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ORIGINAL PAPER

Super Connectivity and Super Edge Connectivity of the Mycielskian of a Graph

Litao Guo · Ruifang Liu · Xiaofeng Guo

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Abstract Mycielski introduced a new graph transformation $\mu(G)$ for graph G, which is called the Mycielskian of G. A graph G is super connected or simply super-κ (resp. super edge connected or super-λ), if every minimum vertex cut (resp. minimum edge cut) isolates a vertex of G. In this paper, we show that for a connected graph G with $|V(G)| \geq 2$, $\mu(G)$ is super-κ if and only if $\delta(G) < 2\kappa(G)$, and $\mu(G)$ is super-λ if and only if $G \ncong K_2$.

Keywords Mycielskian · Super connected · Super edge connected

Mathematics Subject Classification (2010) 05C40

1 Introduction

All graphs considered in this paper are simple, finite and undirected. Unless stated otherwise, we follow Bondy and Murty [2] for terminology and definitions.

Mycielski [6] defined an interesting graph transformation $\mu(G)$. Let $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ whose the vertex set is $V(\mu(G)) = V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E(\mu(G) = E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x is the twin of x') and the vertex u is called the root of $\mu(G)$. For $n \geq 2$, $\mu^n(G) = \mu(\mu^{n-1}(G))$.

L. Guo · X. Guo
School of Mathematical Sciences, Xiamen University, Xiamen, 361005 Fujian, China
e-mail: ltguo@yahoo.cn

R. Liu (✉)
Department of Mathematics, Zhengzhou University, Zhengzhou, 450001 Henan, China
e-mail: lruifang@yahoo.com.cn

Let $G = (V, E)$ be a connected graph, $d_G(v)$ the degree of a vertex v in G (or simply $d(v)$), and $\delta(G)$ the minimum degree of G. The set of neighbors of a vertex v in G is denoted by $N_G(v)$, or briefly, by $N(v)$. More generally for $S \subset V$, $N_G(S) = \{x \mid x \in V\backslash S, x$ is adjacent to a vertex in $S\}$ denotes the neighbor set of S in G, and a vertex x in $N_G(S)$ is also called a neighbor of S. $S' = \{x' : x \in S\}$ is the twin of S. $G - S$ denotes the subgraph of G induced by the vertex set of $V\backslash S$. For $X, Y \subset V$, denote by $[X, Y]$ the set of edges with one end in X and the other in Y.

The connectivity $\kappa(G)$ of a connected graph G is $\min\{|S| : S \subset V$, and $G - S$ is disconnected or reduces to the trivial graph $K_1\}$. The edge connectivity $\lambda(G)$ of a connected graph G is defined similarly. A graph G is super connected, or simply super-κ, if every minimum vertex cut is the set of neighbors of a vertex of G, that is every minimum vertex cut isolates a vertex. Similarly, we can define super-λ graphs.

An obvious inference from the definition of $\mu(G)$ is that $d_{\mu(G)}(x') = d_G(x) + 1$ for all $x \in V$. Consequently, if G is a connected graph, then $\delta(\mu(G)) = \delta(G) + 1$.

Balakrishnan and Francis Raj [1] investigated the vertex connectivity and edge connectivity of $\mu(G)$. We [3] investigated the vertex connectivity and arc connectivity of the Mycielskian of a digraph. In this paper, we study the super connectivity and super edge connectivity of $\mu(G)$. It is proved that for a connected graph G with $|V(G)| \geq 2$, $\mu(G)$ is super-κ if and only if $\delta(G) < 2\kappa(G)$, and $\mu(G)$ is super-λ if and only if $G \ncong K_2$.

2 Super Connectivity of the Mycielskian

Lemma 2.1 (Balakrishnan and Francis Raj [1]) *If G is a connected graph and $0 \leq i < \kappa(G)$, then $\kappa(\mu(G)) = \kappa(G) + i + 1$ if and only if $\delta(G) = \kappa(G) + i$.*

Lemma 2.2 (Balakrishnan and Francis Raj [1]) *If G is a connected graph, then $\kappa(\mu^n(G)) = \kappa(G) + n$ if and only if $\delta(G) = \kappa(G)$.*

Lemma 2.3 (Balakrishnan and Francis Raj [1]) *If G is a connected graph, then* (i) $\kappa(\mu(G)) = 2\kappa(G) + 1$ *if and only if* $\delta(G) \geq 2\kappa(G)$. (ii) $\kappa(\mu(G)) = min\{\delta(G) + 1, 2\kappa(G) + 1\}$.

Theorem 2.4 *For a connected graph G with $|V(G)| \geq 2$, $\mu(G)$ is super-κ if and only if $\delta(G) < 2\kappa(G)$.*

Proof Suppose $\mu(G)$ is super-κ, but $\delta(G) \geq 2\kappa(G)$. By Lemma 2.3, $\kappa(\mu(G)) = 2\kappa(G) + 1$. But since $\mu(G)$ is super-κ, $\kappa(\mu(G)) = \delta(\mu(G)) = \delta(G) + 1$. Therefore we have $\delta(G) = 2\kappa(G)$. Hence, for any minimum vertex cut S of G, $G - S$ has no isolated vertices, and so, for the minimum vertex cut $S \cup S' \cup \{u\}$ of $\mu(G)$, where S' is the twin of S, $\mu(G) - (S \cup S' \cup \{u\})$ also has no isolated vertices, a contradiction.

