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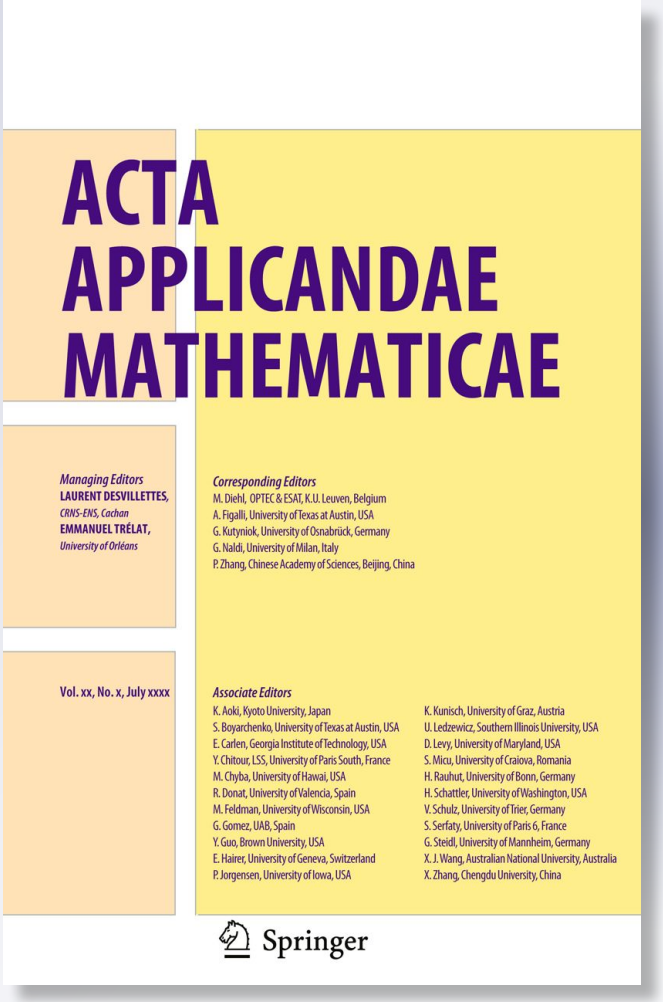

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Asymptotic Profile of Species Migrating on a Growing Habitat

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Abstract This paper deals with a diffusive logistic equation on one dimensional isotropically growing domain. The model equation on growing domains is first presented, and the comparison principle is then proved. The asymptotic behavior of temporal solutions to the reaction-diffusion problem is given by constructing upper and lower solutions. Our result shows that when the domain grows slowly, the species successfully spreads to the whole habitat and stabilizes at a positive steady state, while it dies out in the long run if the domain grows fast. Numerical simulations are also presented to illustrate the analytical result.

Keywords Diffusive logistic equation · Growing domain · Asymptotic profile

Mathematics Subject Classification (2000) 35K57 · 37L15 · 92C15

1 Introduction

As we know, reaction-diffusion systems are used to model many chemical and biological phenomena in the natural world [15]. The diffusive logistic equation is a classical scalar reaction-diffusion equation:

$$\begin{cases} u_t = d\Delta u + u(a - bu), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbf{R}^n$, a, b and d are positive constants. Ecologically, $u(x,t)$ represents the population density, a is the intrinsic growth rate of species, b is the rate of intra-specific competition while d denotes the diffusion coefficient, the homogeneous Dirichlet boundary condition means that the species migrates in a domain surrounded by a hostile environment. Problem (1.1) plays an important role in understanding various population models and some other problems in applied mathematics [8]. It also forms the nucleus of more complex multi-species models in ecology, pattern formation and, most notably, epidemiology [9].

The parabolic problem (1.1) and its generalizations have been extensively studied in literatures [1, 3, 4, 17]. In particular, it is well known that problem (1.1) has only a trivial steady-state solution $u = 0$ when $a \le d\lambda_1(\Omega)$, and a unique nonnegative nontrivial steady-state solution $u^*(x)$ when $a > d\lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ under the Dirichlet boundary conditions on Ω. Moreover, it is further proven that $u(x,t) \to 0$ uniformly if $a \le d\lambda_1(\Omega)$, and when $a > d\lambda_1(\Omega)$, $u(x,t) \to u^*(x)$ uniformly as $t \to \infty$, where $u(x,t)$ is any nonnegative nontrivial solution to system (1.1).

