \frac{b_2 - a_2}{4} < z_2(t) < +\infty \end{aligned}$$

since $b_1 < a_2$. Suppose $y_2(t) > z_1(t)$ is not true for some $t \in (h_0, T]$, let

$$\bar{t} = \inf\{t \in (h_0, T] : \ y_2(t) = z_1(t)\}$$

as the set is not empty. Thus $\rho(\cdot, \bar{t}) = 0$ a.e. on $(y_1(\bar{t}), z_2(\bar{t}))$, then there exists $0 < h(\bar{t}) < h_0$, such that

$$\int_{\bar{t}-h(\bar{t})}^{\bar{t}} \|u(\cdot,\bar{t})\|_{L^\infty(-w(\bar{t}),w(\bar{t}))} < \frac{d}{4},$$

where $w(\bar{t}) = \max\{-y_1(t), z_2(\bar{t})\}$ and

$$d = \min\left\{ \inf_{t\in[0,T]} |y_1(t) - z_1(t)|, \ \inf_{t\in[0,T]} |y_2(t) - z_2(t)| \right\}. \tag{57}$$

Here $d > 0$ because of Remark 5. Then $\rho(\cdot, t) = 0$ a.e. on

$$\left(y_1(\bar{t}) + \frac{d}{4},\ z_2(\bar{t}) - \frac{d}{4} \right)$$

for all $t \in (\bar{t} - h(\bar{t}), \bar{t})$. We can reduce $h(\bar{t})$ so that if $|t - \bar{t}| < h(\bar{t})$, then

$$|z_1(t) - z_1(\bar{t})| < \frac{d}{4} \quad \text{and} \quad |y_2(t) - y_2(\bar{t})| < \frac{d}{4},$$

and then

$$y_1(\bar{t}) + \frac{d}{4} < z_1(\bar{t}) - \frac{d}{4} < z_1(t) \quad \text{and} \quad y_2(t) < y_2(\bar{t}) + \frac{d}{4} < z_2(\bar{t}) - \frac{d}{4}.$$

This shows that $\rho(\cdot,t) = 0$ a.e. on $(z_1(t), y_2(t))$ for all $t \in (\bar{t} - h(\bar{t}), \bar{t})$ which leads to the contradiction since $\bar{t}$ is the minimum. Thus for all $t \in [0,T]$, we have

$$-\infty < y_1(t) < z_1(t) < y_2(t) < z_2(t) < +\infty.$$

Next, we will show (17). We first set

$$R^i = \frac{1}{2}(y_i(t) + z_i(t)), \quad i = 1,2.$$

Moreover, for any $t \in [\frac{h_0}{2}, T)$ one can find a small positive constant $h(t) > 0$ such that if $|s-t| < h(t)$, then

$$|y_i(s) - y_i(t)| + |z_i(s) - z_i(t)| < \frac{d}{4}, \quad i = 1, 2,$$

where $h_0 = h_0(a_i, b_i)$, d are constants defined in (57). Thus let

$$\Omega^i(t) = [R^i(t) - \frac{d}{4}, R^i(t) + \frac{d}{4}] \times [t - h(t), t + h(t)]$$

and

$$V^i = \{(x,t): \; y_i(t) < x < z_i(t), 0 \le t \le T\}$$

then

$$\Omega^i(t) \subset V^i, \quad i = 1, 2.$$

Next, we can cover $[h_0, T]$ by $\bigcup_{t \in [\frac{h_0}{2}, T)} (t - h(t), t + h(t))$, then there exist $\{t_j\}_{j=1}^N$ and $\{h_j\}_{j=1}^N$ such that

$$\begin{cases} 0 < t_1 < t_2 < \cdots < t_N, \\ [h_0, T] \subset \cup_{j=1}^N (t_j - h_j, t_j + h_j), \quad \text{where } h_j = h(t_j), \\ h_0 \in (t_1 - h_1, t_1 + h_1), \\ T \in (t_N - h_N, t_N + h_N) \cap [0, T], \end{cases}$$

