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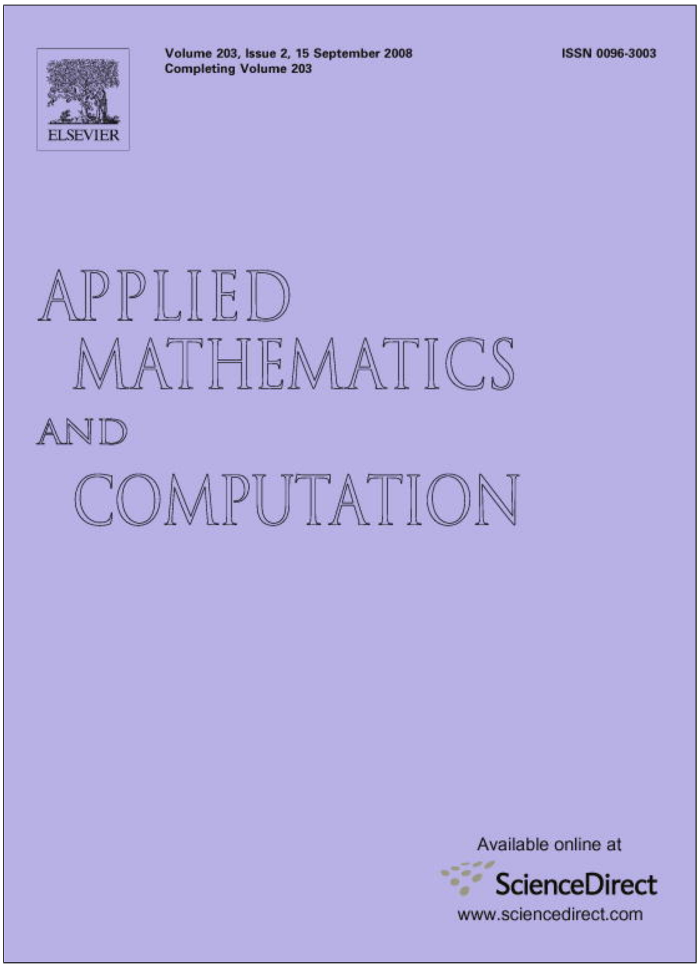
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A unique positive solution for *n*th-order nonlinear impulsive singular integro-differential equations on unbounded domains in Banach spaces ☆

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ABSTRACT

In this paper, the fixed point theory combined with a monotone iterative technique is used to investigate the unique positive solution of a boundary problem for *n*th-order nonlinear impulsive singular integro-differential equations of mixed type on an infinite interval in a Banach space. The conditions for the existence of a unique positive solution are established. In addition, an explicit iterative sequence for approximating the solution of the boundary value problem is derived together with an error estimate. Furthermore, the conditions of the theorems can be easily verified.

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years, because its structure has deep physical interpretation and practical motivation and is based on realistic mathematical models [1–3]. However, the theory for impulsive integro-differential equations in Banach spaces has yet to be developed [4–14]. Most of the previous work in this area only discussed the first-order and second-order equations [4,9–12,14]. Recently, in [6–8], Guo discussed the existence of solutions, multiple solutions and extremal solutions for *n*th-order nonlinear impulsive integro-differential equations with nonsingular arguments in Banach spaces by using the fixed point theory, fixed point index theory and upper and lower solutions together with the monotone iterative technique. Guo also discussed, in [5], the existence of positive solutions for a class of *n*th-order nonlinear impulsive singular integro-differential equations in Banach spaces by means of the fixed point theory for completely continuous operators. The problem of uniqueness of solution has been investigated by many scholars. However, most of the recent work in this area only discussed initial value problems (IVPs) on bounded domains. Recently, for a special case where the IVP has no impulsive and singular arguments, Liu [15] established a unique solution for the IVP by the monotone iterative technique with coupled upper and lower quasi-solutions. A similar conclusion was also obtained by Liu [16]. But one of the requisite assumptions in [5–8,13,15,16] is that the forcing function f in the equation must satisfy some compactness-type conditions, which as we know is difficult and inconvenient to verify in abstract spaces. Recently, Liu got a unique solution for a first-order IVP with no singular arguments by the monotone iterative technique with only an upper (or a lower) solution [14]. The same conclusion was also obtained by Xu in [9,10] and Liu in [11,12] by applying the Banach fixed point theorem.

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Although various results for IVP of nonlinear impulsive integro-differential equations have been obtained, few results are available for the unique solution of boundary value problems (BVPs). For the case where there do not exist impulsive arguments, Guo [6] investigate the unique solution of a BVP for second-order nonlinear nonsingular integro-differential equations of mixed type on an infinite interval in a Banach space. However, there is no result on unique positive solution for the nonlinear impulsive integro-differential equations at the presence of singularities. Therefore, in this paper, we shall use the fixed point theory and monotone iterative technique to investigate the unique positive solution of a BVP on an unbounded domain for a class of nth-order nonlinear impulsive singular integro-differential equations in Banach spaces. We can avoid any compactness-type conditions such as those assumed in Refs. [5–8,15,16] and also the lower or upper solution conditions in [6,7,14–16]. The aim of this paper is to reestablish existence of a unique positive solution, develop an approximation sequence of the solution and derive an error estimate of the approximation sequence for BVP (1.1) under some norm type conditions.

Let E be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leqslant y$ if and only if $y - x \in P$. Let $P_+ = P \setminus \{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. Let $x_i^* \in P_+$ $(i = 0, 1, \dots, n-1)$ and $P_{i\lambda} = \{x \in P : x \geqslant \lambda x_i^*\}$ $(\lambda > 0, i = 0, 1, \dots, n-1)$. When $\lambda = 1$, we write $P_i = P_{i1}$. For details of the cone theory, the reader is referred to Ref. [3].

In this paper, we consider the following BVP for nth-order nonlinear impulsive singular integro-differential equations of mixed type on an unbounded domain in a real Banach space $(E, \|\cdot\|)$:

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t), (Tx)(t), (Sx)(t)), \quad \forall t \in J'_+, \\ \Delta x^{(i)}|_{t=t_k} = I_{ik}(x(t_k), x'(t_k), \dots, x^{(n-1)}(t_k)) \\ \quad (i = 0, 1, \dots, n-1;\ k = 1, 2, 3, \dots), \\ x^{(i)}(0) = x_{0i}\ (i = 0, 1, \dots, n-2), x^{(n-1)}(\infty) = \beta x^{(n-1)}(0), \end{cases} \tag{1.1}$$

where $J = [0, +\infty)$, $0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $J_+ = (0, \infty)$, $J'_+ = J_+ \setminus \{t_1, \dots, t_k, \dots\}$, $f \in C[J_+ \times P_{0\lambda} \times P_{1\lambda} \times \cdots \times P_{(n-1)\lambda} \times P \times P, P]$ for any $\lambda > 0$, $I_{ik} \in C[P_{0\lambda} \times P_{1\lambda} \times \cdots \times P_{(n-1)\lambda}, P]$ for any $\lambda > 0$ $(i = 0, 1, \dots, n-1;\ k = 1, 2, 3, \dots)$, $\beta > 1$, $x^{(n-1)}(\infty) = \lim_{t\to\infty} x^{(n-1)}(t)$ and

