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A New Cost Allocation Approach on One Machine Sequencing Games

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Abstract. One machine sequencing situation is introduced and cooperative games theory is used to allocate cost savings when agents cooperate and gain some cost savings. One machine games is proved to be balanced and have a nonempty core. A new cost allocation, the proportion gain allocation approach, is proposed and proved to give a core allocation for one sequencing games. The results in examples show that the proportion gain allocation approach are fair and reasonable for one machine games.

Introduction

In one machine sequencing situation, a finite number of agents, each having one job, are queued in front of a machine waiting for their jobs to be processed, every agent has due date and penalty for lateness. Agents have linear cost functions associated with the completion time of his job and each group of agents (coalition) is allowed to obtain cost savings by reordering position of agents. There is an initial schedule on agents before the processing of the machine starts, and an optimal schedule of a coalition is an admissible arrangement that maximizes the cost savings of this coalition. By assuming an initial schedule of all agents, two problems need to be solved, the first one is how to find an optimal schedule of all jobs, and the second is that of allocate these cost savings among the parties in a fair way.

To cost allocation problem, cooperative games theory has turned out to be a useful tool for the study of cooperation in sequencing situations. Tijs [1] researches games theory and cost allocation problem, Sharply value method and cost gap allocation method are used to solve cost allocation. Curiel [2,3] proposes sequencing games theory and gives equal gain splitting rule for one machine sequencing situation and provides an axiomatical characterization. Following studies have extended original model by considering ready times [4], due dates [5], precedence relations [6] and controllable processing times [7]. In these papers, the convexity or balancedness of corresponding class of games is established. In [8] proportionate flow shop games are researched, associated games have nonempty core and are convex if initial schedule is in decreasing urgency indices. Calleja [9] considers a class of sequencing situation with two parallel machines in which each agent owns two jobs to be processed, one on each machine, associated games are balanced on some conditions. Zhou [10] concerns a class of flow shop scheduling problem with processing time associated with work stage, corresponding games are proved be balanced, and a cost allocation method is put forward and proved to lead a core allocation. Curiel [11] uses dynamics cooperative games to solve multi-stage sequencing situation and MEGS rule is defined to yield stable allocations.

EGS approach [2] can give a core allocation, which allocated gain obtained by switching neighboring two jobs to associated two agents averagely, sometimes, it is unfair. This paper put forward a new cost allocation approach, the proportion gain allocation approach, to allocating cost savings, which is proved to provide core allocation for one machine sequencing games.

One machine sequencing situation

The set of agents is denoted by $\boldsymbol{N} = \{1,2,\cdots,n\}$, by a bijection $\sigma : \boldsymbol{N} \to \{1,2,\cdots,n\}$, the position of agents in the queue can be described, e.g., $\sigma(i) = j$ means that player *i* is in position *j*, σ_0 is an initial schedule to be processed. The processing times of jobs on machine are denoted by $\boldsymbol{p} = \{p_1, p_2, \cdots, p_n\}$.

For a schedule σ, the completion time $C_\sigma(i)$ of the job of agent *i* is equal to $t_\sigma(i) + p_i$, where $t_\sigma(i)$ is the starting time of the job of agent *i*, and

$$t_\sigma(i) = \begin{cases} 0 & if \quad \sigma(i) = 1 \\ \sum_{j \in P(\sigma,i)} p_j & if \quad \sigma(i) > 1 \end{cases} \tag{1}$$

where $P(\sigma, i)$ is the set of predecessors of *i* with respect to the schedule σ.

The cost of agent *i* are defined as

$$c_\sigma(i) = \alpha_i \cdot C_\sigma(i) \tag{2}$$

where α_i is cost coefficient and $\alpha_i > 0$, the set of cost coefficient is denoted by $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$.

Then one machine sequencing situation can be described as a 4-tuple (N, σ, p, α).

