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Minimal inner-Σ-Ω-groups and their applications

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Abstract We introduce and study the minimal inner-Σ-Ω-groups and the minimal outer-Σ-℧-groups. Then we give some applications and obtain some interesting results, including characterizations of nilpotent, supersolvable, solvable, and p-closed groups in terms of the join of two conjugate cyclic subgroups having the same property.

Keywords nilpotent subgroup, supersolvable subgroup, a pair of conjugate subgroups, minimal inner-Σ-Ω-group

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1 Introduction

All groups in this paper are finite and G always stands for a group. $H < G$ means that H is a proper subgroup of G. The other notations and terminologies in this paper are standard (see [13]).

Let Σ be an abstract group theoretical property, for example, solvability, nilpotency, supersolvability, p-close, etc. Following Chen [7], if all proper subgroups and all proper quotient groups of a group G have the property Σ but G does not have the property Σ, we say that G is an inner-Σ-group and an outer-Σ-group, respectively. If G is both an inner-Σ-group and an outer-Σ-group, then G is called a minimal non-Σ-group. The inner-Σ-groups here are also called minimal non-Σ-groups in [18, p. 258] or [5], or critical groups or S-critical groups for the class of Σ-groups in [10, VII, 6.1] or [6, p. 252], respectively, and the outer-Σ-groups here are just the groups in the boundary or Q-boundary of Σ in [10, III, 2.1] or [6, 2.3.6].

Theorem 1.1. (see [7, Theorem 0.6]) *Let G be a group and $\{1\}$ be a Σ-group. Suppose that G is not a Σ-group. Then there must exist a subgroup G_0 and a quotient group $\overline{G}$ of G such that G_0 and $\overline{G}$ are an inner-Σ-group and an outer-Σ-group, respectively.*

This is an obvious result, but it shows the importance of inner-Σ-groups and outer-Σ-groups in the group theory. The properties of a group may be refracted by its proper subgroups and quotient groups. In fact, in the research of the group theory, subgroups and quotient groups have central importance. The proof of some results in the group theory often tends to the analysis of inner-Σ-groups and outer-Σ-groups. And many problems are often solved by using the analysis of a minimal counterexample. For

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many properties Σ, the minimal counterexample of the assertion "G is a Σ-group" is just an inner-Σ-group or an outer-Σ-group. Hence many people are interested in the research of inner-Σ-groups and outer-Σ-groups. After Miller and Moreno [16] gave the structure of inner-abelian-groups, many people took part in the classification of inner-Σ-groups and outer-Σ-groups for a property Σ. For example, Schmidt [19] and Rédei [17] studied inner-nilpotent groups, and Doerk [9] studied inner-supersolvable group, with a complete description given by Ballester-Bolinches and Esteban-Romero [5]. These results have an independent interest and are also a powerful tool in the theory of groups.

Our purpose in this paper is to generalize the concepts of inner-Σ-groups and outer-Σ-groups and obtain a common way to prove different problems in the theory of groups. We give the concepts of inner-Σ-Ω-groups, outer-Σ-$\mho$-groups and some basic results in Section 2 and obtain some interesting results as applications of inner-Σ-Ω-groups method in Sections 3.

2 Minimal inner-Σ-Ω-groups and outer-Σ-$\mho$-groups

Let G be a group. Ω is a functor that associates with every group G, a family $\Omega(G)$ of subgroups of G such that if $\varphi : G_1 \to G_2$ is an isomorphism of groups, then $H \in \Omega(G_1)$ if and only if $\varphi(H) \in \Omega(G_2)$, and $\mho$ is a functor that associates with every group G, a family $\mho(G)$ of quotients G/N of G such that if $\varphi : G_1 \to G_2$ is an isomorphism of groups, then $G_1/N \in \mho(G_1)$ if and only if $G_2/\varphi(N) \in \mho(G_2)$.

Definition 2.1. (1) *A group G is called an inner-Σ-Ω-group if the following statements hold.*

(i) *G is not a Σ-group but every element of $\Omega(G)$ is a Σ-group.*

(ii) *For every proper subgroup H of G, the condition that every element of $\Omega(H)$ has the property Σ implies that H has the property Σ.*

(2) *A group G is called an outer-Σ-$\mho$-group if the following statements hold.*

(i) *G is not a Σ-group but every element of $\mho(G)$ is a Σ-group.*

(ii) *For every proper quotient group $\overline{G}$ of G, the condition that every element of $\mho(\overline{G})$ has the property Σ implies that $\overline{G}$ has the property Σ.*

(3) *If $\Phi(G) = 1$ and G is an inner-Σ-Ω-group and an outer-Σ-$\mho$-group, then G is called a minimal inner-Σ-Ω-group and a minimal outer-Σ-$\mho$-group, respectively.*

If $\Omega(G)$ (respectively, $\mho(G)$) is empty for every group G and T is an inner-Σ-Ω-group (respectively, an outer-Σ-Ω-group), then we think that every proper subgroup (respectively, every proper quotient group) of T has the property Σ. Obviously, if $\Omega(G)$ (respectively, $\mho(G)$) are the sets of all proper subgroups (respectively, all proper quotient groups) of a group G, then the inner-Σ-Ω-groups (respectively, the outer-Σ-Ω-groups) are exactly the inner-Σ-groups (respectively, the outer-Σ-groups), and the inner-Σ-Ω-group and outer-Σ-$\mho$-group are exactly minimal non-Σ-group.

For example, taking Σ to be supersolvable, we obtain the conceptions of inner-supersolvable, outer-supersolvable, minimal inner-supersolvable group and minimal outer-supersolvable group, respectively.

First, we have also the following obvious result.