$$(Tx)(t) = \int_0^t K(t,s)x(s)\mathrm{d}s, \quad (Sx)(t) = \int_0^\infty H(t,s)x(s)\mathrm{d}s, \tag{1.2}$$

in which $K \in C[D, J]$, $D = \{(t,s) \in J \times J :\ t \geqslant s\}$, $H \in C[J \times J, J]$. $\Delta x^{(i)}|_{t=t_k} = x^{(i)}(t_k^+) - x^{(i)}(t_k^-)$, where $x^{(i)}(t_k^+)$ and $x^{(i)}(t_k^-)$ represent the right and left limits of $x^{(i)}(t)$ $(i = 0, 1, \dots, n-1)$ at $t = t_k$, respectively. BVP (1.1) is singular because we permit that $\|f(t, x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1})\| \to \infty$ as $t \to 0^+$ or $x_i \to \theta^+$ $(i = 0, 1, \dots, n-1)$ ($x_i \to \theta^+$ means $x_i > \theta, x_i \to \theta$), and $\|I_{ik}(x_0, x_1, \dots, x_{n-1})\| \to \infty$ as $x_j \to \theta^+$ $(j = 0, 1, \dots, n-1;\ i = 0, 1, \dots, n-1;\ k = 1, 2, 3, \dots)$.

Let $\mathrm{PC}[J, E] = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+)$ exist for $k = 1, 2, 3, \dots\}$ and $\mathrm{BPC}[J, E] = \{x \in \mathrm{PC}[J, E] : \sup_{t\in J} \mathrm{e}^{-t}\|x(t)\| < \infty\}$. It is easy to see that $\mathrm{BPC}[J, E]$ is a Banach space with norm $\|x\|_{\mathrm{B}} = \sup_{t\in J}(\mathrm{e}^{-t}\|x(t)\|)$.

Let $\mathrm{PC}^{n-1}[J, E] = \{x : x^{(n-1)}(t)$ exist at $t \neq t_k$ and be continuous at $t \neq t_k$ and $x^{(n-1)}(t_k^+)$ and $x^{(n-1)}(t_k^-)$ exist for $k = 1, 2, 3, \dots\}$. For $x \in \mathrm{PC}^{n-1}[J, E]$, as shown in [13], $x^{(i)}(t_k^+)$ and $x^{(i)}(t_k^-)$ exist for $i = 0, 1, 2, \dots, n-2$ and $k = 1, 2, 3, \dots$. Define $x^{(i)}(t_k) = x^{(i)}(t_k^-)$ $(i = 1, 2, \dots, n-1)$. Then $x^{(i)} \in \mathrm{PC}[J, E]$ and, naturally, in (1.1) and in what follows, $x^{(i)}(t_k)$ is understood as $x^{(i)}(t_k^-)$ $(i = 1, 2, \dots, n-1)$.

Let $\mathrm{DPC}^{n-1}[J, E] = \{x \in \mathrm{PC}^{n-1}[J, E] : x^{(i)} \in \mathrm{BPC}[J, E],\ i = 0, 1, \dots, n-1\}$. It is easy to see that $\mathrm{DPC}^{n-1}[J, E]$ is a Banach space with norm $\|x\|_D = \max\{\|x\|_{\mathrm{B}}, \|x'\|_{\mathrm{B}}, \dots, \|x^{(n-1)}\|_{\mathrm{B}}\}$.

Let $\mathrm{BPC}[J, P] = \{x \in \mathrm{BPC}[J, E] :\ x(t) \geqslant \theta\ \forall t \in J\}$ and $\mathrm{DPC}^{n-1}[J, P] = \{x \in \mathrm{DPC}^{n-1}[J, E] :\ x^{(i)}(t) \geqslant \theta\ \forall t \in J, i = 0, 1, \dots, n-1\}$. It is easy to see that $\mathrm{BPC}[J, P]$ is a cone in space $\mathrm{BPC}[J, E]$ and $\mathrm{DPC}^{n-1}[J, P]$ is a cone in space $\mathrm{DPC}^{n-1}[J, E]$.

The function $x \in \mathrm{PC}^{n-1}[J, E] \cap C^n[J'_+, E]$ is called a positive solution of BVP (1.1) if $x^{(i)}(t) > \theta$ $(i = 0, 1, \dots, n-1)$ for $t \in J$ and $x(t)$ satisfies (1.1).

The rest of the paper is organized as follows. In Section 2, we give several important lemmas. The main theorems are formulated and proved in Section 3, followed by an example in Section 4 to demonstrate the application of our results.

2. Preliminaries

For convenience in presentation, we list below some conditions to be used throughout the rest of the paper.

(H_1)

$$k^* = \sup_{t\in J} \int_0^t K(t,s)\mathrm{d}s < \infty, \quad h^* = \sup_{t\in J}\left(\mathrm{e}^{-t}\int_0^t H(t,s)\mathrm{e}^s\mathrm{d}s\right) < \infty$$

and

$$\lim_{t'\to t} \int_0^\infty |H(t',s) - H(t,s)|\mathrm{e}^s\mathrm{d}s = 0 \quad \forall t \in J.$$

(H_2) There exist $a_i \in C[J_+, J]$ $(i = 0, 1, \ldots, n+1)$ such that

$$\|f(t, u_0, u_1, \ldots, u_{n+1}) - f(t, \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n+1})\| \leqslant \sum_{i=0}^{n+1} a_i(t)\|u_i - \bar{u}_i\| \;\; \forall t \in J_+, \;\; u_i \in P_i \;\; (i = 0, 1, \ldots, n-1), \;\; u_n, u_{n+1} \in P,$$

$$a^* = \int_0^\infty \left[\sum_{i=0}^{n-1} a_i(t) + a_n(t)k^* + a_{n+1}(t)h^*\right] \mathrm{e}^t \mathrm{d}t < \infty,$$

$$\tau = \int_0^\infty \sum_{i=0}^{n-1} a_i(t)\|x_i^*\| \mathrm{d}t < \infty, \quad \gamma = \int_0^\infty \|f(t, x_0^*, \ldots, x_{n-1}^*, \theta, \theta)\| \mathrm{d}t < \infty.$$

(H_3) There exist $b_{jkl} \geqslant 0$ $(j, l = 0, 1, \ldots, n-1; \;\; k = 1, 2, 3, \ldots)$ such that $\forall u_i \in P_i$ $(i = 0, 1, \ldots, n-1)$

$$\|I_{jk}(u_0, u_1, \ldots, u_{n-1}) - I_{jk}(\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_{n-1})\| \leqslant \sum_{l=0}^{n-1} b_{jkl}\|u_l - \bar{u}_l\|, \quad b^* = \sum_{k=1}^{\infty}\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} b_{jkl}\mathrm{e}^{t_k} < \infty,$$

$$\lambda = \sum_{k=1}^{\infty}\sum_{j=0}^{n-1}\sum_{l=0}^{n-1} b_{jkl}\|x_l^*\| < \infty, \quad \delta = \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} \|I_{jk}(x_0^*, x_1^*, \ldots, x_{n-1}^*)\| < \infty.$$