We note that all above works were researched on fixed domains, i.e. the domain Ω is independent of time t. A natural question arises: What is the effect on behavior of solution to problem (1.1) when the domain grows? Indeed, domain growth has been suggested as an important mechanism in pattern formation and election, we refer to [2, 5–7, 10, 16] and the references therein for more details. In understanding the effects of incorporating domain growth to pattern formation, the main research method is numerical computation and simulations [12, 13]. To our knowledge, there are few analytical results for local stabilities and Turing patterns on growing domains. Moreover, very little analytical work has been directly carried out to study the asymptotic behavior of solution to problem (1.1) on growing domains. The main reason is that the presence of time-dependent transport coefficients in the model equations which constructed on growing domain leads to difficulty in stability analysis [11]. In this paper, we try to use the method of upper and lower solutions to study the asymptotic behavior of solution to problem (1.1) on a one dimensional growing domain which isotropically grows to infinity. Our results show that when the domain grows slowly to all the new environment, the species will uniformly tend to a positive constant, while it dies out in the long run if the domain grows fast.

The remainder of this paper is organized as follows. In the next section, a general reaction-diffusion equation is first presented on a continuously growing domain, then a diffusive logistic equation on one dimensional isotropically growing domain is derived. In Sect. 3, we investigate the asymptotic behavior of solution for diffusive logistic equation. To illustrate our analytical result, two numerical examples are presented in Sect. 4. Finally, we give a brief discussion in Sect. 5.

2 Model Formulation

In this section, we first present a general reaction-diffusion equation on a growing domain in $\mathbf{R}^n$ and then derive the diffusive logistic model on a isotropically growing domain in one spatial dimension.

As in [7], let $\Omega(t) \subset \mathbf{R}^n$ be a time-varying domain with its growing boundary $\partial\Omega(t)$. For any point $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \Omega(t)$, we assume that $u(x(t),t)$ is the density of a population, at position $x(t)$ and time $t \ge 0$.

According to the principle of mass conservation and Reynolds transport theorem, the evolution equation for reaction diffusion on growing domain $\Omega(t)$ is readily obtained:

$$\frac{\partial u}{\partial t} + \nabla u \cdot \mathbf{a} + u(\nabla \cdot \mathbf{a}) = d\nabla^2 u + f(u) \quad \text{in } \Omega(t), \tag{2.1}$$

where d is the diffusive coefficient of u, $\mathbf{a} = (\dot{x}_1(t), \dot{x}_2(t), \ldots, \dot{x}_n(t))$ is a flow velocity field due to the growth of domain. $f(u)$ is the reaction term. In (2.1), the effect of domain growth is given by introducing two extra terms into the problem: $\nabla u \cdot \mathbf{a}$, an advection term representing the transport of material around $\Omega(t)$ at a rate determined by the flow $\mathbf{a}$ and $(\nabla \cdot \mathbf{a})u$, a dilution term due to local volume expansion [2].

We now look at the partial differential equation (2.1) from a Lagrangian point of view [11, 14]. Let $y_1, y_2, \ldots, y_n$ be fixed cartesian coordinates in fixed domain $\Omega(0)$ such that $x_1(t) = \hat{x}_1(y_1, y_2, \ldots, y_n, t), x_2(t) = \hat{x}_2(y_1, y_2, \ldots, y_n, t), \ldots, x_n(t) = \hat{x}_n(y_1, y_2, \ldots, y_n, t)$. As t varies, the coordinates $x_1, x_2, \ldots, x_n$ change position with time. These positions are then mapped or transformed to a fixed position given by the $y_1, y_2, \ldots, y_n$ coordinates. Under this transformation, we suppose u is mapped into the new function defined as

$$u(x_1(t), x_2(t), \ldots, x_n(t), t) = v(y_1, y_2, \ldots, y_n, t). \tag{2.2}$$

Thus (2.1) can be translated to another form which is defined on the fixed domain $\Omega(0)$ with respect to $y = (y_1, y_2, \ldots, y_n)$. However, the new equation is also more complicated for arbitrary domain growth [14]. To further simplify the model equation (2.1), we assume that domain growth is uniform and isotropic. By isotropic we mean that the boundary curve deforms continuously at the same rate in all directions at all times [11]. In mathematical terms, $x_1(t) = \hat{x}_1(y_1, y_2, \ldots, y_n, t) = \rho(t)y_1, x_2(t) = \hat{x}_2(y_1, y_2, \ldots, y_n, t) = \rho(t)y_2, \ldots, x_n(t) = \hat{x}_n(y_1, y_2, \ldots, y_n, t) = \rho(t)y_n$, where $\rho(t)$ is called growth function subject to $\rho(0) = 1$ and $\dot{\rho}(t) \geq 0$ for all $t > 0$.