Theorem 2.1. *Let G be a group. Suppose that the identity group $\{1\}$ is a Σ-group. Assume that G is not a Σ-group. Then there must exist a subgroup G_0 and a quotient group $\overline{G}$ of G such that G_0 and $\overline{G}$ are an inner-Σ-Ω-group and an outer-Σ-$\mho$-group, respectively.*

Next, we consider the relations between inner-Σ-Ω-groups and inner-Σ-groups and between outer-Σ-$\mho$-groups and outer-Σ-groups. We have the following results.

Theorem 2.2. *Let G be a group. Suppose that $\Omega(H) \subseteq \Omega(G)$ for any subgroup H of G. Then G is an inner-Σ-Ω-group if and only if G is an inner-Σ-group; moreover, G is a minimal inner-Σ-Ω-group if and only if G is a minimal inner-Σ-group.*

Proof. The sufficiency of the theorem is obvious. We only need to prove the necessity of the theorem. Let G be an inner-Σ-Ω-group, H be an arbitrary proper subgroup of G. For $A \in \Omega(H)$, since $\Omega(H) \subseteq \Omega(G)$, A is a Σ-group. By Definition 2.1, H is a Σ-group. By the arbitrariness of H, G is an inner Σ-group. □

Theorem 2.3. *Let G be a group. Suppose that $\mho(\overline{G}) \subseteq \mho(G)$ for any quotient group $\overline{G}$ of G. Then G is an outer-Σ-$\mho$-group if and only if G is an outer-Σ-group; moreover, G is a minimal outer-Σ-$\mho$-group if and only if G is a minimal outer-Σ-group.*

Proof. We only need to prove the necessity of the theorem. Let G be an outer-Σ-$\mho$-group, N be an arbitrary non-trivial normal subgroup of G and $\overline{G} = G/N$. For every element $\overline{G}/\overline{A}$ of $\mho(\overline{G})$, since $\mho(\overline{G}) \subseteq \mho(G)$, $\overline{G}/\overline{A}$ is a Σ-group. By the definition of outer-Σ-$\mho$-groups, $\overline{G}$ is a Σ-group. Hence G is an outer Σ-group. $\square$

Let Σ be an abstract group theoretical property. The property Σ is called subgroup closed (respectively, quotient group closed) if a group G has the property Σ, it implies that every subgroup (respectively, every quotient group) of G has the property Σ. $\Omega(H) \preceq \Omega(G)$ means that for any $L \in \Omega(H)$, there exists an element $\widetilde{L} \in \Omega(G)$ such that $L \leqslant \widetilde{L}$.

Theorem 2.4. *Let G be a group and Σ be a subgroup closed. Suppose that $\Omega(H) \preceq \Omega(G)$ for any subgroup H of G. Then G is an inner-Σ-Ω-group if and only if G is an inner-Σ-group; moreover, G is a minimal inner-Σ-Ω-group if and only if G is a minimal inner-Σ-group.*

Proof. The sufficiency of the theorem is obvious. We only need to prove the necessity of the theorem. Let G be an inner-Σ-Ω-group, and H be an arbitrary proper subgroup of G. For each element A of $\Omega(H)$, since $\Omega(H) \preceq \Omega(G)$, there exists $L \in \Omega(G)$ such that $A \leqslant L$. Since Σ is a subgroup closed, A is a Σ-group. By Definition 2.1, H is a Σ-group. By the arbitrariness of H, G is an inner-Σ-group. $\square$

Theorem 2.5. *Let G be a group and Σ be a quotient group closed. Suppose that G is an inner-Σ-Ω-group, $\Phi(G) = 1$ and $\Omega(H) \subseteq \Omega(G)$ for any subgroup H of G, then G is also an outer-Σ-group.*

Proof. Suppose that $1 < N \lhd G$ such that $G/N = \overline{G}$ is not a Σ-group. Let S be a subgroup of G such that $G = \langle S, N\rangle$. Then $G = SN$ and $G/N = SN/N \cong S/S \cap N$ is not a Σ-group. Since Σ is a quotient group closed, S is not a Σ-group. If $S < G$, since G is an inner-Σ-Ω-group and $\Omega(S) \subseteq \Omega(G)$, it is easy to get that S is a Σ-group, a contradiction. Hence $S = G$ and so $N \leqslant \Phi(G) = 1$, a contradiction. This contradiction shows that $\overline{G}$ is a Σ-group. Hence G is an outer-Σ-group. $\square$

Corollary 2.1. *Suppose that G is an inner-nilpotent-Ω-group and $\Omega(H) \preceq \Omega(G)$ for any subgroup H of G. Then*

(1) *$|G| = p^a q^b$, where p and q are primes.*

(2) *G has a normal Sylow q-subgroup Q; if $q > 2$, then $\exp(Q) = q$ and if $q = 2$, then $\exp(Q) \leqslant 4$; G has a cyclic Sylow p-subgroup $P = \langle a\rangle$.*

(3) *Let $c \in Q$, then c is a generator of Q if and only if $[c, a] \neq 1$.*

(4) *If c is a generator of Q, then $[c, a] = c^{-1}c^a$ is also a generator of Q.*

(5) *If c is a generator of Q, then $Q = \langle c, c^a, \ldots, c^{a^{p-2}}, c^{a^{p-1}}\rangle$, namely, $Q = \langle [c, a], [c, a]^a, \ldots, [c, a]^{a^{p-1}}\rangle$.*

Proof. By Theorem 2.4, G is an inner-nilpotent group. By [7, Theorem 1.1], it is easy to see that the result is true. $\square$

By Theorem 2.4, if G is a minimal inner-supersolvable-Ω-group and $\Omega(H) \preceq \Omega(G)$ for any subgroup H of G, then G is also a minimal inner-supersolvable group. By [7, pp. 49–51, Theorem 7.3], the minimal inner-supersolvable groups belong to six classes. In this paper, we denote by G_t a generic group in the t-th class. Then G_t may be described in the following Corollary 2.2.