(H_4) There exist $0 < t_* < t^* < \infty$ and $\sigma \in C[I, J]$ $(I = [t_*, t^*])$ such that

$$f(t, x_0, x_1, \ldots, x_{n-1}, x_n, x_{n+1}) \geqslant \sigma(t)x_{n-1}^* \quad \forall t_* \leqslant t \leqslant t^*, \quad x_i \geqslant x_i^* \; (i = 0, 1, \ldots, n-1), \quad x_n \geqslant \theta, \quad x_{n+1} \geqslant \theta,$$

and

$$\int_{t_*}^{t^*} \sigma(s)\mathrm{d}s \geqslant \beta - 1.$$

Lemma 2.1 [8]. *If condition* (H_1) *is satisfied, then the operator* T *and* S *defined by* (1.2) *are bounded linear operators from* $\mathrm{BPC}[J, E]$ *into* $\mathrm{BPC}[J, E]$ *and* $\|T\| \leqslant k^*, \;\; \|S\| \leqslant h^*$. *Moreover,* $T(\mathrm{BPC}[J, P]) \subset \mathrm{BPC}[J, P], \;\; S(\mathrm{BPC}[J, P]) \subset \mathrm{BPC}[J, P]$.

Lemma 2.2 [6]. *If* $x \in \mathrm{PC}^{n-1}[J, E] \cap [J'_+, E]$ *and* $\int_0^\infty \|x^{(n)}(t)\|\mathrm{d}t < \infty$, *then*

$$x(t) = \sum_{j=0}^{n-1} \frac{t^j}{j!}x^{(j)}(0) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}x^{(n)}(s)\mathrm{d}s + \sum_{0<t_k<t}\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{j!}[x^{(j)}(t_k^+) - x^{(j)}(t_k)] \quad \forall t \in J. \tag{2.1}$$

In what follows, we write $Q = \{x \in \mathrm{DPC}^{n-1}[J, P] : x^{(i)}(t) \geqslant x_i^* \; \forall t \in J, i = 0, 1, \ldots, n-1\}$. Evidently, Q is a closed convex set in space $\mathrm{DPC}^{n-1}[J, E]$.

Lemma 2.3. *If conditions* (H_1), (H_2) *and* (H_3) *are satisfied, then*

$$\int_0^\infty \|f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s))\|\mathrm{d}s \leqslant a^*\|x\|_D + \tau + \gamma \quad \forall x \in Q \tag{2.2}$$

and

$$\sum_{k=1}^{\infty}\sum_{j=0}^{n-1} \|I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k))\| \leqslant b^*\|x\|_D + \lambda + \delta \quad \forall x \in Q. \tag{2.3}$$

Proof. By (H_2) and (H_3), we have

$$\|f(t, u_0, u_1, \ldots, u_{n+1})\| \leqslant \sum_{i=0}^{n+1} a_i(t)\|u_i\| + \sum_{i=0}^{n-1} a_i(t)\|x_i^*\| + \|f(t, x_0^*, x_1^*, \ldots, x_{n-1}^*, \theta, \theta)\|$$

and

$$\sum_{j=0}^{n-1} \|I_{jk}(u_0, u_1, \ldots, u_{n-1})\| \leqslant \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} b_{jkl}(\|u_l\| + \|x_l^*\|) + \sum_{j=0}^{n-1} \|I_{jk}(x_0^*, x_1^*, \ldots, x_{n-1}^*)\|.$$

So, for $x \in Q$, Lemma 2.1 implies that

$$\begin{aligned}
\|f(t,x(t),x'(t),\ldots,x^{(n-1)}(t),(Tx)(t),(Sx)(t))\| &\leqslant \sum_{i=0}^{n-1} a_i(t)e^t\frac{\|x^{(i)}(t)\|}{e^t} + a_n(t)e^t\frac{\|(Tx)(t)\|}{e^t} + a_{n+1}(t)e^t\frac{\|(Sx)(t)\|}{e^t} \\
&\quad + \sum_{i=0}^{n-1} a_i(t)\|x_i^*\| + \|f(t,x_0^*,x_1^*,\ldots,x_{n-1}^*,\theta,\theta)\| \\
&\leqslant \sum_{i=0}^{n-1} a_i(t)e^t\|x^{(i)}\|_B + a_n(t)e^t k^*\|x\|_B + a_{n+1}(t)e^t h^*\|x\|_B + \sum_{i=0}^{n-1} a_i(t)\|x_i^*\| \\
&\quad + \|f(t,x_0^*,x_1^*,\ldots,x_{n-1}^*,\theta,\theta)\| \\
&\leqslant \left[\sum_{i=0}^{n-1} a_i(t) + k^*a_n(t) + h^*a_{n+1}(t)\right]e^t\|x\|_D + \sum_{i=0}^{n-1} a_i(t)\|x_i^*\| \\
&\quad + \|f(t,x_0^*,x_1^*,\ldots,x_{n-1}^*,\theta,\theta)\| \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{n-1}\|I_{jk}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k))\| &\leqslant \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} b_{jkl}\left(e^{t_k}\|\frac{x^{(l)}(t_k)}{e^{t_k}}\| + \|x_l^*\|\right) + \sum_{j=0}^{n-1}\|I_{jk}(x_0^*,x_1^*,\ldots,x_{n-1}^*)\| \\
&\leqslant \sum_{j=0}^{n-1}\sum_{l=0}^{n-1} b_{jkl}(e^{t_k}\|x\|_D + \|x_l^*\|) + \sum_{j=0}^{n-1}\|I_{jk}(x_0^*,x_1^*,\ldots,x_{n-1}^*)\|. \qquad (2.5)
\end{aligned}$$

It follows from (2.4) and (2.5) that (2.2) and (2.3) hold, and Lemma 2.3 is proved. □

Remark 2.1. If conditions $(H_1), (H_2)$ and (H_3) are satisfied, then, for any $x \in Q$, the infinite integral

$$\int_0^\infty f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))ds$$

and the infinite series

$$\sum_{k=1}^{\infty}\sum_{j=0}^{n-1} I_{jk}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k))$$

are convergent.

Lemma 2.4. *Let conditions (H_1)–(H_4) be satisfied. Then $x \in Q \cap C^n[J'_+,E]$ is a solution of BVP (1.1) if and only if $x \in Q$ is a solution of the following impulsive integral equation:*

$$\begin{aligned}
x(t) = \sum_{j=0}^{n-2}\frac{t^j}{j!}x_{0j} &+ \frac{t^{n-1}}{(\beta-1)(n-1)!}\left\{\int_0^\infty f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))ds + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k))\right\} \\
&+ \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))ds \\
&+ \sum_{0<t_k<t}\sum_{j=0}^{n-1}\frac{(t-t_k)^j}{j!}I_{jk}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k)) \quad \forall t \in J. \qquad (2.6)
\end{aligned}$$

Proof. If $x \in Q \bigcap C^n[J'_+,E]$ is a solution of BVP (1.1), then by Lemma 2.3, we have

$$\int_0^\infty \|x^{(n)}(t)\|dt = \int_0^\infty \|f(t,x(t),x'(t),\ldots,x^{(n-1)}(t),(Tx)(t),(Sx)(t))\|dt \leqslant a^*\|x\|_D + \tau + \gamma < \infty. \qquad (2.7)$$