Now suppose $\delta(G) < 2\kappa(G)$ but $\mu(G)$ is not super-κ. Then, by Lemma 2.1, $\kappa(\mu(G)) = \delta(G) + 1$. Then there is a minimum vertex cut S of $\mu(G)$ with $|S| = \kappa(\mu(G)) = \delta(G) + 1 \leq 2\kappa(G)$ such that $\mu(G) - S$ is not connected but has no isolated vertex.

Case 1 $|V \cap S| < \kappa(G)$. Then $G - (V \cap S)$ is connected, and each vertex of $V'\backslash S$ is adjacent to at least $\kappa(G)$ vertices of V and so is adjacent to at least one vertex in $V\backslash S$. Thus $\mu(G) - S$ is connected, which is impossible.

Case 2 $|V \cap S| \geq \kappa(G)$.

Subcase 2.1 $u \notin S$. Then $(V'\backslash S) \cup \{u\}$ induces a star in $\mu(G) - S$, say S^*. In addition, since $|S| = \kappa(\mu(G)) = \delta(G) + 1 \leq 2\kappa(G)$, we have $|V' \cap S| \leq \kappa(G)$.

If $|V' \cap S| < \kappa(G)$, then each vertex in $V\backslash S$ is adjacent to at least one vertex in $V'\backslash S$ (that is, in S^*), and so $\mu(G) - S$ is connected, a contradiction.

Otherwise, $|V' \cap S| = \kappa(G)$, and so $|V \cap S| = \kappa(G)$. Then $|S| = 2\kappa(G) = \delta(G)+1$. If $\kappa(G) > 1$, then $\kappa(G) < \delta(G)$. So any vertex in $V\backslash S$ is adjacent to at least one vertex of $V'\backslash S$, and so $\mu(G) - S$ is connected, a contradiction. In the other case, $\kappa(G) = \delta(G) = 1$, and $|S| = 2$. Let $S = \{x, y'\}$, $x \in V$, and $y' \in V'$. Then, for any vertex $z \in V - x$, either z is adjacent to a vertex in $V' - y'$ or z is adjacent to only the vertex y' in V'. For the latter, the twin y of y' must be not equal to x (otherwise $\mu(G) - S$ would have an isolated vertex z, contradicting our assumption). Thus zyz' is a path in $\mu(G) - S$, that is, z is connected by the path zyz' to one vertex z' in S^*. This means that $\mu(G) - S$ is connected, again a contradiction.

Subcase 2.2 $u \in S$. Then $|V' \cap S| \leq 2\kappa(G) - |V \cap S| - 1$, and every vertex z' in $V'\backslash S$ is adjacent to at least one vertex in $V\backslash S$ (otherwise, S would isolate z', a contradiction).

If $G - (V \cap S)$ is connected, then $\mu(G) - S$ is connected, a contradiction. Hence $G - (V \cap S)$ is not connected. Let C_i, C_j be any two connected components of $G - (V \cap S)$. Each of $V(C_i)$ and $V(C_j)$ has at least $\kappa(G)$ neighbors in $V \cap S$, and so both $V(C_i)$ and $V(C_j)$ have at least $2\kappa(G) - |V \cap S|$ common neighbors in $V \cap S$. Let T be the set of the common neighbors of $V(C_i)$ and $V(C_j)$ in $V \cap S$, and let T' be the twin of T. Then $|T'| = |T| \geq 2\kappa(G) - |V \cap S| > |V' \cap S|$, and so there is a vertex in T' adjacent to a vertex of C_i and also a vertex of C_j, implying that C_i and C_j are contained in a common connected component of $\mu(G) - S$ and so do all connected components of $G - (V \cap S)$. Since every vertex z' in $V'\backslash S$ is adjacent to at least one vertex in $V\backslash S$, the graph $\mu(G) - S$ is connected, which is impossible. □

Corollary 2.5 *If T is a tree with $|V(T)| \geq 2$, then $\mu(T)$ is super-κ.*

Corollary 2.6 *If G is an edge transitive connected graph with $|V(G)| \geq 2$, then $\mu(G)$ is super-κ.*

Proof If G is an edge transitive graph, then $\delta(G) = \kappa(G) < 2\kappa(G)$(we can see [4]), and so $\mu(G)$ is super-κ by Theorem 2.4. □

Lemma 2.7 (Mader [5]) *If G is a connected graph which is vertex transitive and K_4-free, then $\delta(G) = \kappa(G)$.*

Corollary 2.8 *If G is a nontrivial connected graph which is vertex transitive and K_4-free, then $\mu(G)$ is super-κ.*

Corollary 2.9 *If G is a connected graph with $|V(G)| \geq 2$ and $\delta(G) = \kappa(G)$, then $\mu^n(G)$ is super-κ.*