The total costs of agents are defined as

$$c_N(\sigma) = \sum_{i \in N} c_\sigma(i) \tag{3}$$

Let $\Pi(N)$ be the set of all schedules of the agents, $\hat{\sigma}$ is an optimal schedule that minimizes the total costs, so

$$c_N(\hat{\sigma}) = \min_{\sigma \in \Pi(N)} c_N(\sigma) \tag{4}$$

It follows from Smith rule [12] that an optimal schedule is arranged according to decreasing urgency indices in one sequencing situation, and urgency indices are defined as $\alpha_i p_i^{-1}$.

Example 1 *Let* (N, σ, p, α) *be one sequencing situation with* $\boldsymbol{N} = \{1,2,3,4,5,6,7\}$, $\sigma_0 =$ *(1 2 3 4 5 6 7),* $\boldsymbol{p} = \{30,50,40,70,40,50,60\}$, $\boldsymbol{\alpha} = \{0.7,0.4,0.5,0.9,0.6,0.8,0.3\}$.

In initial schedule σ_0, the completion time can be calculated, $C_{\sigma_0}(1) = 30$, $C_{\sigma_0}(2) = 80$, $C_{\sigma_0}(3) = 120$, $C_{\sigma_0}(4) = 190$, $C_{\sigma_0}(5) = 230$, $C_{\sigma_0}(6) = 280$, $C_{\sigma_0}(7) = 340$, the total costs of agents are 748.

According to Smith rule, optimal schedule $\hat{\sigma}$ = (1 6 5 4 3 2 7), jobs are reordered according to decreasing $\alpha_i \cdot p_i^{-1}$ coefficient. The completion time can be calculated, $C_{\hat{\sigma}}(1) = 30$, $C_{\hat{\sigma}}(2) = 280$, $C_{\hat{\sigma}}(3) = 230$, $C_{\hat{\sigma}}(4) = 190$, $C_{\hat{\sigma}}(5) = 120$, $C_{\hat{\sigma}}(6) = 80$, $C_{\hat{\sigma}}(7) = 340$, the total costs of agents are 657, total saving costs are 91 in optimal schedule $\hat{\sigma}$.

One machine sequencing games

A cooperative games is a pair (*N*, *v*) where *N* is a finite set (of agents) and $v: 2^N \to \boldsymbol{R}$ is a mapping with *v* (ϕ) = 0. The mapping *v* assigns to each coalition $S \subset N$ the worth of the coalition *v*(*S*). A cooperative games (*N*, *v*) is called super-additive if for all coalitions $S, T \subset N$

$$v(S) + v(T) \le v(S \cup T) \quad \text{Whenever } S \cap T \subset \phi \tag{5}$$

The precondition of cooperative games is the formation of coalition, and how to divide the total profit *v*(*N*) among the agents is very important. An allocation of the amount *v*(*N*) can be described by a vector $x \in R^N$ with $x(N) = \sum_{i \in N} x(i) = v(N)$. The quantity x_i is the amount allocated to agent *i*. The core *C*(*v*) of a cooperative game (*N*, *v*) is defined as the set of efficient allocations for which no coalition has an incentive to split off from the grand coalition, i.e., for all $S \subset N$

$$C(v) = \left\{ x \in R^N \middle| \sum_{i \in N} x(i) = v(N), \quad \sum_{i \in S} x(i) \ge v(S) \right\} \tag{6}$$

the core of cooperative games can be empty, games with nonempty core are called balanced. The Sharply value [13] is one of the famous solution concepts in cooperation games theory.

A coalition S is called connected with respect to σ if for all $i, j \in S$ and $k \in N$ such that $\sigma(i) < \sigma(k) < \sigma(j)$ it holds that $k \in S$. For a coalition S, $S \setminus \sigma$ is the set of all maximally connected components of S with respect to σ. Notice that $S \setminus \sigma$ is a partition of S. A cooperative game (N, v) is called σ_0-component additive if it satisfies the following three conditions: (a) $v(i) = 0$ for all $i \in N$; (b) (N, v) is super-additive; (c) $v(S) = \sum_{T \in S\setminus\sigma} v(T)$. Le Breton [14] showed that σ_0-component additive games are balanced.