So, (2.1) holds. Differentiating (2.1), we can get

$$x^{(n-1)}(t) = x^{(n-1)}(0) + \int_0^t x^{(n)}(s)ds + \sum_{0<t_k<t}\left[x^{(n-1)}(t_k^+) - x^{(n-1)}(t_k)\right] \quad \forall t \in J. \qquad (2.8)$$

Substituting (1.1) into (2.1) and (2.8), we have

$$x(t) = \sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0j} + \frac{t^{n-1}}{(n-1)!} x^{(n-1)}(0) + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s$$
$$+ \sum_{0<t_k<t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \forall t \in J, \tag{2.9}$$

and

$$x^{(n-1)}(t) = x^{(n-1)}(0) + \int_0^t f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s + \sum_{0<t_k<t} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \forall t \in J. \tag{2.10}$$

Letting $t \to \infty$ in both sides of (2.10) and using the conclusion in Remark 2.1, we obtain

$$x^{(n-1)}(\infty) = x^{(n-1)}(0) + \int_0^\infty f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \forall t \in J. \tag{2.11}$$

Using the relation $x^{(n-1)}(\infty) = \beta x^{(n-1)}(0)$, we get

$$x^{(n-1)}(0) = \frac{1}{\beta - 1} \left\{ \int_0^t f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \right\}. \tag{2.12}$$

Now, substituting (2.12) into (2.9), we see that $x(t)$ satisfies Eq. (2.6). Conversely, if $x \in Q$ is a solution of Eq. (2.6), then, direct differentiation of (2.6) gives

$$x^{(i)}(t) = \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} x_{0j} + \frac{t^{n-1-i}}{(\beta-1)(n-1-i)!} \left\{ \int_0^\infty f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s \right.$$
$$\left. + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \right\} + \frac{1}{(n-1-i)!} \int_0^t (t-s)^{n-1-i} f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s$$
$$+ \sum_{0<t_k<t} \sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \forall t \in J \ (i = 0, 1, \ldots, n-2), \tag{2.13}$$

$$\begin{aligned} x^{(n-1)}(t) = &\frac{1}{\beta - 1} \left\{ \int_0^\infty f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s \right. \\ &\left. + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \right\} \\ &+ \int_0^t f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s \\ &+ \sum_{0<t_k<t} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \forall t \in J. \end{aligned} \tag{2.14}$$

$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t), (Tx)(t), (Sx)(t)) \quad \forall t \in J'_+. \tag{2.15}$$

So, $x \in C^n[J'_+, E]$ and, by (2.13)–(2.15), it is easy to see that $x(t)$ satisfies (1.1). Then, Lemma 2.4 is proved. □

Consider an operator A defined by

$$\begin{aligned} (Ax)(t) = &\sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0j} + \frac{t^{n-1}}{(\beta-1)(n-1)!} \\ &\times \left\{ \int_0^\infty f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \right\} \\ &+ \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \mathrm{d}s \\ &+ \sum_{0<t_k<t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) \quad \forall t \in J. \end{aligned} \tag{2.16}$$

It is easy to verify that $x \in Q \cap C^n[J'_+, E]$ is a positive solution of BVP (1.1) if and only if $x \in Q$ is a fixed point of the operator A.

Lemma 2.5. *If conditions* $(H_1)-(H_4)$ *are satisfied, then the operator A defined by (2.16) is an operator from Q into Q.*

Proof. Let $x \in Q$. Differentiation of Eq. (2.16) gives

$$\begin{aligned}(Ax)^{(i)}(t) &= \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!}x_{0j} + \frac{t^{n-1-i}}{(\beta-1)(n-1-i)!} \\ &\times \left\{ \int_0^\infty f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k)) \right\} \\ &+ \frac{1}{(n-1-i)!}\int_0^t (t-s)^{n-1-i} f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))\mathrm{d}s \\ &+ \sum_{0<t_k<t}\sum_{j=i}^{n-1} \frac{(t-t_k)^{j-i}}{(j-i)!} I_{jk}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k)) \quad \forall t \in J \ (i=0,1,\ldots,n-2),\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}(Ax)^{(n-1)}(t) &= \frac{1}{\beta-1}\left\{ \int_0^\infty f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{k=1}^{\infty} I_{(n-1)k}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k)) \right\} \\ &+ \int_0^t f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))\mathrm{d}s + \sum_{0<t_k<t} I_{(n-1)k}(x(t_k),x'(t_k),\ldots,x^{(n-1)}(t_k)) \quad \forall t \in J.\end{aligned} \tag{2.18}$$

It follows from (2.2), (2.3), (2.17) and (2.18) that

$$\begin{aligned}\|(Ax)^{(i)}(t)\| &\leqslant \max\{\|x_{0i}\|,\ldots,\|x_{0(n-2)}\|\}\mathrm{e}^t + \frac{\mathrm{e}^t}{\beta-1}\{a^*\|x\|_D + \tau + \gamma + b^*\|x\|_D + \lambda + \delta\} + \mathrm{e}^t(a^*\|x\|_D + \tau + \gamma) \\ &+ \mathrm{e}^t(b^*\|x\|_D + \lambda + \delta) \quad \forall t \in J \ (i=0,1,\ldots,n-1),\end{aligned} \tag{2.19}$$

which implies

$$\|(Ax)^{(i)}\|_B \leqslant \max\{\|x_{0i}\| : i=0,1,\ldots,n-2\} + \frac{\beta}{\beta-1}(\tau+\gamma+\lambda+\delta) + \frac{\beta}{\beta-1}(a^*+b^*)\|x\|_D \quad (i=0,1,\ldots,n-1). \tag{2.20}$$

Hence, by (2.16) and (2.20), we have that $A : \mathrm{DPC}^{n-1}[J,P] \to \mathrm{DPC}^{n-1}[J,P]$ and

$$\|Ax\|_D \leqslant \max\{\|x_{0i}\| : i=0,1,\ldots,n-2\} + \frac{\beta}{\beta-1}(\tau+\gamma+\lambda+\delta) + \frac{\beta}{\beta-1}(a^*+b^*)\|x\|_D \quad \forall\, x \in \mathrm{DPC}^{n-1}[J,P]. \tag{2.21}$$

By (2.17), we get

$$(Ax)^{(i)}(t) \geqslant x_{0i} \geqslant x_i^* \quad \forall t \in J \ (i=0,1,\ldots,n-2). \tag{2.22}$$

Moreover, (2.18) and condition (H_4) imply

$$(Ax)^{(n-1)}(t) \geqslant \frac{1}{\beta-1}\int_{t_*}^{t^*} f(s,x(s),x'(s),\ldots,x^{(n-1)}(s),(Tx)(s),(Sx)(s))\mathrm{d}s \geqslant \frac{1}{\beta-1}\left(\int_{t_*}^{t^*}\sigma(s)\mathrm{d}s\right)x_{n-1}^* \geqslant x_{n-1}^* \quad \forall t \in J. \tag{2.23}$$