Corollary 2.2. *Suppose that G is a minimal inner-supersolvable-Ω-group and $\Omega(H) \preceq \Omega(G)$ for any subgroup H of G. Then G is isomorphic to a group G_i in one of the following classes, where $1 \leqslant i \leqslant 6$.*

(I) *G_1 is a minimal nonabelian group and $|G_1| = pq^\beta$, where $p \nmid q - 1$, $\beta \geqslant 2$.*

(II) *$G_2 = \langle a, c_1\rangle$ and $|G_2| = p^\alpha r^p$ and $p^{\alpha-1} \| r - 1$, where $\alpha \geqslant 2$, $a^{p^\alpha} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_i c_j = c_j c_i$; $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p - 1$; $c_p^a = c_1^t$, the exponent of t modulo r is $p^{\alpha-1}$.*

(III) *$G_3 = \langle a, b, c_1\rangle$ and $|G_3| = 8r^2$ and $4 \mid r - 1$. $a^4 = c_1^r = c_2^r = 1$, $a^2 = b^2$, $ba = a^{-1}b$, $c_1^a = c_2$, $c_2^a = c_1^{-1}$, $c_1^b = c_1^s$, $c_2^b = c_2^s$, the exponent of s modulo r is 4.*

(IV) $G_4 = \langle a, b, c_1\rangle$ *and* $|G_4| = p^{\alpha+\beta}r^p$ *and* $p^{\max\{\alpha,\beta\}} \mid r-1$, *where* $\beta \geqslant 2$. $a^{p^\alpha} = b^{p^\beta} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_ic_j = c_jc_i$, $ab = b^{1+p^{\beta-1}}a$; $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^a = c_1^t$, $c_i^b = c_i^{u^{1+ip^{\beta-1}}}$, $i = 1, 2, \ldots, p$; *the exponents of* t *and* u *modulo* r *are* $p^{\alpha-1}$ *and* p^β, *respectively.*

(V) $G_5 = \langle a, b, c, c_1\rangle$ *and* $|G_5| = p^{\alpha+\beta+1}r^p$ *and* $p^{\max\{\alpha,\beta\}} \mid r-1$. $a^{p^\alpha} = b^{p^\beta} = c^p = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_ic_j = c_jc_i$, $ba = abc$, $ca = ac$, $cb = bc$, $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^a = c_1^t$, $c_i^c = c_i^u$, $c_i^b = c_i^{vu^{p-i+1}}$, *the exponents of* t, v *and* u *modulo* r *are* $p^{\alpha-1}$, p^β *and* p, *respectively.*

(VI) $G_6 = \langle a, b, c_1\rangle$, $|G_6| = p^\alpha qr^p$ *and* $p^\alpha q \mid r-1$, $p \mid q-1$, $\alpha \geqslant 1$. $a^{p^\alpha} = b^q = c_1^r = c_2^r = \ldots = c_p^r = 1$; $c_ic_j = c_jc_i$, $i, j = 1, 2, \ldots, p$; $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^a = c_1^t$; $b^a = b^u$, $c_i^b = c_i^{v^{u^{i-1}}}$, $i = 1, 2, \ldots, p$; *the exponent of* t *and* v *modulo* r *are* $p^{\alpha-1}$ *and* q, *respectively; the exponent of* u *modulo* q *is* p.

Lemma 2.1. *Let* G *be a group. Suppose that* G *has a unique minimal normal subgroup* N *and* N *is solvable. If* $\Phi(G) = 1$, *then*

(a) $G = AN$ *and* $A \cap N = 1$, *where* A *is a maximal subgroup of* G.

(b) *Assume that* G' *is nilpotent. Then* A *is a cyclic subgroup.*

Proof. By [7, p. 34, Main Lemma], the result is true. □

3 The join of a pair of conjugate subgroups

In [2], Aschbacher and Guralnick proved that any group G is generated by a pair of conjugate solvable subgroups and so a group G is determined by some join $\langle H, H^g\rangle$. This motivates the search for group properties Σ which can be characterised by the fact that every subgroup generated by two conjugate subgroups satisfies the property Σ. And this result also tells us that we should consider some special subgroups.

Let G be a group. The famous Baer theorem [3] shows that $x \in O_p(G)$ if and only if $\langle x, x^g\rangle$ is a p-subgroup for every $g \in G$, where x is a p-element of the group G. In this paper, we investigate the properties of G from the properties of the subgroups which are generated by two conjugate elements. We get many interesting results and improve also some results of Baer and Thompson.

Let $\Omega_2(G) = \{\langle a, a^g\rangle \mid a, g \in G\}$. Then $\Omega_2(G) = \{\langle A, A^g\rangle \mid g \in G$, A is a cyclic subgroup of $G\}$. We have the following Proposition 3.1.

Proposition 3.1. *Let* G *be a group and* Σ *a group theoretical property.*

(1) *Suppose that* Σ *is a subgroup closed and* $H \leqslant G$. *If every element in* $\Omega_2(G)$ *has the property* Σ, *then for any* $K \in \Omega_2(H)$, K *also has the property* Σ.

(2) *Suppose that* Σ *is a quotient group closed and* $N \lhd G$. *If every element in* $\Omega_2(G)$ *has the property* Σ, *then for any* $K \in \Omega_2(G/N)$, K *also has the property* Σ.