and $\cup_{j=1}^N \Omega_j^i \subset V^i$, $\Omega_j^i \cap \Omega_{j+1}^i \neq \emptyset$, where

$$\Omega_j^i(t) = \left[R_j^i(t) - \frac{d}{4}, R_j^i(t) + \frac{d}{4} \right] \times [t_j - h_j, t_j + h_j]$$

with $R_j^i = R^i(t_j)$, $j = 1, 2, \cdots, N$, $i = 1, 2$. Now denote Ω_0^i by

$$\Omega_0^i = \left[a_i + \frac{b_i - a_i}{4}, b_i - \frac{b_i - a_i}{4} \right] \times [0, h_0],$$

then $\Omega_0^i \subset V^i$, $i = 1, 2$. Choose $\phi^0 \in C_0^\infty(\mathbb{R})$ such that $0 \le \phi^0 \le 1$,

$$\phi^0(x) = \begin{cases} 0, & x \in (-\infty, a_1 + \frac{b_1 - a_1}{4}] \\ 1, & x \in [b_1 - \frac{b_1 - a_1}{4}, a_2 + \frac{b_2 - a_2}{4}] \\ 0, & x \in [b_2 - \frac{b_2 - a_2}{4}, +\infty). \end{cases}$$

It follows from (5) that

$$\int_{\mathbb{R}} \rho \phi^0 \Big|_0^t = \int_0^t \int_{\mathbb{R}} (\rho u \phi_x^0 + \rho \phi_t^0), \quad \text{for all } t \in [0, h_0],$$

thus

$$\int_{z_1(t)}^{y_2(t)} \rho = \int_{b_1}^{a_2} \rho(0,x) dx, \quad \text{for all } t \in [0, h_0].$$

Now for $j = 1$, $t_1 - h_1 \in [0, h_0]$, then

$$\int_{z_1(t_1 - h_1)}^{y_2(t_1 - h_1)} \rho = \int_{b_1}^{a_2} \rho(0,x) dx, \quad \text{for all } t \in [0, h_0].$$

Define $\phi^1 \in C_0^\infty(\mathbb{R})$ by $0 \le \phi^1(x) \le 1$ and

$$\phi^1(x) = \begin{cases} 0, & x \in (-\infty, R_1^1 - \frac{d}{4}] \\ 1, & x \in [R_1^1 + \frac{d}{4}, R_1^2 - \frac{d}{4}] \\ 0, & x \in [R_1^2 + \frac{d}{4}, +\infty). \end{cases}$$

Thus (5) shows that

$$\int_{\mathbb{R}} \rho\phi^1 \Big|_{t_1-h_1}^{t} = \int_{t_1-h_1}^{t} \int_{\mathbb{R}} (\rho u \phi_x^1 + \rho\phi_t^1), \quad \text{for all } t \in [t_1 - h_1, t_1 + h_1],$$

that is

$$\begin{aligned} \int_{z_1(t)}^{y_2(t)} \rho(x,t)dx &= \int_{z_1(t_1-h_1)}^{y_2(t_1-h_1)} \rho(x, t_1 - h_1)dx \\ &= \int_{b_1}^{a_2} \rho(0,x)dx, \quad \text{for all } t \in [t_1 - h_1, t_1 + h_1]. \end{aligned}$$

Repeating the above process shows that

$$\int_{z_1(t)}^{y_2(t)} \rho(x,t)dx = \int_{b_1}^{a_2} \rho(0,x)dx, \quad \text{for all } t \in [0, T].$$