Let (N, σ, p, α) be one sequencing situation, σ_0 is an initial schedule, $A(S)$ is the set of admissible schedules for coalition S, one machine sequencing games are defined by

$$v(S) = \max_{\sigma \in A(S)} \left\{ \sum_{i \in S} \left(c_{\sigma_0}(i) - c_{\sigma}(i) \right) \right\} \tag{7}$$

Agents rearrange in $A(S)$, and at last, the value of coalition S equals maximal cost savings that the coalition can obtain in admissible rearrangements.

Theorem 1 *One sequencing games are* σ_0*-component additive games, and hence balanced.*

Proof: Let (N, σ, p, α) be one sequencing situation and (N, v) be the associated one sequencing games.

Obviously, a coalition including only a player can not save cost. So $v(i) = 0$ for all $i \in N$.

Let $S, T \subset N$ and $S \cap T \subset \phi$, assume σ_0 is an initial schedule, optimal schedules are σ_S and σ_T with respect to coalition S and T, then

$$\begin{aligned} v(S) + v(T) &= \sum_{i \in S} \left(c_{\sigma_0}(i) - c_{\sigma_S}(i) \right) + \sum_{i \in T} \left(c_{\sigma_0}(i) - c_{\sigma_T}(i) \right) \\ &= \sum_{i \in S \cup T} \left(c_{\sigma_0}(i) - c_{\sigma_{(S \cup T)}}(i) \right) \leq \max_{\sigma \in A(S \cup T)} \sum_{i \in S \cup T} \left(c_{\sigma_0}(i) - c_{\sigma}(i) \right) = v(S \cup T) \end{aligned}$$

Note that $A(S \cup T)$ is the set of admissible schedules for coalition $S \cup T$ and $\sigma_{(S \cup T)}$ is a schedule which satisfies

$$\sigma_{(S \cup T)}(i) = \begin{cases} \sigma_S(i) & \text{if } i \in S \\ \sigma_T(i) & \text{if } i \in T \\ \sigma_0(i) & \text{if } i \in N \setminus (S \cup T) \end{cases}$$

So one sequencing games (N, v) are super-additive.

Let $T \in S \setminus \sigma_0$ and $T = \{T_1, T_1, \cdots T_s\}$, then

$$\begin{aligned} v(S) &= \max_{\sigma \in A(S)} \sum_{i \in S} \left(c_{\sigma_0}(i) - c_{\sigma}(i) \right) \\ &= \max_{\sigma_1 \in A(T_1)} \sum_{i \in T_1} \left(c_{\sigma_0}(i) - c_{\sigma_1}(i) \right) + \max_{\sigma_2 \in A(T_2)} \sum_{i \in T_2} \left(c_{\sigma_0}(i) - c_{\sigma_2}(i) \right) + \cdots + \max_{\sigma_s \in A(T_s)} \sum_{i \in T_s} \left(c_{\sigma_0}(i) - c_{\sigma_s}(i) \right) \\ &= v(T_1) + v(T_2) + \cdots + v(T_s) = \sum_{T \in S \setminus \sigma} v(T) \end{aligned}$$

One sequencing games (N, v) satisfy condition (a)-(c) of σ_0-component additive games given in previous section, so one sequencing games are σ_0-component additive, and hence balanced.

Cost allocation approach

According to Smith rule, optimal schedule can be obtained by reordering jobs according to decreasing urgency indices. EGS approach [2] allocate cost savings obtained by switching neighboring two jobs to associated two agents averagely, sometimes, it is unfair. We propose a cost allocation approach, the proportion gain allocation (PGA) approach, and show that it gives a core allocation for one sequencing games.

Let i and j be two neighbors with i standing in front of j, the cost saving obtained by switching i and j is denoted by g_{ij}, and

$$g_{ij} = \max(0, p_i \alpha_j - p_j \alpha_i) \tag{8}$$

If $F(\sigma_0, i)$ is defined as the set of followers of *i* with respect to the schedule σ_0, then the total cost savings are

$$v(N) = \sum_{i\in N}\sum_{j\in F(\sigma_0,i)} g_{ij} \tag{9}$$

Let (N, σ, p, α) be one sequencing situation, and (N, v) be the associated one sequencing games, σ_0 is an initial schedule, PGA approach is defined by

$$PGA_i(v) = \sum_{k\in P(\sigma_0,i)} \frac{c_{\sigma_0}(i)}{c_{\sigma_0}(k)+c_{\sigma_0}(i)}\cdot g_{ki} + \sum_{j\in F(\sigma_0,i)} \frac{c_{\sigma_0}(i)}{c_{\sigma_0}(i)+c_{\sigma_0}(j)}\cdot g_{ij} \tag{10}$$