Proof. (1) is clear. We only need to prove (2). Let H/N be a cyclic subgroup of G/N, then $H = \langle xN\rangle$. By the hypothesis, $\langle x, x^g\rangle$ has the property Σ for all $g \in G$. Since $\langle H/N, (H/N)^{gN}\rangle = \langle x, x^g\rangle N/N \cong \langle x, x^g\rangle/\langle x, x^g\rangle \cap N$ for all $g \in G$, the group $\langle H/N, (H/N)^{gN}\rangle$ has the property Σ. Hence (2) is true. □

Lemma 3.1. *Let* G *be a group. Suppose that* G *is isomorphic to an inner-nilpotent group or* G_2. *Then there exist a cyclic subgroup* H *of* G *and* $g \in G$ *such that* $\langle H, H^g\rangle = G$.

Proof. Assume that G is isomorphic to an inner-nilpotent group. Using the same description as in Corollary 2.1, $G = PQ$, where $P = \langle a\rangle$. We choose $H = P$ and $g = c$, where c is a generator of Q. Then $(a^{-1})^c \in \langle H, H^c\rangle$, so $[c, a] = (a^{-1})^c a \in \langle H, H^c\rangle$, thus a, $[c, a]$, $[c, a]^a$, $\ldots$, $[c, a]^{a^{p-1}}$ belong to $\langle H, H^c\rangle$. By Corollary 2.1 (4) and (5), we have $\langle H, H^c\rangle = G$.

Assume that G is isomorphic to G_2. We use the same description as in Corollary 2.2. Let $P \in \mathrm{Syl}_p(G)$. We choose $H = P = \langle a\rangle$ and $g = c_1$, then $(a^{-1}a^{c_1})^{-1} = c_1^{-1}c_2 \in \langle H, H^{c_1}\rangle$, so $(c_1^{-1}c_2)^a = c_2^{-1}c_3 \in \langle H, H^{c_1}\rangle$, thus $c_1^{-1}c_2c_2^{-1}c_3 = c_1^{-1}c_3$ belongs to $\langle H, H^{c_1}\rangle$. Similarly, $c_1^{-1}c_4$, $c_1^{-1}c_5$, $\ldots$, $c_1^{-1}c_p$ belong to $\langle H, H^{c_1}\rangle$. Then $(c_1^{-1}c_p)^a = c_2^{-1}c_1^t \in \langle H, H^{c_1}\rangle$, so $c_1^{-1}c_2c_2^{-1}c_1^t = c_1^{t-1} \in \langle H, H^{c_1}\rangle$. Since the exponent of

t modulo r is $p^{\alpha-1}$ and $\alpha \geqslant 2$, we get $(t-1,r)=1$, thus $c_1 \in \langle H, H^{c_1}\rangle$. Hence a, c_1, …, c_p belong to $\langle H, H^{c_1}\rangle$. We have $\langle H, H^{c_1}\rangle = G$. Thus this result is true. □

3.1 Nilpotent groups and an improvement of Baer's theorem

Let $\mathcal{Z}$ be a complete set of Sylow subgroups of a group G, that is, for each prime p dividing the order of G, $\mathcal{Z}$ contains exactly one Sylow p-subgroup of G (see [1]). Let $\mathcal{Z}\cap E=\{P\cap E\,|\,P\in\mathcal{Z}\}$. Let $\mathbf{S}$ be a class of groups. If there is not the section in a group G to be isomorphic to a member of $\mathbf{S}$, then G is called $\mathbf{S}$-free. Let $\mathbf{F}_1$ denote the set of the inner-abelian groups of order pq^r described in [7, Theorem 1.5].

Theorem 3.1. *Let G be a group and $\mathcal{Z}$ be a complete set of Sylow subgroups of G. Suppose that E is a normal subgroup of G such that G/E is nilpotent and G is $\mathbf{F}_1$-free. If for every cyclic subgroup C of every Sylow subgroup of E contained in a member of $\mathcal{Z}\cap E$, $\langle C, C^g\rangle$ is nilpotent for all $g\in G$, then G is nilpotent.*

Proof. Assume that the result is false, and let G be a counterexample with least $|G|+|E|$.

Let $H<G$. Of course, H is $\mathbf{F}_1$-free. Obviously, $H/H\cap E \cong HE/E$ is nilpotent. Let $K=H\cap E$ and K_p be a Sylow p-subgroup of K. Then $\overline{\mathcal{Z}}=\{K_p \mid p\in\pi(K)\}$ is a complete set of Sylow subgroups of K. Assume that C is a cyclic subgroup of K_p. Since $K\leqslant E$, there exists a $x\in E$ such that $C^x\leqslant P\cap E$ where $P\in\mathcal{Z}$. By the hypothesis, $\langle C^x, C^{xg}\rangle$ is nilpotent for all $g\in G$, then $\langle C, C^{g^{x^{-1}}}\rangle$ is nilpotent for all $g\in G$. One can check easily that $\tau: G\to G$ defined by $\tau(g)=g^{x^{-1}}$, where $g\in G$, is a bijective map. Since $g^{x^{-1}}$ runs over G as g does for a fixed x, we get $\langle C, C^u\rangle$ is nilpotent for all $u\in G$. Hence H and its normal subgroup K satisfy the hypothesis. By the minimal choice of $|G|+|E|$, H is nilpotent. By Corollary 2.1, we may assume that $G=P^*Q$, where Q is a normal Sylow q-subgroup of G and P^* is a cyclic Sylow p-subgroup of G.