Next we give the diffusive logistic equation on one dimensional isotropically growing domain. Let $\Omega(t) = (-R(t), R(t))$, where $R(t)$ representing the domain size at time t, and $x(t) \in \Omega(t)$ can be described as follows:

$$x(t) = \rho(t)y, \quad y \in (-1, 1). \tag{2.3}$$

Clearly, the transformation (2.3) changes the growing interval $(-R(t), R(t))$ to the fixed interval $(-1, 1)$.

By (2.3), we have $\mathbf{a} = \dot{x}(t) = \dot{\rho}(t)y = \frac{\dot{\rho}}{\rho}x, \nabla \cdot \mathbf{a} = \frac{\dot{\rho}}{\rho}$. If denote $v(y,t) = u(x(t), t)$ $(= u(\rho(t)y, t))$, then $\frac{\partial v}{\partial t} = \nabla u \cdot \mathbf{a} + \frac{\partial u}{\partial t}, \nabla^2 u = \frac{1}{\rho^2(t)}\frac{\partial^2 v}{\partial y^2}$. So (2.1) becomes

$$\frac{\partial v}{\partial t} = \frac{d}{\rho^2(t)}\frac{\partial^2 v}{\partial y^2} - \frac{\dot{\rho}(t)}{\rho(t)}v + f(v), \quad -1 < y < 1,\ t > 0. \tag{2.4}$$

Taking $f(v) = v(a - bv)$ and considering the Dirichlet boundary condition, we then have the following diffusive logistic problem on the growing domain $\Omega(t)$:

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{d}{\rho^2(t)}\frac{\partial^2 v}{\partial y^2} - \frac{\dot{\rho}(t)}{\rho(t)}v + v(a - bv), & -1 < y < 1,\ t > 0, \\ v(-1, t) = v(1, t) = 0, & t > 0, \\ v(y, 0) = v_0(y)(= u_0(y)), & -1 < y < 1. \end{cases} \tag{2.5}$$

3 Asymptotic Behavior of Temporal Solution

In this section, we assume that the size of $\Omega(t)$ tends to infinity, that is $R(t) \to \infty$, as $t \to \infty$. According to (2.3), we correspondingly assume the growth function $\rho(t)$ is continuous differentiable on $[0, +\infty)$ and satisfies

$$\rho(0) = 1, \qquad \dot{\rho}(t) > 0, \qquad \lim_{t\to\infty} \rho(t) = +\infty, \qquad \lim_{t\to\infty} \frac{\dot{\rho}(t)}{\rho(t)} = k. \tag{3.1}$$

A biologically reasonable example of $\rho(t)$ is $\rho(t) = \exp(kt)$ or $\rho(t) = 1 + mt$ [16].

Under the above assumptions, we investigate the asymptotical behavior of the solution of (2.5). Firstly we give the following definition of upper and lower solutions.

Definition 3.1 A functions $\tilde{v}(y,t)$ is called an upper solution of (2.5) if $\tilde{v} \in C^{2,1}((-1,1) \times (0,\infty)) \cap C([-1,1] \times [0,+\infty))$ and satisfies

$$\begin{cases} \tilde{v}_t \geq \frac{d}{\rho^2(t)}\tilde{v}_{yy} - \frac{\dot{\rho}(t)}{\rho(t)}\tilde{v} + \tilde{v}(a - b\tilde{v}), & -1 < y < 1,\ t > 0, \\ \tilde{v}(-1,t) \geq 0, \qquad \tilde{v}(1,t) \geq 0, & t > 0, \\ \tilde{v}(y,0) \geq v_0(y), & -1 < y < 1. \end{cases} \tag{3.2}$$

Similarly, $\hat{v}(y,t) \in C^{2,1}((-1,1) \times (0,+\infty)) \cap C([-1,1] \times [0,+\infty))$ is called a low solution of (2.5) if it satisfies all the reversed inequalities in (3.2).