It follows from (2.21)–(2.23) that $Ax \in Q$ and Lemma 2.5 is proved. □

3. Main results

Theorem 3.1. *Let conditions* $(H_1)-(H_4)$ *be satisfied. Assume that*

$$\alpha = \frac{\beta}{\beta-1}(a^*+b^*) < 1, \tag{3.1}$$

then BVP (1.1) has a unique positive solution $\bar{x} \in Q \cap C^n[J'_+, E]$; *moreover, for any* $x_0 \in Q$ *there exists a monotone iterative sequence* $\{x_m(t)\}$ *such that* $x_m^{(i)}(t) \to \bar{x}^{(i)}(t)$ *as* $m \to \infty$ $(i = 0, 1, \ldots, n-1)$ *uniformly on* $J_r = [0, r]$ *for any* $r > 0$ *and* $x_m^{(n)}(t) \to \bar{x}^{(n)}(t)$ *as* $m \to \infty$ *for any* $t \in J'_+$, *where*

$$\begin{aligned} x_m(t) = {} & \sum_{j=0}^{n-2} \frac{t^j}{j!} x_{0j} + \frac{t^{n-1}}{(\beta-1)(n-1)!} \left\{ \int_0^\infty f(s, x_{m-1}(s), x'_{m-1}(s), \ldots, x_{m-1}^{(n-1)}(s), (Tx_{m-1})(s), (Sx_{m-1})(s)) \mathrm{d}s \right. \\ & \left. + \sum_{k=1}^{\infty} I_{(n-1)k}(x_{m-1}(t_k), x'_{m-1}(t_k), \ldots, x_{m-1}^{(n-1)}(t_k)) \right\} \\ & + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, x_{m-1}(s), x'_{m-1}(s), \ldots, x_{m-1}^{(n-1)}(s), (Tx_{m-1})(s), (Sx_{m-1})(s)) \mathrm{d}s \\ & + \sum_{0<t_k<t} \sum_{j=0}^{n-1} \frac{(t-t_k)^j}{j!} I_{jk}(x_{m-1}(t_k), x'_{m-1}(t_k), \ldots, x_{m-1}^{(n-1)}(t_k)) \quad \forall t \in J \ (m = 1, 2, 3, \ldots). \end{aligned} \tag{3.2}$$

In addition, there exists an error estimate for the approximation sequence

$$\|x_m - \bar{x}\|_D \leqslant \frac{\alpha^m}{1-\alpha} \|x_1 - x_0\|_D. \tag{3.3}$$

Proof. By (2.17) and (2.18), we find

$$\begin{aligned} \|(Ax)^{(i)}(t) - (Ay)^{(i)}(t)\| \leqslant {} & \frac{\mathrm{e}^t}{\beta-1} \int_0^\infty \{\|f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \\ & - f(s, y(s), y'(s), \ldots, y^{(n-1)}(s), (Ty)(s), (Sy)(s))\|\} \mathrm{d}s \\ & + \frac{\mathrm{e}^t}{\beta-1} \sum_{k=1}^{\infty} \{\|I_{(n-1)k}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) - I_{(n-1)k}(y(t_k), y'(t_k), \ldots, y^{(n-1)}(t_k))\|\} \\ & + \int_0^t \mathrm{e}^{t-s} \{\|f(s, x(s), x'(s), \ldots, x^{(n-1)}(s), (Tx)(s), (Sx)(s)) \\ & - f(s, y(s), y'(s), \ldots, y^{(n-1)}(s), (Ty)(s), (Sy)(s))\|\} \mathrm{d}s \\ & + \sum_{0<t_k<t} \sum_{j=i}^{n-1} \mathrm{e}^{t-t_k} \{\|I_{jk}(x(t_k), x'(t_k), \ldots, x^{(n-1)}(t_k)) - I_{jk}(y(t_k), y'(t_k), \ldots, y^{(n-1)}(t_k))\|\} \\ & \forall x, y \in Q \ (i = 0, 1, \ldots, n-1). \end{aligned} \tag{3.4}$$

By (2.2), (2.3), (3.4) and conditions (H_1)–(H_3), we obtain

$$\begin{aligned} \left\| \frac{(Ax)^{(i)}(t) - (Ay)^{(i)}(t)}{\mathrm{e}^t} \right\| \leqslant {} & \frac{1}{\beta-1} \int_0^\infty \left\{ \sum_{i=0}^{n-1} a_i(s) \mathrm{e}^s \frac{\|x^{(i)}(s) - y^{(i)}(s)\|}{\mathrm{e}^s} + a_n(s) \mathrm{e}^s \frac{\|(Tx)(s) - (Ty)(s)\|}{\mathrm{e}^s} \right. \\ & \left. + a_{n+1}(s) \mathrm{e}^s \frac{\|(Sx)(s) - (Sy)(s)\|}{\mathrm{e}^s} \right\} \mathrm{d}s + \frac{1}{\beta-1} \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \left\{ b_{(n-1)kl} \mathrm{e}^{t_k} \frac{\|x^{(l)}(t_k) - y^{(l)}(t_k)\|}{\mathrm{e}^{t_k}} \right\} \\ & + \int_0^t \left\{ \sum_{i=0}^{n-1} a_i(s) \mathrm{e}^{-s} \|x^{(i)}(s) - y^{(i)}(s)\| + a_n(s) \mathrm{e}^{-s} \|(Tx)(s) - (Ty)(s)\| + a_{n+1}(s) \mathrm{e}^{-s} \|(Sx)(s) - (Sy)(s)\| \right\} \mathrm{d}s \\ & + \sum_{0<t_k<t} \sum_{j=i}^{n-1} \sum_{l=0}^{n-1} b_{jkl} \mathrm{e}^{-t_k} \|x^{(l)}(t_k) - y^{(l)}(t_k)\| \\ \leqslant {} & \frac{1}{\beta-1} \int_0^\infty \left\{ \sum_{i=0}^{n-1} a_i(s) \mathrm{e}^s \|x^{(i)} - y^{(i)}\|_{\mathrm{B}} + a_n(s) \mathrm{e}^s k^* \|x - y\|_{\mathrm{B}} + a_{n+1}(s) \mathrm{e}^s h^* \|x - y\|_{\mathrm{B}} \right\} \mathrm{d}s \\ & + \frac{1}{\beta-1} \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \{ b_{(n-1)kl} \mathrm{e}^{t_k} \|x^{(l)} - x^{(l)}\|_{\mathrm{B}} \} \\ & + \int_0^\infty \left\{ \sum_{i=0}^{n-1} a_i(s) \|x^{(i)} - y^{(i)}\|_{\mathrm{B}} + a_n(s) k^* \|x - y\|_{\mathrm{B}} + a_{n+1}(s) h^* \|x - y\|_{\mathrm{B}} \right\} \mathrm{d}s \\ & + \sum_{k=1}^{\infty} \sum_{j=i}^{n-1} \sum_{l=0}^{n-1} b_{jkl} \|x^{(l)} - y^{(l)}\|_{\mathrm{B}} \quad \forall t \in J, \ \ x, y \in Q \ (i = 0, 1, \ldots, n-1), \end{aligned} \tag{3.5}$$

which implies

$$\|(Ax)^{(i)}-(Ay)^{(i)}\|_B\leqslant\frac{\beta}{\beta-1}(a^*+b^*)\|x-y\|_D\quad(i=0,1,\ldots,n-1).$$