Suppose that N is a normal subgroup of G. We shall prove that $(G/N, EN/N)$ satisfies the hypothesis. Clearly, $(G/N)/(EN/N)\cong G/EN$ is nilpotent and G/N is $\mathbf{F}_1$-free. Let H/N be a cyclic subgroup of a Sylow subgroup of $EN/N\cap \mathcal{Z}N/N$. Then we may assume that $H=\langle xN\rangle$ and $\langle x\rangle$ is a cyclic subgroup of a Sylow subgroup in $E\cap\mathcal{Z}$. By the hypothesis, $\langle x, x^g\rangle$ is nilpotent for any $g\in G$. Since $\langle H/N, (H/N)^{gN}\rangle = \langle x, x^g\rangle N/N \cong \langle x, x^g\rangle/\langle x, x^g\rangle\cap N$, the group $\langle H/N, (H/N)^{gN}\rangle$ is a nilpotent subgroup of G/N for all $g\in G$. Then $(G/\Phi(G), E\Phi(G)/\Phi(G))$ satisfies the hypothesis of the theorem. The minimality of $|G|+|E|$ implies that $G/\Phi(G)$ is nilpotent and so is G, a contradiction. Thus $\Phi(G)=1$. By [7, Theorem 1.5], $G\in\mathbf{F}_1$, again, a contradiction. This shows that there exists no counterexample, therefore the result is true. □

Remark 3.1. The condition of "G is $\mathbf{F}_1$-free" cannot be removed. For example, let $G=S_3$. We choose $E=A_3$, then the pair (S_3, A_3) satisfies the hypothesis of Theorem 3.1. Nevertheless, S_3 is not nilpotent.

Corollary 3.1. *Let G be a group. Then G is nilpotent if and only if for every cyclic subgroup H of G with prime power order, $\langle H, H^g\rangle$ is nilpotent for all $g\in G$.*

Proof. We only need to prove the sufficiency. Suppose that G is a minimal counterexample. Then for every proper subgroup H of G, since every element in $\Omega_2(H)$ is nilpotent, H is nilpotent. Hence G is an inner-nilpotent group. By Lemma 3.1, there exists a cyclic subgroup H and $g\in G$ such that $G=\langle H, H^g\rangle$, thus G is nilpotent by the hypothesis, a contradiction. This contradiction completes the proof of this corollary. □

3.2 Supersolvable groups and the join of a pair of conjugate cyclic subgroups

Recall that Baer in [4] proved: A group is supersolvable if and only if every two elements generate a supersolvable subgroup. But we find that "every two elements" can not be replaced by "every two conjugate elements" in Baer's result. For example, every two conjugate elements of G_3 generate a supersolvable subgroup but G_3 is not supersolvable. But we can get the following theorems.

Theorem 3.2. *Let G be a group and assume that G' is nilpotent. Then G is supersolvable if and only if for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is supersolvable for all $g \in G$.*

Proof. We only need to prove the sufficiency. Suppose that G is a minimal counterexample. Then for every proper subgroup H of G, since every element of $\Omega_2(H)$ is supersolvable, H is supersolvable. Hence G is a inner-supersolvable group. Suppose that $\Phi(G) \neq 1$. By Proposition 3.1, every element in $\Omega_2(G/\Phi(G))$ is supersolvable. By the minimality of G, $G/\Phi(G)$ is supersolvable. Hence G is supersolvable, a contradiction. Thus $\Phi(G) = 1$ and G is a minimal inner-supersolvable group. Hence G is one of the groups listed in Corollary 2.2. Since G' is nilpotent, by Lemma 2.1, the complement of a Sylow r-subgroup in G is cyclic. Then G is isomorphic to G_1 or G_2. By Lemma 3.1, there exist a cyclic subgroup H and $g \in G$ such that $G = \langle H, H^g\rangle$. By the hypothesis, G is supersolvable, a contradiction. This contradiction completes the proof of this theorem. □

Theorem 3.3. *Let G be G_l-free for $3 \leqslant l \leqslant 6$. If for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is supersolvable for all $g \in G$, then G is supersolvable.*

Proof. By the proof of Theorem 3.2, if G is a minimal counterexample, then G is a minimal inner-supersolvable group. Hence G is one of the groups listed in Corollary 2.2. By Lemma 3.1, the result is true. □

In [14], Janko and Newman proved a finite group all of whose proper two-generator subgroups have cyclic commutator subgroups which have an ordered Sylow tower. If we consider the join of a pair of conjugate subgroups, then we can extend this result to the following.

Theorem 3.4. *Let G be a group. If for every cyclic subgroup H of G, $\langle H, H^g\rangle'$ is a cyclic subgroup for all $g \in G$, then G is supersolvable.*

Proof. Suppose that G is a minimal counterexample to our theorem. Then by Proposition 3.1, it is easy to see that G is a minimal inner-supersolvable-Ω_2-group. By Corollary 2.2, one of the following occurs.

(1) G is isomorphic to G_1 or G_2.

By Lemma 3.1, there exist a cyclic subgroup H and $g \in G$ such that $G = \langle H, H^g\rangle$. Then, by the hypothesis, $G' = \langle H, H^g\rangle'$ is cyclic, so G is supersolvable, a contradiction.

(2) G is isomorphic to G_3.

Let $H = \langle a\rangle$. By the hypothesis, $\langle H, H^g\rangle'$ is a cyclic subgroup for all $g \in G$. In particular, we choose $g = c_1$ and let $T = \langle a, a^{c_1}\rangle$. Then T' is a cyclic subgroup. So $a^{-1}a^{c_1} = (c_1^{-1})^a c_1 = c_2^{-1}c_1 \in T$, $(c_2^{-1}c_1)^a = c_1c_2 \in T$, thus $c_2^{-1}c_1c_1c_2 = (c_1)^2 \in T$. Since $(2, r) = 1$, we have $\{c_1, c_2\} \subset T$. Then $[a, c_1] = c_2^{-1}c_1 \in T'$, $[a, c_2] = c_1c_2 \in T'$, so $c_2^{-1}c_1c_1c_2 = (c_1)^2 \in T'$. Since $(2, r) = 1$, we get $c_1 \in T'$, then c_2 also belongs to T'. Hence $R \leqslant T'$, where $R \in \mathrm{Syl}_r(G)$ is an elementary abelian r-subgroup of G, contrary to the condition that T' is cyclic.