To show our main result, we need the following lemmas.

Lemma 3.1 (Comparison Principle) *Let $v(y,t)$ be a solution of (2.5), $\tilde{v}(y,t)$ and $\hat{v}(y,t)$ are upper and lower solutions of (2.5) respectively, then $\hat{v}(y,t) \leq v(y,t) \leq \tilde{v}(y,t)$ in $[-1,1] \times [0,+\infty)$.*

Proof Define $w = \tilde{v} - v$, and it is easy to see that $w(y,t)$ satisfies

$$\begin{cases} w_t \geq \frac{d}{\rho^2(t)}w_{yy} + (a - b(\tilde{v} + v) - \frac{\dot{\rho}}{\rho})w, & -1 < y < 1,\ t > 0, \\ w(-1,t) \geq 0, \qquad w(1,t) \geq 0, & t > 0, \\ w(y,0) \geq 0, & -1 < y < 1. \end{cases} \tag{3.3}$$

Applying the maximum principle leads to

$$w(y,t) \geq 0, \quad -1 \leq y \leq 1,\ t \geq 0,$$

that is $\tilde{v}(y,t) \geq v(y,t),\ -1 \leq y \leq 1,\ t \geq 0$. Similarly, $\hat{v}(y,t) \leq v(y,t)$ can be proved. □

Lemma 3.2 *Let $v(y,t)$ be a nonnegative nontrivial solution of the following problem*

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}v_{yy} - \frac{\dot{\rho}(t)}{\rho(t)}v + v(a - bv), & -1 < y < 1,\ t > 0, \\ v(-1,t) = v(1,t) = 0, & t > 0, \\ v(y,0) = v_0(y) \geq 0, & -1 < y < 1. \end{cases} \tag{3.4}$$

If $v_0(y) \in C^2[-1,1],\ v_0(-1) = v_0(1) = v_{0,yy}(-1) = v_{0,yy}(1) = 0$ and $v_{0,yy}(y) \leq 0$ in $[-1,1]$, then $v(y,t) \in C^{2,1}([-1,1] \times [0,+\infty))$ and

$$v_{yy}(y,t) \leq 0 \quad \textit{for } y \in (-1,1),\ t > 0.$$

Proof Since the initial function v_0 is smooth and satisfies the consistency condition:

$$\frac{d}{\rho^2(0)}v_{0,yy}(y) - \frac{\dot{\rho}(0)}{\rho(0)}v_0(y) + v_0(a - bv_0)(y) = 0 \quad \text{for } y = -1, 1,$$

then the standard parabolic regularity theory shows that the solution $v(y,t) \in C^{2,1}([-1,1] \times [0,+\infty))$. Denote $w = v_{yy}$, then it satisfies

$$w_t \le \frac{d}{\rho^2(t)} w_{yy} + \left(-\frac{\dot{\rho}(t)}{\rho(t)} + a - 2bv \right) w. \tag{3.5}$$

In addition, the condition $v_{yy}(y,0) \le 0$ implies $w(y,0) \le 0$. Since $v(-1,t) = 0$, we have $w(-1,t) = v_{yy}(-1,t) = \frac{\rho^2(t)}{d}(v_t + \frac{\dot{\rho}(t)}{\rho(t)} v - v(a-bv))(-1,t) = 0$. Similarly, $w(1,t) = 0$. Using the comparison principle gives that $w(y,t) \le 0$ for $y \in (-1,1),\ t > 0$, which implies that $v_{yy}(y,t) \le 0$ for $y \in (-1,1),\ t > 0$. □

Lemma 3.3 *Let $v(t)$ be the solution to the following problem*:

$$\begin{cases} v' = -\frac{\dot{\rho}(t)}{\rho(t)} v + v(a-bv), \\ v(0) \ge 0, \end{cases} \tag{3.6}$$

where $\rho(t)$ is continuous differentiable on $[0,+\infty)$ and satisfies $\lim_{t\to\infty} \frac{\dot{\rho}(t)}{\rho(t)} = k$.