Hence

$$\|Ax-Ay\|_D\leqslant\alpha\|x-y\|_D\quad\forall\, x,y\in Q. \tag{3.6}$$

Since $\alpha<1$ on account of (3.1) and Q is a closed convex set in space $DPC^{n-1}[J,E]$, the Banach fixed point theorem and Lemma 2.5 imply that A has a unique fixed point $\bar{x}$ in Q, and for any $x_0\in Q$, $\|x_m-\bar{x}\|_D\to 0$ as $m\to\infty$, where $x_m=Ax_{m-1}$ $(m=1,2,3,\ldots)$. By Lemma 2.4, this unique fixed point $\bar{x}$ is the unique solution of BVP (1.1) in $Q\cap C^n[J'_+,E]$. By (3.6), we have

$$\|x_m-x_{m-1}\|_D\leqslant\alpha^{m-1}\|x_1-x_0\|_D\ (m=1,2,3,\ldots),$$

and so,

$$\|x_k-x_m\|_D\leqslant\|x_k-x_{k-1}\|_D+\cdots+\|x_{m+1}-x_m\|_D\leqslant(\alpha^{k-1}+\cdots+\alpha^m)\|x_1-x_0\|_D=\frac{\alpha^m(1-\alpha^{k-m})}{1-\alpha}\|x_1-x_0\|_D,\quad 1\leqslant m\leqslant k. \tag{3.7}$$

Letting $k\to\infty$ in both sides of (3.7), we have that (3.3) holds. By the definition of the norm in $DPC^{n-1}[J,E]$, we have

$$\|x_m^{(i)}(t)-\bar{x}^{(i)}(t)\|\leqslant e^t\|x_m-\bar{x}\|_D\quad\forall t\in J\ (i=0,1,\ldots,n-1;\ m=1,2,3,\ldots),$$

and therefore, the iterative sequence $\{x_m(t)\}$ satisfies that $x_m^{(i)}(t)\to\bar{x}^{(i)}(t)$ as $m\to\infty$ $(i=0,1,\ldots,n-1)$ uniformly on $J_r=[0,r]$ for any $r>0$. Let $t\in J_+$ be arbitrarily fixed. By (3.3), we have

$$H(t,s)\|x_m(s)-y(s)\|\leqslant\frac{\alpha^m}{1-\alpha}H(t,s)e^s\|x_1-x_0\|_D. \tag{3.8}$$

By (3.8) and the dominated convergence theorem, we have

$$\|(Sx_m)(t)-(S\bar{x})(t)\|\leqslant\int_0^\infty H(t,s)\|x_m(s)-\bar{x}(s)\|ds\to 0,\quad\text{as } m\to\infty,$$

which implies that

$$(Sx_m)(t)\to(S\bar{x})(t)\quad\text{as } m\to\infty \text{ for any } t\in J. \tag{3.9}$$

Differentiating both sides of (3.2), we have for any $t\in J'_+$ $(m=1,2,3,\ldots)$

$$x_m^{(n)}(t)=f(t,x_{m-1}(t),x'_{m-1}(t),\ldots,x_{m-1}^{(n-1)}(t),(Tx_{m-1})(t),(Sx_{m-1})(t)),$$

which implies by virtue of (3.9) that

$$\lim_{m\to\infty}x_m^{(n)}(t)=f(t,\bar{x}(t),\bar{x}'(t),\ldots,\bar{x}^{(n-1)}(t),(T\bar{x})(t),(S\bar{x})(t))=\bar{x}^{(n)}(t)\quad\forall t\in J'_+.$$

Hence, Theorem 3.1 is proved. □

Theorem 3.2. *Let conditions* (H_1)–(H_4) *and inequality* (3.1) *be satisfied. Denote by* $\bar{x}(t)$ *and* $\bar{y}(t)$ *the unique solutions in* $Q\cap C^n[J'_+,E]$ *of BVP* (1.1) *and the following BVP*

$$\begin{cases} y^{(n)}(t)=f(t,y(t),y'(t),\ldots,y^{(n-1)}(t),(Ty)(t),(Sy)(t))\quad\forall t\in J'_+,\\ \Delta y^{(i)}|_{t=t_k}=I_{ik}(y(t_k),x'(t_k),\ldots,y^{(n-1)}(t_k))\\ \quad(i=0,1,\ldots,n-1;k=1,2,3,\ldots),\\ y^{(i)}(0)=y_{0i}\ (i=0,1,\ldots,n-2),\quad y^{(n-1)}(\infty)=\beta y^{(n-1)}(0), \end{cases} \tag{3.10}$$

respectively, where $y_{0i}\geqslant x_i^*$ $(i=0,1,\ldots,n-2)$. *Then*

$$\|\bar{x}-\bar{y}\|_D\leqslant(1-\alpha)^{-1}\max\{\|x_{0i}-y_{0i}\|:i=0,1,\ldots,n-1\}. \tag{3.11}$$

Proof. By Lemma 2.4, $\bar{x}(t),\bar{x}^{(i)}(t)$ $(i=1,\ldots,n-2)$, $\bar{x}^{(n-1)}(t)$ satisfy (2.6), (2.13) and (2.14), respectively, and $\bar{y}(t)$ satisfies

$$\begin{aligned}\bar{y}^{(i)}(t)={}&\sum_{j=i}^{n-2}\frac{t^{j-i}}{(j-i)!}y_{0j}+\frac{t^{n-1-i}}{(\beta-1)(n-1-i)!}\Bigg\{\int_0^\infty f(s,\bar{y}(s),\bar{y}'(s),\ldots,\bar{y}^{(n-1)}(s),(T\bar{y})(s),(S\bar{y})(s))ds\\&+\sum_{k=1}^\infty I_{(n-1)k}(\bar{y}(t_k),\bar{y}'(t_k),\ldots,\bar{y}^{(n-1)}(t_k))\Bigg\}+\frac{1}{(n-1-i)!}\int_0^t(t-s)^{n-1-i}f(s,\bar{y}(s),\bar{y}'(s),\ldots,\bar{y}^{(n-1)}(s),(T\bar{y})(s),(S\bar{y})(s))ds\\&+\sum_{0<t_k<t}\sum_{j=i}^{n-1}\frac{(t-t_k)^{j-i}}{(j-i)!}I_{jk}(\bar{y}(t_k),\bar{y}'(t_k),\ldots,\bar{y}^{(n-1)}(t_k))\quad\forall t\in J\ (i=0,1,\ldots,n-2),\end{aligned} \tag{3.12}$$

and

$$\bar{y}^{(n-1)}(t) = \frac{1}{\beta-1}\left\{\int_0^{\infty} f(s,\bar{y}(s),\bar{y}'(s),\ldots,\bar{y}^{(n-1)}(s),(T\bar{y})(s),(S\bar{y})(s))ds + \sum_{k=1}^{\infty} I_{(n-1)k}(\bar{y}(t_k),\bar{y}'(t_k),\ldots,\bar{y}^{(n-1)}(t_k))\right\}$$
$$+\int_0^t f(s,\bar{y}(s),\bar{y}'(s),\ldots,\bar{y}^{(n-1)}(s),(T\bar{y})(s),(S\bar{y})(s))ds + \sum_{0<t_k<t} I_{(n-1)k}(\bar{y}(t_k),\bar{y}'(t_k),\ldots,\bar{y}^{(n-1)}(t_k)) \quad \forall t\in J. \tag{3.13}$$