(3) G is isomorphic to G_i, where $i \in \{4, 5, 6\}$.

Let $H = \langle a\rangle$. By the hypothesis, $\langle H, H^g\rangle'$ is a cyclic subgroup for all $g \in G$. In particular, we choose $g = c_1$ and let $T = \langle a, a^{c_1}\rangle$. Then T' is a cyclic subgroup.

(I) Assume that $p \geqslant 3$. Then $a^{-1}a^{c_1} = (c_1^{-1})^a c_1 = c_2^{-1}c_1 \in T$, $(c_2^{-1}c_1)^a = c_3^{-1}c_2 \in T$, thus $c_2^{-1}c_1c_3^{-1}c_2 = c_3^{-1}c_1 \in T$. Hence $[a, c_2^{-1}c_1] = c_2^{-2}c_1c_3 \in T'$, $[a, c_3^{-1}c_1] = c_2^{-1}c_3^a c_1 c_3^{-1} \in T'$. Let $S = \langle c_2^{-2}c_1c_3, c_2^{-1}c_3^a c_1c_3^{-1}\rangle$. Then $S \leqslant T'$. If $p > 3$, then $c_3^a = c_4$. It is clear that S is an elementary abelian r-subgroup of G. If $p = 3$, then $c_3^a = c_1^t$, we also get that S is an elementary abelian r-subgroup of G. If not, there exists i such that $(c_2^{-2}c_1c_3)^i = c_2^{-1}c_1^{t+1}c_3^{-1}$, where i is non-zero integer and $-r < i < r$. Then $r \mid i + 1$, $r \mid 1 - 2i$ and $r \mid i - 1 - t$, so $r \mid 3i$. Since $p \mid r - 1$, we get $r \nmid 3$, so $r \mid i$, a contradiction. Thus S is an elementary abelian r-subgroup of G and $S \leqslant T'$, contrary to the condition that T' is cyclic.

(II) Assume that $p = 2$ and $\alpha \geqslant 2$. Then we have $c_1c_2^{-1} \in T$ and $(c_1c_2^{-1})^a = c_2c_1^{-t} \in T$, so $c_1c_2^{-1}c_2c_1^{-t} = c_1^{1-t} \in T$. Since the exponent of t modulo r is $p^{\alpha-1}$ and $\alpha \geqslant 2$, we get $r \nmid t - 1$, that is, $(1 - t, r) = 1$, thus $c_1 \in T$. Hence a, c_1, c_2 belong to T. Therefore, $T = HR$, where $R \in \mathrm{Syl}_r(G)$ and R is an elementary abelian r-subgroup of G. Then $[a, c_1] = c_2^{-1}c_1 \in T'$, $[a, c_2] = (c_2^a)^{-1}c_2 = (c_1)^{-t}c_2 \in T'$.

Thus $(c_1)^{-t}c_2c_2^{-1}c_1 = (c_1)^{1-t} \in T'$, so c_1, c_2 belong to T', hence $R \leqslant T'$, contrary to the assumption that T' is cyclic.

(III) Assume that $p = 2$ and $\alpha = 1$. We choose $H = \langle b \rangle$, $g = c_1^{-1}c_2^{-1}$.

If $G \cong G_4$, then $l = b^{-1}b^{c_1^{-1}c_2^{-1}} = c_1^{u^{1+p^{\beta-1}}-1}c_2^{u^{1+2p^{\beta-1}}-1} \in \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Suppose that $\langle l \rangle \lhd \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Then $l^b = l^m$, where $0 \leqslant m \leqslant r-1 (m \in \mathbb{Z})$, so

$$(c_1^{u^{1+p^{\beta-1}}-1})^b(c_2^{u^{1+2p^{\beta-1}}-1})^b = c_1^{(u^{1+p^{\beta-1}}-1)m}c_2^{(u^{1+2p^{\beta-1}}-1)m},$$
$$c_1^{(u^{1+p^{\beta-1}}-1)(u^{1+p^{\beta-1}}-m)} = c_2^{(u^{1+2p^{\beta-1}}-1)(m-u^{1+2p^{\beta-1}})},$$

thus $r \mid (u^{1+p^{\beta-1}}-1)(u^{1+p^{\beta-1}}-m)$ and $r \mid (u^{1+2p^{\beta-1}}-1)(m-u^{1+2p^{\beta-1}})$. Since the exponent of u modulo r is p^β, we have $u^{p^\beta} \equiv 1 \pmod r$ and $r \nmid u$. Since $\beta \geqslant 2$, we have $1+p^{\beta-1} < p^\beta$, so $r \nmid u^{1+p^{\beta-1}}-1$, thus $r \mid u^{1+p^{\beta-1}}-m$. If $r \mid u^{1+2p^{\beta-1}}-1$, then $r \mid u^{1+p^\beta}-u^{p^\beta} = u^{p^\beta}(u-1)$, so $r \mid u-1$, contrary to the condition that the exponent of u modulo r is p^β. Thus $r \mid m - u^{1+2p^{\beta-1}}$, hence $r \mid u^{1+p^\beta} - u^{1+p^{\beta-1}} = u^{1+p^{\beta-1}}(u^{p^{\beta-1}(p-1)}-1)$, so $r \mid u^{p^{\beta-1}(p-1)}-1$, contrary to the condition that the exponent of u modulo r is p^β. Hence $\langle l \rangle \ntrianglelefteq \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Therefore, b, c_1, c_2 belong to $\langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Using the same method as in (II), we get a contradiction.