(1) *If $k < a$, then $\lim_{t\to\infty} v(t) = \frac{a-k}{b}$.*
(2) *If $k > a$, then $\lim_{t\to\infty} v(t) = 0$.*

Proof (1) $\lim_{t\to\infty} \frac{\dot{\rho}(t)}{\rho(t)} = k$ implies that for any $\varepsilon > 0$, we can find $T_1 > 0$ such that $k - \varepsilon \le \frac{\dot{\rho}(t)}{\rho(t)} \le k + \varepsilon$ for $t \ge T_1$. Let $\bar{v}(t)$ be the solution of problem

$$\begin{cases} \bar{v}' = -(k-\varepsilon)\bar{v} + \bar{v}(a - b\bar{v}), \quad t > T_1, \\ \bar{v}(T_1) = v(T_1). \end{cases} \tag{3.7}$$

It follows from the comparison principle that $v(t) \le \bar{v}(t)$ for $t > T_1$. By solving the problem (3.7), we have

$$\bar{v}(t) = \frac{a-k+\varepsilon}{b} e^{(a-k+\varepsilon)(t-T_1)} \left[e^{(a-k+\varepsilon)(t-T_1)} - 1 + \frac{a-k+\varepsilon}{bv(T_1)} \right]^{-1}. \tag{3.8}$$

It is easy to see that $\lim_{t\to\infty} \bar{v}(t) = \frac{a-k+\varepsilon}{b}$ when $a - k > 0$. Thus we have $\limsup_{t\to\infty} v(t) \le \frac{a-k}{b}$ since that ε can be chosen sufficiently small.

Similarly, if we denote $\hat{v}(t)$ be the solution to the following problem

$$\begin{cases} \hat{v}' = -(k+\varepsilon)\hat{v} + \hat{v}(a - b\hat{v}), \quad t > T_1, \\ \hat{v}(T_1) = v(T_1), \end{cases} \tag{3.9}$$

then $v(t) \ge \hat{v}(t)$ and $\lim_{t\to\infty} \hat{v}(t) = \frac{a-k-\varepsilon}{b}$. It follows that $\liminf_{t\to\infty} v(t) \ge \frac{a-k}{b}$. So we obtain $\lim_{t\to\infty} v(t) = \frac{a-k}{b}$ if $k < a$.

(2) If $k > a$, then we can choose $\varepsilon > 0$ sufficiently small such that $a - k + \varepsilon < 0$. Analogously, by arguments similar to those in the proof of (1), we see that $v(t) \le \bar{v}(t)$ for $t > T_1$ and $\lim_{t\to\infty} \bar{v}(t) = 0$. In addition, $\hat{v}(t) = 0$ is a lower solution of problem (3.6). Thus we have $\lim_{t\to\infty} v(t) = 0$ when $k > a$. □

Theorem 3.1 *Suppose that the growth function $\rho(t)$ satisfies (3.1). If $k < a$, then the solution of problem (2.5) satisfies $v(y,t) \to \frac{a-k}{b}$ uniformly in any compact subset of $(-1,1)$ as $t \to \infty$.*

Proof Let $\lambda_1 (= (\frac{\pi}{2})^2)$ be the principal eigenvalue of Laplace operator under the homogenous Dirichlet boundary condition and ϕ $(= \sin \frac{\pi}{2} y)$ be the corresponding eigenfunction of λ_1.

On one hand, the following problem

$$\begin{cases} \hat{v}_t = \frac{d}{\rho^2(t)} \hat{v}_{yy} - \frac{\dot{\rho}(t)}{\rho(t)} \hat{v} + \hat{v}(a - b\hat{v}), & -1 < y < 1,\ t > 0, \\ \hat{v}(-1, t) = 0, \qquad \hat{v}(1, t) = 0, & t > 0, \\ \hat{v}(y, 0) = \delta\phi(y), & -1 < y < 1, \end{cases} \tag{3.10}$$

admits a unique solution $\hat{v}(y, t)$, where δ is a positive constant. First take δ sufficiently small such that $\delta\phi(y) \le v_0(y)$, then $\hat{v}(y, t)$ is an lower solution of (2.5). It follows from the comparison principle that $\hat{v}(y, t) \le v(y, t)$ for $y \in (-1, 1),\ t > 0$.

Since $\lim_{t\to\infty} \rho(t) = +\infty$, then for any $L > \sqrt{\frac{d}{a-k}} \frac{\pi}{2}$, there exists a $T_2 > 0$ such that $\rho(t) > L$ for $t \ge T_2$.