Proof Since $\delta(G) = \kappa(G) < 2\kappa(G)$, by Lemma 2.2, $\delta(\mu^n(G)) = \kappa(\mu^n(G)) = \kappa(G) + n < 2\kappa(\mu^n(G))$ for $n = 1, 2, \ldots$. By Theorem 2.4, the graph $\mu^n(G)$ is super-κ for $n = 1, 2, \ldots$. □

3 Super Edge Connectivity of the Mycielskian

Lemma 3.1 (Balakrishnan and Francis Raj [1]) *If G is a connected nontrivial graph, then $\lambda(\mu(G)) = \delta(G) + 1 = \delta(\mu(G))$.*

Theorem 3.2 *For a connected graph G with $|V(G)| \geq 2$, $\mu(G)$ is super-λ if and only if $G \ncong K_2$.*

Proof If $G = K_2$, then $\mu(G) = C_5$ and we can easily see that $\mu(G)$ is not super-λ. Thus, the necessity is proved. Now we prove the sufficiency.

We assume that $\mu(G)$ is not super-λ. According to Lemma 3.1, $\lambda(\mu(G)) = \delta(G) + 1 = \delta(\mu(G))$. There exists a minimum edge cut F of $\mu(G)$ with $|F| = \delta(G) + 1$ such that $\mu(G) - F$ is not connected but has no isolated vertex.

Let U be the set of edges incident with u in $\mu(G)$. For $X \subseteq V$ and $Y' \subseteq V'$, let $[X, Y']$ denote the set of the edges with one end vertex in X and the other end vertex in Y', and let $[u, Y']$ denote the set of the edges with one end vertex u and the other end vertex in Y'.

Claim 1 *Let G_i be a connected component of $G - (E \cap F)$. Then there is a path from u to a vertex of G_i in $\mu(G) - F$.*

Proof Let $X'_i \subseteq V'$ be the set of the neighbors of $V(G_i)$ in V', and $U'_i = [u, X'_i]$. We have that $|X'_i| = |U'_i| \geq \delta(G)$, $1 \leq |V(G_i)| \leq |V|$ (equation in the last inequality holds only if $G - (E \cap F)$ is connected).

Suppose in $\mu(G) - F$ there is no path from u to a vertex of G_i. We consider the following cases.

Case 1 $G - (E \cap F)$ is not connected. Then $|E \cap F| \geq 1$ and $|([V(G_i), X'_i] \cup U'_i) \cap F| \leq \delta(G)$. Since $|X'_i| \geq \delta(G)$ and there is no path from u to a vertex of G_i in $\mu(G) - F$, $|E \cap F| = 1$ and $|X'_i| = |([V(G_i), X'] \cup U'_i) \cap F| = \delta(G)$.

If $|V(G_i)| \geq 2$, then G_i contains an edge and $|X'_i| \geq \delta(G) + 1$, a contradiction. Hence $|V(G_i)| = 1$ and $\delta(G) = 1$. Let $V(G_i) = \{x_i\}$, y_j is the unique neighbor of x_i in G. Then $F = \{x_i y_j, u y'_j\}$. Since $G \ncong K_2$, $|V(G)| \geq 3$, and $d_G(y_j) \geq 2$, the vertex y'_j is adjacent to at least one vertex in $G - x_i$. Thus $\mu(G) - F$ would be connected, contradicting our assumption.

Case 2 $G - (E \cap F)$ is connected. Then $|V(G_i)| = |V(G)| = \delta(G) + 1$ since there is no path from u to a vertex of G_i in $\mu(G) - F$. Hence, G is a complete graph. Since $G \ncong K_2$, we have $\delta(G) \geq 2$. To separate paths from u to a vertex of V, F must be equal to U. This contradicts the fact that $\mu(G) - F$ has no isolated vertex.

Claim 1 is thus proved.

Now we can proceed with the proof of Theorem 3.2.

By Claim 1, in $\mu(G) - F$ there is a path from u to a vertex of any connected component G_i of $G - (E \cap F)$. So $V \cup \{u\}$ are contained in a same component of $\mu(G) - F$. On the other hand, choose $x' \in V'$. Since $\mu(G) - F$ has no isolated vertex, either x' is adjacent to a vertex of V in $\mu(G) - F$ or $x'u \notin F$, so that also x' is in the component of $\mu(G) - F$ containing u. Thus $\mu(G) - F$ is connected, a contradiction.

The proof is thus complete. □

Corollary 3.3 *If G is a connected graph with $|V(G)| \geq 3$, then $\mu^n(G)$ is super-λ.*

Proof is by induction on n.

Furthermore, we can generalize Theorem 3.2 to the following:

Theorem 3.4 *Let G be a graph in which every connected component has at least two vertices. Then $\mu(G)$ is super-λ if and only if $G \ncong nK_2 \cup G'$, where every connected component of G' has at least three vertices and $n \geq 1$.*

By Theorem 3.2, it is easy to see that $\mu(G)$ is super-λ if and only if any connected component of G is not isomorphic to K_2.

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