We finish the proof of Theorem 1.4. □

Remark 6. If we choose the smooth function $\bar{\rho}(x)$ in Remark 1 to be monotone and assume $(\rho - \bar{\rho}(x)) \in L^\infty([0,T], L^\alpha(\mathbb{R}))$ for some $\alpha > 1$, then we can bound $(y_2(t) - z_1(t))$ from below. To see this, from (17) we have

$$\int_{z_1(t)}^{y_2(t)} (\rho(x,t) - \bar{\rho}(x))dx + \int_{z_1(t)}^{y_2(t)} \bar{\rho}(x)dx = \int_{b_1}^{a_2} \rho(0,x)dx,$$

and

$$\begin{aligned} \int_{z_1(t)}^{y_2(t)} (\rho(x,t) - \bar{\rho}(x))dx &\le \int_{z_1(t)}^{y_2(t)} |\rho(x,t) - \bar{\rho}(x)|dx \\ &\le \|\rho - \bar{\rho}\|_{L^\infty([0,T];L^\alpha(\mathbb{R}))} (y_2(t) - z_1(t))^{\frac{\alpha-1}{\alpha}} \\ &= C(y_2(t) - z_1(t))^{\frac{\alpha-1}{\alpha}}, \end{aligned}$$

where $C = \|\rho - \bar{\rho}\|_{L^\infty([0,T];L^\alpha(\mathbb{R}))}$. Then

$$\int_{b_1}^{a_2} \rho(0,x)dx \le C(y_2(t) - z_1(t))^{\frac{\alpha-1}{\alpha}} + \max\{\rho_+, \rho_-\}(y_2(t) - z_1(t))$$

for all $t \in [0,T]$. If there exist some $t \in [0,T]$ such that $y_2(t) - z_1(t) < 1$, then

$$\left(\frac{\int_{b_1}^{a_2} \rho(0,x)}{C'} \right)^{\frac{\alpha}{\alpha-1}} \le y_2(t) - z_1(t) < 1,$$

where $C' = \max\{C, \rho_+, \rho_-\}$. Otherwise, $(y_2(t) - z_1(t)) \ge 1$, then

$$\frac{\int_{b_1}^{a_2} \rho(0,x)dx}{C'} \le y_2(t) - z_1(t).$$

Thus

$$y_2(t) - z_1(t) \geq \min\left\{\left(\frac{\int_{b_1}^{a_2}\rho(0,x)}{C'}\right)^{\frac{\alpha}{\alpha-1}}, 1\right\},$$

where $C' = \max\{\|\rho - \bar{\rho}\|_{L^\infty([0,T];L^\alpha(\mathbb{R}))}, \rho_+, \rho_-\}$ for all $t \in [0,T]$ with $T > 0$.

Acknowledgments. The authors would like to express their sincere thanks to Professor Zhouping Xin for his interests in this work. Zhen would like to thank Professor Tao Luo for stimulating discussions. The first author is supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology grant 2012047640. The second author is supported by the National Science Foundation of China (NSFC) grant 11171223 and the Fundamental Research Funds for the Central Universities grant 2012121005.

REFERENCES

[1] A. Amosov and A. Zlotnik, *A semidiscrete method for solving equations of the one dimensional motion of a non homogeneous viscous heat conducting gas with nonsmooth data*, Izv. Vyssh. Uchebn. Zaved. Mat., **41** (1997), 3–19.

[2] D. Bresch and B. Desjardins, *On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids*, J. Math. Pures Appl., **87** (2007), 57–90.

[3] T. Chang and L. Hsiao, "The Riemann Problem and Interaction of Waves in Gas Dynamics," Longman Scientific and Technical, Harlow; copublished in the United States with John Wiley and Sons, Inc., New York, 1989.

[4] G. Chen, D. Hoff and K. Trivisa, *Global solutions of the compressible Navier-Stokes equations with large discontinuous initial data*, Comm. Partial Differential Equations, **25** (2000), 2233–2257.

[5] G. Chen and J. Glimm, *Global solutions to the compressible Euler equations with geometrical structure*, Comm. Math. Phys., **180** (1996), 153–193.

[6] Y. Cho and H. Kim, *Existence results for viscous polytropic fluids with vacuum*, J. Differential Equations, **228** (2006), 377–411.