If $G \cong G_5$, then $l = b^{-1}b^{c_1^{-1}c_2^{-1}} = c_1^{vu^p-1}c_2^{vu^{p-1}-1} \in \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Suppose that $\langle l \rangle \lhd \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Then $l^b = l^m$, where $0 \leqslant m \leqslant r-1 (m \in \mathbb{Z})$, so

$$(c_1^{vu^p-1})^b(c_2^{vu^{p-1}-1})^b = c_1^{(vu^p-1)m}c_2^{(vu^{p-1}-1)m},$$
$$c_1^{(vu^p-1)(vu^p-m)} = c_2^{(vu^{p-1}-1)(m-vu^{p-1})},$$

thus $r \mid (vu^p-1)(vu^p-m)$ and $r \mid (vu^{p-1}-1)(m-vu^{p-1})$. Since the exponents of v and u modulo r are p^β and p, we have $v^{p^\beta} \equiv 1 (\mathrm{mod}\ r)$ and $u^p \equiv 1 (\mathrm{mod}\ r)$, respectively. If $r \mid vu^p-1$, then $r \mid vu^p-1-(u^p-1)$, that is, $r \mid u^p(v-1)$, so $r \mid v-1$, contrary to the condition that the exponent of v modulo r is p^β. Thus $r \mid vu^p - m$. If $r \mid vu^{p-1}-1$, then $r \mid u^p - 1 - (vu^{p-1}-1) = u^{p-1}(u-v)$, that is, $r \mid u-v$, so $r \mid v^p - 1$, thus $\beta = \alpha = 1$, we have that the complement of a Sylow r-subgroup in G is abelian, a contradiction. Hence $r \mid m - vu^{p-1}$, thus $r \mid vu^p - vu^{p-1} = vu^{p-1}(v-1)$, so $r \mid v-1$, again, a contradiction. Hence $\langle l \rangle \ntrianglelefteq \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Therefore, b, c_1, c_2 belong to $\langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Using the same method as in (II), we get a contradiction.

If $G \cong G_6$, then $l = b^{-1}b^{c_1^{-1}c_2^{-1}} = c_1^{v-1}c_2^{v^u-1} \in \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Suppose that $\langle l \rangle \lhd \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$, then $l^b = l^m$ where $0 \leqslant m \leqslant r-1 (m \in \mathbb{Z})$, so

$$(c_1^{v-1})^b(c_2^{v^u-1})^b = c_1^{(v-1)m}c_2^{(v^u-1)m},$$
$$c_1^{(v-1)(v-m)} = c_2^{(v^u-1)(m-v^u)},$$

thus $r \mid (v-1)(v-m)$ and $r \mid (v^u-1)(m-v^u)$. Since the exponent of v modulo r is q, we have $v^q \equiv 1(\mathrm{mod}\ r)$, $r \nmid v$ and $r \mid v-m$. If $r \mid v^u-1$, then $q \mid u$, contrary to the condition that the exponent of u modulo q is p. So $r \mid v^u - m$, thus $r \mid v^u - v = v(v^{u-1}-1)$. As before, $q \mid u-1$, again a contradiction. Hence $\langle l \rangle \ntrianglelefteq \langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Therefore, b, c_1, c_2 belong to $\langle b, b^{c_1^{-1}c_2^{-1}}\rangle$. Using the same method as in (II), we get a contradiction.

These contradictions complete the proof of this theorem. □

3.3 An improvement of Thompson's theorem on the solvability of a group

In 1968, Thompson obtained the following theorem as a corollary of his classification of minimal simple groups in [21].

Theorem 3.5. (J. G. Thompson) *A group is solvable if and only if every two elements generate a solvable subgroup.*

A direct proof not using the classification of minimal simple groups has been obtained by Flavell in [11]. In this section, we shall get an extension of Thompson's result above under a weaker condition, that is, we have the following theorem.

Theorem 3.6. *Let G be a group. Then G is solvable if and only if for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is solvable for all $g \in G$.*

Proof. The necessity of the theorem is obvious. We only need to prove the sufficiency of the theorem. Now suppose that G is a minimal counterexample to our theorem. Then by Proposition 3.1, it is easy to get that G is a minimal inner-solvable-Ω_2-group. By Theorem 2.2, G is a simple group in which every proper subgroup is solvable. Thus G is one of the minimal simple subgroups listed in [7, Theorem 5.1], this means that one of the following occurs.

(1) $G = PSL(2, p)$, $p > 3$, $5 \nmid p^2 - 1$, $|G| = \frac{1}{2}p(p^2 - 1)$.

By Dickson's theorem [13, II, 8.27], G has a dihedral maximal subgroup T of order $p + 1$. Let H be a cyclic maximal subgroup of T with order $\frac{1}{2}(p + 1)$. If $g \in G \setminus N_G(H)$, then $H < \langle H, H^g\rangle \leqslant G$. If $\langle H, H^g\rangle \neq G$, then there exists a maximal subgroup S of G such that $H < \langle H, H^g\rangle \leqslant S$. By Dickson's theorem, if S contains a cyclic subgroup of order $\frac{1}{2}(p + 1)$, then S is a dihedral subgroup of G of order $p+1$. Since there is the unique cyclic subgroup of order $\frac{1}{2}(p+1)$ in a dihedral group of order $p+1$, we have $H^g = H$, thus $g \in N_G(H)$, a contradiction. This implies that $G = \langle H, H^g\rangle$ is solvable, a contradiction.

(2) $G = PSL(2, 2^q)$, q is a prime, $|G| = 2^q(2^{2q} - 1)$.