Set $T_0 = \max\{T_1, T_2\}$, where T_1 is taken as in the proof of Lemma 3.3. Since $\hat{v}_{yy}(y, 0) = \delta\phi''(y) = -\lambda_1 \delta\phi(y) \le 0$, it follows from Lemma 3.2 that $\hat{v}_{yy}(y, t) \le 0$ for $y \in (-1, 1)$, $t > 0$. Therefor $\hat{v}(y, t)$ satisfies

$$\hat{v}_t \ge \frac{d}{L^2} \hat{v}_{yy} - (k + \varepsilon)\hat{v} + \hat{v}(a - b\hat{v}), \quad -1 < y < 1, t > T_0. \tag{3.11}$$

Now consider the following problem

$$\begin{cases} v_t = \frac{d}{L^2} v_{yy} + (a - k - \varepsilon)v - bv^2, & -1 < y < 1,\ t > T_0, \\ v(-1, t) = v(1, t) = 0, & t > T_0, \\ v(y, T_0) = \delta\phi(y), & -1 < y < 1. \end{cases} \tag{3.12}$$

Clearly, (3.12) admits a unique solution $\hat{v}_\varepsilon(y, t)$. Using the comparison principle yields that $\hat{v}_\varepsilon(y, t) \le \hat{v}(y, t)$, and therefore,

$$\hat{v}_\varepsilon(y, t) \le \hat{v}(y, t) \le v(y, t), \quad -1 < y < 1, t > T_0. \tag{3.13}$$

Since that $L > \sqrt{\frac{d}{a-k}} \frac{\pi}{2}$, we can choose sufficiently small $\varepsilon > 0$ such that $a - k - \varepsilon > d(\frac{\pi}{2L})^2$. It is well-known that $\hat{v}_\varepsilon(y, t) \to v^*(y)$ uniformly in $[-1, 1]$ as $t \to \infty$, where v^* is the unique positive solution of

$$\begin{cases} -\frac{d}{L^2} v_{yy} = (a - k - \varepsilon)v - bv^2, & -1 < y < 1, \\ v(-1) = v(1) = 0. \end{cases} \tag{3.14}$$

It follows that $\liminf_{t\to\infty} v(y, t) \ge v^*(y)$ uniformly in $[-1, 1]$. Now for any fixed $L > \sqrt{\frac{d}{a-k}} \frac{\pi}{2}$, we define z and w by $z = Ly$, $w(z, t) = v(\frac{z}{L}, t)$, then the problem (3.14) becomes

$$\begin{cases} -dw_{zz} = (a - k - \varepsilon)w - bw^2, & -L < z < L, \\ w(-L) = w(L) = 0. \end{cases} \tag{3.15}$$

Let $w^*(z)$ denote the unique positive solution of the problem (3.15). Using Lemma 2.2 of [8], we easily see that $w^*(z) \to \frac{a-k-\varepsilon}{b}$ as $L \to +\infty$ uniformly in any compact subset of $(-\infty, +\infty)$. That is $v^*(y) \to \frac{a-k-\varepsilon}{b}$ as $L \to +\infty$ uniformly in any compact subset of $(-1, 1)$. Therefore $\liminf_{t\to\infty} v(y, t) \ge \frac{a-k}{b}$ in any compact subset of $(-1, 1)$.

On the other hand, let $\tilde{v}(y,t)$ denote the unique solution of the problem:

$$\begin{cases} \tilde{v}_t = \frac{d}{\rho^2(t)}\tilde{v}_{yy} - \frac{\dot{\rho}(t)}{\rho(t)}\tilde{v} + \tilde{v}(a - b\tilde{v}), & -1 < y < 1,\ t > T_0, \\ \tilde{v}(-1,t) = 0, \qquad \tilde{v}(1,t) = 0, & t > T_0, \\ \tilde{v}(y,0) = M\phi(y), & -1 < y < 1, \end{cases} \tag{3.16}$$

where M is a sufficiently large constant. It follows from the comparison principle that $v(y,t) \le \tilde{v}(y,t)$ for $-1 < y < 1, t > T_0$. Note that $\tilde{v}_{yy}(y,0) = M\phi''(y) = -\lambda_1 M\phi(y) \le 0$, we have $\tilde{v}_{yy}(y,t) \le 0$ for $y \in (-1,1),\ t > 0$ according to Lemma 3.2. So $\tilde{v}(y,t)$ satisfies

$$\tilde{v}_t \le -\frac{\dot{\rho}(t)}{\rho(t)}\tilde{v} + \tilde{v}(a - b\tilde{v}), \quad -1 < y < 1,\ t > T_0. \tag{3.17}$$