[7] R. Duan and Y. Zhao, *A note on the non-formation of vacuum states for compressible Navier-Stokes equations*, J. Math. Anal. Appl., **311** (2005), 744–754.

[8] D. Fang and T. Zhang, *Compressible Navier-Stokes equations with vacuum state in one dimension*, Comm. Pure Appl. Anal., **3** (2004), 675–694.

[9] E. Feireisl, "Dynamics of Viscous Compressible Fluids," Oxford University Press, Oxford, 2004.

[10] H. Fujita-Yashima, M. Padula and A. Novotny, *équation monodimensionnelle d'un gaz vizqueux et calorifére avec des conditions initiales moins restrictives*, Ric. Mat., **42** (1993), 199–248.

[11] D. Hoff, *Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data*, J. Differential Equations, **95** (1992), 33–74.

[12] D. Hoff, *Discontinuous solutions of the Navier-Stokes equations for compressible flow*, Arch. Ration. Mech. Anal., **114** (1991), 15–46.

[13] D. Hoff and D. Serre, *The failure of continuous dependence on initial data for the Navier-Stokes equations of compressible flow*, SIAM J. Appl. Math., **51** (1991), 887–898.

[14] D. Hoff and J. Smoller, *Non-formation of vacuum states for compressible Navier-Stokes equations*, Comm. Math. Phys., **216** (2001), 255–276.

[15] X. Huang and J. Li, *Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations*, preprint, `arXiv:1107.4655`.

[16] X. Huang, J. Li and Z. Xin, *Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations*, Comm. Pure Appl. Math., **65** (2012), 549–585.

[17] S. Jiang and P. Zhang, *On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations*, Comm. Math. Phys., **215** (2001), 559–581.

[18] S. Jiang and P. Zhang, *Global weak solutions to the Navier-Stokes equations for a 1D viscous polytropic ideal gas*, Quart. Appl. Math., **61** (2003), 435–449.

[19] Y. Kanel, *The Cauchy problem for equations of gas dynamics with viscosity*, Siberian Math. J., **20** (1979), 208–218.

[20] A. Kazhikhov, *On the Cauchy problem for the equations of a viscous gas*, Siberian Math. J., **23** (1982), 44–49.

[21] A. Kazhikhov and V. Shelukhin, *Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas*, J. Appl. Math. Mech., **41** (1977), 273–282.

[22] H. Li, J. Li and Z. Xin, *Vanishing of vacuum states and blow-up phenomena of the compressible Navier-Stokes equations*, Comm. Math. Phys., **281** (2008), 401–444.

[23] P. Lions, "Mathematical Topics in Fluid Mechanics. II. Compressible Models," The Clarendon Press, Oxford University Press, New York, 1998.

[24] T. Luo, Z. Xin and T. Yang, *Interface behavior of compressible Navier-Stokes equations with vacuum*, SIAM J. Math. Anal., **31** (2000), 1175–1191.

[25] Z. Luo, *Local existence of classical solutions to the two-dimensional viscous compressible flows with vacuum*, Comm. Math. Sci., **10** (2012), 527–554.

[26] A. Matsumura and T. Nishida, *The initial value problem for the equations of motion of viscous and heat-conductive gases*, Math. Kyoto Univ., **20** (1980), 67–104.

[27] J. Nash, *Le problème de Cauchy pour les équations différentielles d'un fluide général*, Bull. Soc. Math. France., **90** (1962), 487–497.

[28] J. Serrin, *On the uniqueness of compressible fluid motions*, Arch. Ration. Mech. Anal., **3** (1959), 271–288.

[29] Z. Xin, *Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density*, Comm. Pure Appl. Math., **51** (1998), 229–240.

[30] Z. Xin and H. Yuan, *Vacuum state for spherically symmetric solutions of the compressible Navier-Stokes equations*, J. Hyperbolic Differential Equations, **3** (2006), 403–442.

Received June 2012; revised March 2013.

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