Arguing similar to (3.5), for any $t \in J$ and $i = 0,1,\ldots,n-1$, we have from (2.6), (2.13), (2.14), (3.12) and (3.13) that

$$\begin{aligned}\left\|\frac{\bar{x}^{(i)}(t)-\bar{y}^{(i)}(t)}{e^t}\right\| &\leqslant \max\{\|x_{0i}-y_{0i}\| : i=0,1,\ldots,n-2\} + \frac{1}{\beta-1}\int_0^{\infty}\{\|f(s,\bar{x}(s),\bar{x}'(s),\ldots,\bar{x}^{(n-1)}(s),(T\bar{x})(s),(S\bar{x})\\ &\quad\times(s)) - f(s,\bar{y}(s),\bar{y}'(s),\ldots,\bar{y}^{(n-1)}(s),(T\bar{y})(s),(S\bar{y})(s))\|\}ds + \frac{1}{\beta-1}\\ &\quad\times\sum_{k=1}^{\infty}\{\|I_{(n-1)k}(\bar{x}(t_k),\bar{x}'(t_k),\ldots,\bar{x}^{(n-1)}(t_k)) - I_{(n-1)k}(\bar{y}(t_k),\bar{y}'(t_k),\ldots,\bar{y}^{(n-1)}(t_k))\|\}\\ &\quad+\int_0^t e^{-s}\{\|f(s,\bar{x}(s),\bar{x}'(s),\ldots,\bar{x}^{(n-1)}(s),(T\bar{x})(s),(S\bar{x})(s)) - f(s,\bar{y}(s),\bar{y}'(s),\ldots,\bar{y}^{(n-1)}(s),(T\bar{y})\\ &\quad\times(s),(S\bar{y})(s))\|\}ds + \sum_{0<t_k<t}\sum_{j=i}^{n-1} e^{-t_k}\{\|I_{jk}(\bar{x}(t_k),\bar{x}'(t_k),\ldots,\bar{x}^{(n-1)}(t_k))\\ &\quad- I_{jk}(\bar{y}(t_k),\bar{y}'(t_k),\ldots,\bar{y}^{(n-1)}(t_k))\|\}\\ &\leqslant \max\{\|x_{0i}-y_{0i}\| : i=0,1,\ldots,n-2\} + \alpha\|\bar{x}-\bar{y}\|_D,\end{aligned} \tag{3.14}$$

which implies that

$$\|\bar{x}-\bar{y}\|_D \leqslant \max\{\|x_{0i}-y_{0i}\| : i=0,1,\ldots,n-2\} + \alpha\|\bar{x}-\bar{y}\|_D.$$

So, (3.11) holds and the theorem is proved. □

Remark 3.1. In [5], by requiring that f satisfies some noncompact measure conditions and P is a normal cone, the author establishes the existence of positive solutions for BVP (1.1). In this paper, we do not impose any compactness condition on f, but we also obtain the unique positive solution of BVP (1.1).

Remark 3.2. Form (3.11) we know that $x^{(i)}(t) \to y^{(i)}(t)$ $(i = 0,1,\ldots,n-1)$ uniformly on any finite sub-interval of J when $x_{0i} \to y_{0i}$ $(i = 0,1,\ldots,n-2)$. This means that, when conditions (H_1)–(H_4) and inequality (3.1) are satisfied, the unique solution of BVP (1.1) in $Q \cap C^n[J'_+, E]$ is continuously dependent on the boundary values x_{0i} $(i = 0,1,\ldots,n-2)$.

Remark 3.3. In [5], the author obtains only the existence of positive solutions for BVP (1.1). In this paper, we not only establish the conditions for the existence of a unique positive solution for the BVP, but also develop an iterative sequence for approximating the solution and give an error estimate for the approximation. The iterative sequence $\{x_m\}$ defined by (3.2) is expressed explicitly, which is an important improvement of existing results.

Remark 3.4. For the special case where IVP (1.1) has no singularities and $J = [0,a]$ and the condition $x^{(n-1)}(\infty) = \beta x^{(n-1)}(0)$ is replaced by $x^{(n-1)}(0) = x_{0(n-1)}$, papers [9–12] also get a unique solution.

4. Example

Example 4.1. Consider the infinite system of scalar second-order impulsive singular integro-differential equations

$$\begin{cases} x_n''(t) = \frac{e^{-2t}}{60M\sqrt{t}}\left(\frac{360M}{n^2} + x_{n+1}(t) + x_n'(t) + \frac{1}{n^2x_n(t)} + \frac{1}{16n^4x_{2n}'(t)}\right)\\ \qquad + \frac{1}{10^n}e^{-2t}\left[1+\left(\int_0^t e^{-(t+1)s}x_n(s)ds\right)^2\right]^{-1}\\ \qquad + \frac{1}{100}e^{-3t}\int_0^{\infty} te^{-2s}\sin^2(t-s)x_{2n}(s)ds\\ \qquad\quad \forall\ 0<t<\infty, t\neq k,\ k=1,2,3,\ldots,\\ \triangle x_n|_{t=k} = \frac{100^{-k}}{e^k}\left(x_{2n}(k) + x_{n+1}'(k) + \frac{1}{n^2x_n(k)}\right),\ k=1,2,3,\ldots,\\ \triangle x_n'|_{t=k} = \frac{200^{-k}}{e^k}\left(x_{n+1}(k) + x_{2n}'(k) + \frac{1}{n^4x_n'(k)}\right),\ k=1,2,3,\ldots,\\ x_n(0) = \frac{1}{n},\ 2x_n'(\infty) = 3x_n'(0),\ n=1,2,3,\ldots, \end{cases} \tag{4.1}$$

where $M = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt > 0$. The infinite system has a unique positive solution $\{x_n(t)\}$ satisfying $x_n(t) \geqslant \frac{1}{n}$, and $x'_n(t) \geqslant \frac{1}{n^2}$ for $0 \leqslant t < \infty$ $(n = 1, 2, 3, \ldots)$. This unique solution can be obtained by taking limit of an iterative sequence.