Let $\bar{v}(t)$ denote the positive solution of following problem

$$\begin{cases} \bar{v}' = -\frac{\dot{\rho}(t)}{\rho(t)}\bar{v} + \bar{v}(a - b\bar{v}), & t > T_0, \\ \bar{v}(0) = M. \end{cases} \tag{3.18}$$

Using again the comparison principle, we have $\tilde{v}(y,t) \le \bar{v}(t)$ for $-1 < y < 1,\ t > T_0$. It follows from Lemma 3.3 that $\lim_{t\to\infty} \bar{v}(t) = \frac{a-k}{b}$. So we deduce $\limsup_{t\to\infty} v(y,t) \le \frac{a-k}{b}$ uniformly for $y \in (-1,1)$. Therefore we have $\lim_{t\to\infty} v(y,t) = \frac{a-k}{b}$ uniformly for y in any compact subset of $(-1,1)$. This completes the proof. □

Theorem 3.2 *Suppose that the growth function $\rho(t)$ satisfies (3.1), If $k > a$, then the solution of problem (2.5) satisfies $v(y,t) \to 0$ uniformly for $y \in (-1,1)$ as $t \to \infty$.*

Proof Clearly, $\hat{v} = 0$ is a lower solution of (2.5). We define $\tilde{v}(y,t)$ and $\bar{v}(t)$ as in the proof of Theorem 3.1. So

$$v(y,t) \le \tilde{v}(y,t) \le \bar{v}(t) \tag{3.19}$$

holds for $(y,t) \in (-1,1) \times (T_0, +\infty)$. By Lemma 3.3, we have $\bar{v}(t) \to 0$ if $k > a$. Using (3.19) yields $\limsup_{t\to\infty} v(y,t) \le 0$ uniformly for $y \in (-1,1)$. So we obtain that $v(y,t) \to 0$ uniformly for $y \in (-1,1)$ as $t \to \infty$ if $k > a$. □

Remark 3.1 From the proofs of Theorems 3.1 and 3.2, we can see that, if $k < a$, then the solution to the corresponding problem of (2.1) satisfies $u(x,t) \to \frac{a-k}{b}$ uniformly in any compact subset of $(-\infty, +\infty)$ as $t \to \infty$. If $k > a$, then $u(x,t) \to 0$ uniformly for $x \in (-\infty, +\infty)$ as $t \to \infty$.

Remark 3.2 Theorems 3.1 and 3.2 show that when the habitat increases to infinity, for the fixed birth rate a, if k is small, then the species will uniformly tend to a positive constant; if k is large, then the species will be extinct.

4 Numerical Simulations

In this section, we present numerical simulations to illustrate our theoretical analysis using Matlab in one dimensional space.

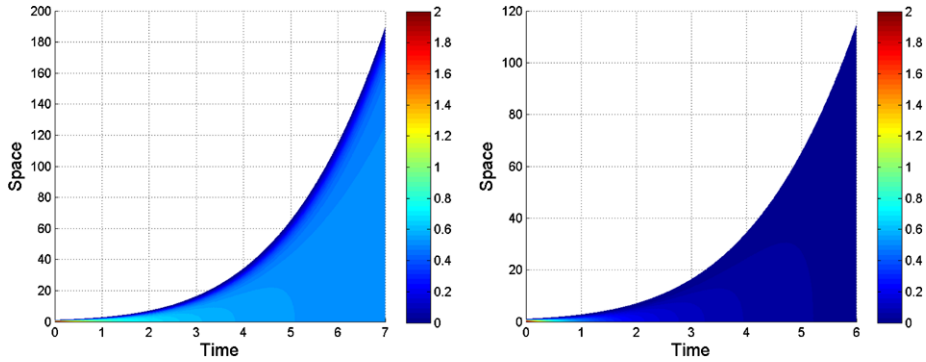

Fig. 1 Numerical simulations of the asymptotic behavior of the solution to problem (2.5) corresponding to the growth function $\rho(t) = \exp(kt)$. *Left*: $k = 1 < a = 1.5$. *Right*: $k = 2 > a = 1.5$