By Dickson's theorem, G has a maximal subgroup T, where T is a dihedral group of order $2(2^q + 1)$. We choose the maximal cyclic subgroup H of T with order $2^q + 1$ and $g \in G \setminus N_G(H)$. Using the same method as in (1), we also get a contradiction.

(3) $G = PSL(2, 3^q)$, q is an odd prime, $|G| = \frac{1}{2}3^q(3^{2q} - 1)$.

By Dickson's theorem, G has a dihedral maximal subgroup T of order $3^q + 1$. We choose the maximal cyclic subgroup H of T with order $\frac{1}{2}(3^q + 1)$ and $g \in G \setminus N_G(H)$. Using the same method as in (1), we still get a contradiction.

(4) $G = PSL(3, 3)$, $|G| = 2^4 \cdot 3^3 \cdot 13$.

Let $H \in \mathrm{Syl}_{13}(G)$ and $T = N_G(H)$. By [8, p. 13], it is easy to see that T is a maximal subgroup of G of order 39, and if $g \in G$ with $|g| = 2$, then $\langle H, H^g\rangle = G$. Hence G is solvable, a contradiction.

(5) $G = S_Z(2^q)$, q is an odd prime, $|G| = (2^{2q} + 1)2^{2q}(2^q - 1)$.

By [20], G has a maximal subgroup T of order $2^2(2^q + 2^{\frac{q+1}{2}} + 1)$. We choose a cyclic subgroup H of T with order $2^q + 2^{\frac{q+1}{2}} + 1$ and $g \in G \setminus N_G(H)$, then $N_G(H) = T$, $H < \langle H, H^g\rangle \leqslant G$. If $\langle H, H^g\rangle \neq G$, then there exists a maximal subgroup S of G such that $\langle H, H^g\rangle \leqslant S$. By [20], it is easy to see that if S contains a cyclic subgroup of order $2^q + 2^{\frac{q+1}{2}} + 1$, then S is isomorphic to T and so $S = N_G(H) = T$, a contradiction. This implies that $G = \langle H, H^g\rangle$ is solvable, a contradiction.

These contradictions complete the proof of this theorem. □

Remark 3.2. As pointed out by the referee, Gordeev, et al. [12] established the assertion of Theorem 3.6. But the proofs follow a completely different approach, and their proof relies on the classification of finite simple groups.

Naturally, we would like to put forward the following.

Problem 3.1. Could you prove Theorem 3.6 without using the classification of minimal simple groups?

3.4 *p*-closed groups and the join of a pair of conjugate cyclic subgroups

To conclude, we investigate the properties of a group in which every pair of conjugate elements generates a p-closed subgroup.

Theorem 3.7. *Let G be a group. Then G is 2-closed if and only if for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is 2-closed for all $g \in G$.*

Proof. We only need to prove the sufficiency. Now suppose that G is a minimal counterexample to our theorem. Then by Proposition 3.1, it is easy to prove that G is an inner-2-closed-Ω_2-group. By Theorem 2.2, we have the G is an inner-2-closed group in which every proper subgroup is 2-closed. By [7, p. 23, Theorem 4.5], $G = \langle a, b\rangle$ and $a^{2^\alpha} = b^q = 1$, $b^a = b^{-1}$, q is an odd prime. By the hypothesis, $\langle a, a^g\rangle$ is 2-closed for all $g \in G$. We choose $g = b^{-1}$, then $[a, b^{-1}] = a^{-1}a^{b^{-1}} \in \langle a, a^{b^{-1}}\rangle$. Since $[a, b^{-1}] = b^{-2}$, $(2, q) = 1$, we have $q \mid |\langle a, a^{b^{-1}}\rangle|$. Thus $2^\alpha q \mid |\langle a, a^{b^{-1}}\rangle|$, hence we get $G = \langle a, a^{b^{-1}}\rangle$ is 2-closed, a contradiction. This contradiction completes the proof of this theorem. □

Theorem 3.8. *Let G be a group. Then G is 3-closed if and only if for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is 3-closed for all $g \in G$.*

Proof. We only need to prove the sufficiency. Now suppose that G is a minimal counterexample to our theorem. Then by Proposition 3.1, it is easy to prove that G is an inner-3-closed-Ω_2-group. By Theorem 2.2, G is an inner-3-closed group. By [7, p. 24, Theorem 4.6], one of the following occurs.

(1) G is solvable.

In this case, by [7, p. 24, Theorem 4.6], G is an inner-nilpotent group of order $3^\alpha q^\beta$. By Lemma 3.1, we have $G = \langle H, H^g\rangle$, where H is a cyclic subgroup of G and $g \in G$, then $G = \langle H, H^g\rangle$ is 3-closed by the hypothesis, a contradiction.

(2) G is non-solvable, then $G \cong PSL(2, 2^p)$, p is an odd prime.

In this case, by the proof of Theorem 3.6, there exist a cyclic subgroup H of G and $g \in G$ such that $G = \langle H, H^g\rangle$. Thus G is 3-closed, contrary to the simplicity of G. □

Theorem 3.9. *Let G be a group. Then G is 5-closed if and only if for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is 5-closed for all $g \in G$.*

Proof. We only need to prove the sufficiency. Suppose that G is a minimal counterexample to our theorem. Then by Proposition 3.1, it is easy to prove that G is an inner-5-closed-Ω_2-group. By Theorem 2.2, G is an inner-5-closed group. By [15, Theorem 2], one of the following occurs.

(1) G is solvable, then G is an inner-nilpotent group of order $5^\alpha q^\beta$.

(2) G is non-solvable, then $G \cong PSL(2, 5)$ or $G \cong S_Z(2^q)$, q is an odd prime.

Using the similar discussion as in Theorem 3.6 and Theorem 3.8, this theorem is true. □

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