Theorem 2 *The PGA approach gives a core allocation of one sequencing game.*

Proof: Let (N, σ, p, α) be one sequencing situation, and (*N*, *v*) be the associated one sequencing games, σ_0 is an initial schedule, and $\hat{\sigma}$ is optimal schedule.

According to prior definition:

$$\sum_{i\in N} PGA_i(v) = \sum_{i\in N}\sum_{j\in F(\sigma_0,i)} g_{ij} = v(N)$$

Let $S \in 2^N$, then

$$\sum_{i\in S} PGA_i(v) = \sum_{i\in S}\left(\sum_{k\in P(\sigma_0,i)} \frac{c_{\sigma_0}(i)}{c_{\sigma_0}(i)+c_{\sigma_0}(k)}\cdot g_{ki} + \sum_{j\in F(\sigma_0,i)} \frac{c_{\sigma_0}(i)}{c_{\sigma_0}(i)+c_{\sigma_0}(j)}\cdot g_{ij}\right)$$

$$\geq \sum_{i\in S}\left(\sum_{k\in P(\sigma_0,i),k\in S} \frac{c_{\sigma_0}(i)}{c_{\sigma_0}(i)+c_{\sigma_0}(k)}\cdot g_{ki} + \sum_{j\in F(\sigma_0,i),j\in S} \frac{c_{\sigma_0}(i)}{c_{\sigma_0}(i)+c_{\sigma_0}(j)}\cdot g_{ij}\right) = v(S)$$

The PGA approach satisfies conditions of core allocation (Eq.6 in previous section) for one sequencing games, so it gives a core allocation.

Example 2 *Using PGA approach find core allocation for one sequencing games in Example 1.*

In initial schedule, cost of all agents $\boldsymbol{c}$ = (21,32,60,171,138,224,102), total cost of all agents is 748. We have found optimal schedule $\hat{\sigma}$ = (1 6 5 4 3 2 7) and saving total costs are 91 in optimal schedule $\hat{\sigma}$. We can calculate the gain obtained by switching neighboring two jobs

$$g_{12}=0, g_{13}=0, g_{14}=0, g_{15}=0, g_{16}=0, g_{17}=0, g_{23}=9, g_{24}=17, g_{25}=14, g_{26}=20, g_{27}=0$$

$$g_{34}=1, g_{35}=4, g_{36}=7, g_{37}=0, g_{45}=6, g_{46}=11, g_{47}=0, g_{56}=2, g_{57}=0, g_{67}=0$$

Gains allocated to agents are

$$PGA_1(v) = \sum_{k\in P(\sigma_0,1)} \frac{c_{\sigma_0}(1)}{c_{\sigma_0}(k)+c_{\sigma_0}(1)}\cdot g_{k1} + \sum_{j\in F(\sigma_0,1)} \frac{c_{\sigma_0}(1)}{c_{\sigma_0}(1)+c_{\sigma_0}(j)}\cdot g_{1j} = 0,$$

$$PGA_2(v)=10.9, PGA_3(v)=8.8, PGA_4(v)=23.1, PGA_5(v)=17.6, PGA_6(v)=30.5, PGA_7(v)=0$$

Clearly, $\sum_{i\in N} PGA_i(v) = \sum_{i\in N}\sum_{j\in F(\sigma_0,i)} g_{ij} = v(N) = 91$.

Conclusions

In this paper one machine sequencing situation and one machine games are discussed. We prove that one machine games are balanced. We propose the proportion gain allocation approach to allocate cost savings and prove that it can give core allocation for one sequencing games. Some research results can be extended to others sequencing situation where cost savings can be obtained by cooperation and associated cost allocation problem.

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