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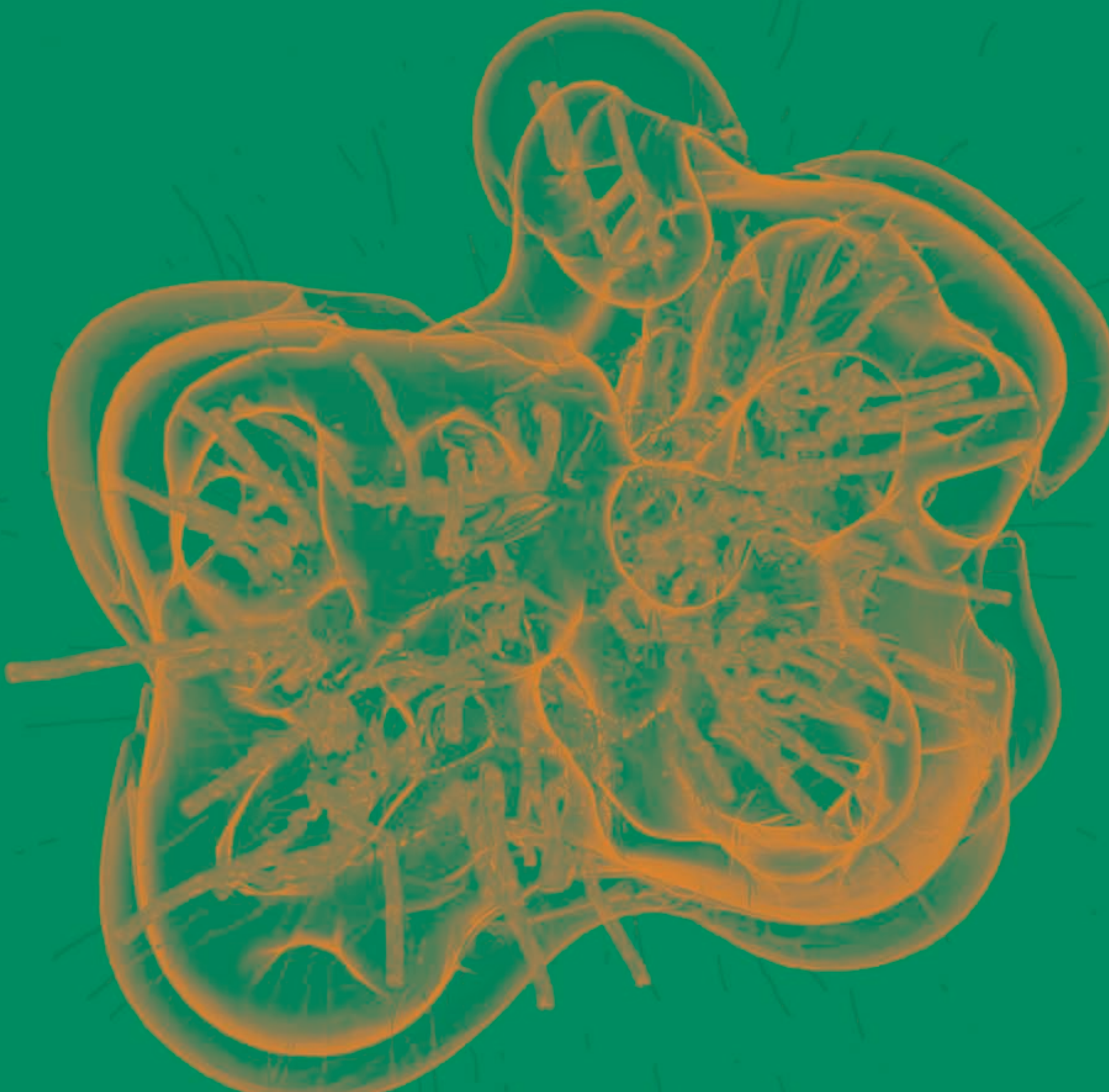

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Delay-dependent stability of neural networks of neutral type with time delay in the leakage term

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Abstract

This paper studies the global asymptotic stability of neural networks of neutral type with mixed delays. The mixed delays include constant delay in the leakage term (i.e. 'leakage delay'), time-varying delays and continuously distributed delays. Based on the topological degree theory, Lyapunov method and linear matrix inequality (LMI) approach, some sufficient conditions are derived ensuring the existence, uniqueness and global asymptotic stability of the equilibrium point, which are dependent on both the discrete and distributed time delays. These conditions are expressed in terms of LMI and can be easily checked by the MATLAB LMI toolbox. Even if there is no leakage delay, the obtained results are less restrictive than some recent works. It can be applied to neural networks of neutral type with activation functions without assuming their boundedness, monotonicity or differentiability. Moreover, the differentiability of the time-varying delay in the non-neutral term is removed. Finally, two numerical examples are given to show the effectiveness of the proposed method.

Mathematics Subject Classification: 92B20, 34D23

1. Introduction

For two decades, dynamic behaviours of delayed neural networks have been investigated widely due to their potential applications in many fields such as pattern recognition, image processing, fixed-point computation and synchronization problem [1–4]. Many interesting results have been obtained; see [5–16], where [5–8] discussed the case of constant delays, [9–13] studied

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the case of time-varying delays and [14–16] considered the case of continuously distributed delays.