Proof. Let $E = c_0 = \{x = (x_1, \ldots, x_n, \ldots) : x_n \to 0\}$ with norm $\|x\| = \sup_n |x_n|$, and $P = \{x = (x_1, \ldots, x_n, \ldots) \in c_0 : x_n \geqslant 0, n = 1, 2, 3, \ldots\}$. Then, infinite system (4.1) can be regarded as a BVP of form (1.1) in E with $n = 2$. In this situation, $u = (u_1, \ldots, u_n, \ldots), v = (v_1, \ldots, v_n, \ldots), w = (w_1, \ldots, w_n, \ldots), z = (z_1, \ldots, z_n, \ldots)$, $K(t,s) = e^{-(t+1)s}$, $H(t,s) = te^{-2s}\sin^2(t-s)$, $x_{00} = (1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots)$, $\beta = \frac{3}{2}$, $f = (f_1, \ldots, f_n, \ldots)$ and $I_{ik} = (I_{ik1}, \ldots, I_{ikn}, \ldots)$ $(i = 0, 1)$, in which

$$f_n(t,u,v,w,z) = \frac{e^{-2t}}{60M\sqrt{t}}\left(\frac{360M}{n^2} + u_{n+1} + v_n + \frac{1}{n^2 u_n} + \frac{1}{16n^4 v_{2n}}\right) + \frac{1}{10^n} e^{-2t}(1 + w_n^2)^{-1} + \frac{1}{100} e^{-3t} z_{2n} \tag{4.2}$$

and

$$I_{0kn}(u,v) = \frac{100^{-k}}{e^k}\left(u_{2n} + v_{n+1} + \frac{1}{n^2 u_n}\right), \tag{4.3}$$

$$I_{1kn}(u,v) = \frac{200^{-k}}{e^k}\left(u_{n+1} + v_{2n} + \frac{1}{n^4 v_n}\right). \tag{4.4}$$

Let $x_0^* = x_{00}$ and $x_1^* = (1, \frac{1}{4}, \ldots, \frac{1}{n^2}, \ldots)$. Then $P_{0\lambda} = \{x = (x_1, \ldots, x_n, \ldots) : x_n \geqslant \frac{\lambda}{n}, n = 1, 2, 3, \ldots\}$ and $P_{1\lambda} = \{x = (x_1, \ldots, x_n, \ldots) : x_n \geqslant \frac{\lambda}{n^2}, n = 1, 2, 3, \ldots\}$ for $\lambda > 0$. It is clear, $f \in C[J_+ \times P_{0\lambda} \times P_{1\lambda} \times P \times P, P], I_{ik} \in C[P_{0\lambda} \times P_{1\lambda}, P]$ for any $\lambda > 0$ $(i = 0, 1; k = 1, 2, 3, \ldots)$. It is easy to see that

$$k^* = \sup_{0\leqslant t} \int_0^t e^{-(t-1)s} ds = \sup_{0\leqslant t} \frac{1}{t+1}(1 - e^{-(t+1)t}) \leqslant 1, \quad h^* = \sup_{0\leqslant t}\{te^{-t}\int_0^\infty e^{-s}\sin^2(t-s)ds\} \leqslant \sup_{0\leqslant t}(te^{-t}) = e^{-1}$$

and

$$\int_0^\infty |H(t',s) - H(t,s)|e^s ds \leqslant \int_0^\infty \left[|(t'-t)\sin^2(t'-s)| + t|\sin^2(t'-s) - \sin^2(t-s)|\right] e^{-s} ds \leqslant (1+2t)|t'-t| \to 0,\ t' \to t \quad (t \in J).$$

So, condition (H_1) is satisfied. By (4.2), we have

$$f_n(t,u,v,w,z) \geqslant \frac{1}{n^2\sqrt{t}} 6e^{-2t}, \quad t \in J_+,\ u \in P_0,\ v \in P_1,\ w \in P,\ z \in P\ (n = 1, 2, 3, \ldots),$$
$$\int_{\frac{1}{2}}^1 \frac{6e^{-2t}}{\sqrt{t}} dt \geqslant 6\int_{\frac{1}{2}}^1 e^{-2t} dt = 3(e^{-1} - e^{-2}) > \frac{1}{2} = \beta - 1.$$

So, condition (H_4) is satisfied for $t_* = \frac{1}{2}, t^* = 1$ and $\sigma(t) = \frac{6e^{-2t}}{\sqrt{t}}$. We see from (4.2) that, for any $t \in J_+,\ u \in P_0,\ v \in P_1$ $(P_i = P_{i1}\ (i = 0, 1))$, $w \in P, z \in P$,

$$\begin{aligned}|f_n(t,u,v,w,z) - f_n(t,\bar{u},\bar{v},\bar{w},\bar{z})| \leqslant{}& \frac{e^{-2t}}{60M\sqrt{t}}\left(|u_{n+1} - \bar{u}_{n+1}| + |v_n - \bar{v}_n| + \frac{|u_n - \bar{u}_n|}{n^2 u_n \bar{u}_n} + \frac{|v_{2n} - \bar{v}_{2n}|}{16n^4 v_{2n}\bar{v}_{2n}}\right)\\ &+ \frac{e^{-2t}}{10^n}|(1 + w_n^2)^{-1} - (1 + \bar{w}_n^2)^{-1}| + \frac{e^{-3t}}{100}|z_{2n} - \bar{z}_{2n}|, \quad n = 1, 2, 3, \ldots,\end{aligned}$$

and therefore for any $t \in J_+,\ u \in P_0,\ v \in P_1,\ w, z \in P$

$$\|f(t,u,v,w,z) - f(t,\bar{u},\bar{v},\bar{w},\bar{z})\| \leqslant \frac{e^{-2t}}{30M\sqrt{t}}\|u - \bar{u}\| + \frac{e^{-2t}}{30M\sqrt{t}}\|v - \bar{v}\| + \frac{e^{-2t}}{10}\|w - \bar{w}\| + \frac{e^{-3t}}{100}\|z - \bar{z}\|,$$

hence (H_2) is satisfied for $a_0(t) = a_1(t) = \frac{e^{-2t}}{30M\sqrt{t}}$, $a_2(t) = \frac{e^{-2t}}{10}$, $a_3(t) = \frac{e^{-3t}}{100}$ with

$$a^* \leqslant \int_0^\infty \left(\frac{e^{-t}}{15M\sqrt{t}} + \frac{e^{-t}}{10} + \frac{e^{-2t}}{100e}\right) dt = \frac{1}{6} + \frac{1}{200e}$$

and $\tau = \frac{1}{15}, \gamma = 6M + \frac{14}{5}$.

Similarly, it is easy to get for any $u_0 \in P_0, u_1 \in P_1$,

$$\begin{aligned}\|I_{0k}(u_0,u_1) - I_{0k}(\bar{u}_0,\bar{u}_1)\| &\leqslant \frac{2}{(100e)^k}\|u - \bar{u}\| + \frac{1}{(100e)^k}\|v - \bar{v}\|, \quad \|I_{1k}(u_0,u_1) - I_{0k}(\bar{u}_0,\bar{u}_1)\|\\ &\leqslant \frac{2}{(200e)^k}\|u - \bar{u}\| + \frac{1}{(200e)^k}\|v - \bar{v}\|,\end{aligned}$$

so, (H_3) is satisfied for $b_{0k0} = \frac{2}{(100e)^k}, b_{0k1} = \frac{1}{(100e)^k}$, $b_{1k0} = \frac{2}{(200e)^k}$, $b_{1k1} = \frac{1}{(200e)^k}$ with $b^* = \frac{1}{33} + \frac{3}{199}$, $\lambda = 3(\frac{1}{100e-1} + \frac{1}{200e-1})$, $35em\delta = \frac{7}{4}(\frac{1}{100e-1} + \frac{1}{200e-1})$. On the other hand, we can obtain

$$\frac{\beta}{\beta-1}(a^*+b^*)<0.75$$

i.e. inequality (3.1) is satisfied. Hence, our conclusion follows from Theorem 3.1 immediately. □

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