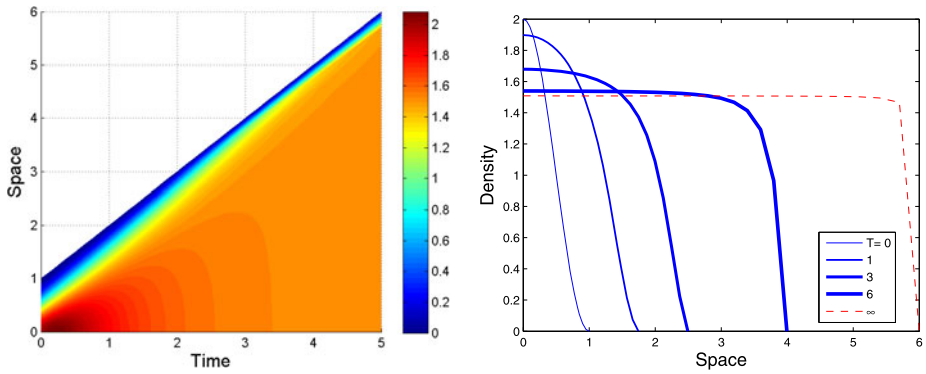

Fig. 2 (Color online) Numerical simulations of the asymptotic behavior of the solution to problem (2.5) corresponding to the growth function $\rho(t) = 1+t$. *Left*: The developing process of domain growth corresponding to the growth function $\rho(t) = 1 + t$. *Right*: Convergence of temporal solutions to the positive constant 1.5 (*red dashed line*)

Regarding the domain growth and symmetry, we choose the growing domain $[0, R(t))$, $t \geq 0$. For any compact subset $[0, x(t)]$ $(= [0, \rho(t)y], 0 \leq y \leq 1)$, we choose exponential growth function $\rho(t) = \exp(kt), k > 0$, or linear growth function $\rho(t) = 1 + mt, m > 0$. It is easy to see that domain grows from initial size $\rho(0) = 1$ to the final size $\rho_\infty = +\infty$.

Firstly we choose parameters in system (2.5): $a = 1.5$, $b = 1$, $d = 1$, and the growth function $\rho(t) = e^t$ $(k = 1 < a)$, then according to Theorem 3.1, solution $v(y, t)$ to problem (2.5) asymptotically converges to $\frac{a-k}{b} = 0.5$ uniformly in any compact subset of $(-1, 1)$ as $t \to \infty$. If we choose $\rho(t) = e^{2t}$, then $k = 2 > a$. By Theorem 3.2, solution $v(y, t)$ to problem (2.5) asymptotically converges to 0 uniformly for $y \in (-1, 1)$ as $t \to \infty$. These are showed in Fig. 1, where the process of domain growth is presented. If we choose another growth function $\rho(t) = 1 + t$, then $k = 0 < a$. By Theorem 3.2, we know that $v(y, t)$ asymptotically converges to $\frac{a}{b} = 1.5$, see Fig. 2.

5 Discussion

We investigated a single species diffusive logistic model on a growing domain which grows isotropically to infinity. In order to simplify the reaction-diffusion equation (RDE) on a continuously growing domain, we assumed that the domain growth is isotropic, and then it was transformed into RDE on fixed domain in the Lagrangian coordinate system [7]. But this transformation bring about time-dependence diffusive coefficient and dilution terms which make it difficult to carry out stability analysis. To our knowledge, most works in studying pattern formation in literatures were done through numerical simulations [6, 7, 12, 13, 16]. In this paper, we succeeded carrying out the analysis of asymptotic behavior of the solution to system (2.5) using the method of upper and lower solutions. Numerical simulations are also given. Our results show that if the relative growth rate of domain is small, then the species will spread and tend to a positive constant state, if the relative growth rate of domain is large, then the species will be extinct. Ecologically, it implies that slow growth of domain takes a positive effect on spreading of the species while fast growth of domain takes a negative effect on the existence of the species.

Domain growth is not a new but interesting topic attracting a lots of attention. However, whether our method can be applied to reaction-diffusion systems modelling two or more species in high dimensional space also need to further investigation. From both theoretical and applied points of view, this provides challenging work in the future.

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