Recently, Gopalsamy [17] initially investigated the bidirectional associative memory (BAM) neural networks with constant delays in the leakage terms as follows:

$$\left.\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t-\tau_i^{(1)}) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\sigma_j^{(2)})) + I_i \\ \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} &= -b_i y_i(t-\tau_i^{(2)}) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_j^{(1)})) + J_i\end{aligned}\right\}, \qquad t>0,\ i=1,2,\ldots,n, \tag{I}$$

where the first terms on the right-hand side of model (I) are variously known as forgetting or leakage terms, and time delays $\tau_i^{(1)}$ and $\tau_i^{(2)}$ are called leakage delays or forgetting delays, see [17–19]. Such time delays in the leakage terms are difficult to handle and have been rarely considered in the literature. The author presented several sufficient conditions to guarantee the existence-uniqueness, global asymptotic stability and global exponential stability of equilibrium point for model (I) via the Lyapunov–Kravsovskii functionals, M-matrices method and some analysis techniques, see [17] for detailed information. Inspired by this work, Peng [20] further investigated the BAM neural networks with continuously distributed delays in the leakage terms, and obtained some conditions for the existence and global attractiveness of periodic solutions via Lyapunov functional and analysis theory. As we know, linear matrix inequality (LMI) techniques have been successfully used to tackle various dynamical behaviours of delayed neural networks [6, 21, 28]. Such type of results not only can be easily verified via the MATLAB LMI toolbox but also reflect the neuron's inhibitory and excitatory effects on neural networks. However, hardly any work has been done so far on the stability of neural networks with leak delays via the LMI approach, which remains an interesting research topic.

On the other hand, a type of time delay, namely neutral-type time delays, has recently drawn much research attention [22–27, 36–38]. In fact, many practical delay systems can be modelled as differential systems of neutral type, whose differential expression includes not only the derivative term of the current state but also the derivative term of the past state, such as partial element equivalent circuits and transmission lines in electrical engineering, controlled constrained manipulators in mechanical engineering and population dynamics, see [37, 38]. Moreover, it has been shown that the existing neural network models in many cases cannot characterize the properties of a neural reaction process precisely due to the complicated dynamic properties of the neural cells in the real world, and it is natural and necessary that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [26, 39]. To date, there have been many results on dynamical analysis of neural networks of neutral type by using the Lyapunov–Krasovskii functional and LMI; see [22–30, 34, 35] and references therein. For instance, Park and Kwon [25] studied the global asymptotic stability of delayed cellular neural networks of neutral type with interval time-varying delays. In [30], Zhu *et al* investigated the robust stability of Hopfield neural networks of neutral type with time-varying delays. However, it is worth pointing out that the given criteria in [22–30, 34, 35] have been based on the following assumptions: (i) the activation functions are bounded, monotonic or differentiable; (ii) the time delays are constant delays or time-varying delays which are continuously differentiable and their derivatives are bounded. Such undesirable assumptions restrict the application of those results to real problems. Hence, there is still enough room to develop novel stability conditions for improvement.

Inspired by the above discussion, in this paper, we consider a class of neural networks of neutral type with mixed delays. The mixed delays include leakage delay, time-varying delays and continuously distributed delays. By constructing a proper Lyapunov–Krasovskii functional and employing topological degree theory and LMI techniques, some delay-dependent sufficient conditions ensuring the existence, uniqueness and global asymptotic stability of the equilibrium point are derived, which are expressed in terms of LMI and can be easily checked by the MATLAB LMI toolbox. The obtained results require neither the differentiability of time-varying delays in the non-neutral term nor the boundedness, monotonicity or differentiability of the activation functions, and are dependent on the leakage delay, time-varying delays and continuously distributed delays. Moreover, even if there is no leakage delay, the obtained results are less restrictive than some recent works [22–30, 34, 35]. Finally, two numerical examples are given to show the effectiveness of the proposed method.

2. Preliminaries

Notation. Let $\mathbb{R}$ ($\mathbb{R}^+$) denote the set of (positive) real numbers, $\mathbb{Z}_+$ denote the set of positive integers and $\mathbb{R}^n$ denote the n-dimensional real spaces equipped with the Euclidean norm $||\cdot||$, $\mathscr{A} > 0$ or $\mathscr{A} < 0$ denote that the matrix $\mathscr{A}$ is a symmetric and positive definite or negative definite matrix. The notation $\mathscr{A}^{\mathrm{T}}$ and $\mathscr{A}^{-1}$ mean the transpose of $\mathscr{A}$ and the inverse of a square matrix. $\lambda_{\max}(\mathscr{A})$ or $\lambda_{\min}(\mathscr{A})$ denotes the maximum eigenvalue or the minimum eigenvalue of matrix $\mathscr{A}$. I denotes the identity matrix with appropriate dimensions and $\Lambda = \{1, 2, \ldots, n\}$. $[\cdot]^*$ denotes the integer function. For any interval $J \subseteq \mathbb{R}$, set $V \subseteq \mathbb{R}^k(1 \leqslant k \leqslant n), C(J, V) = \{\phi : J \to V \text{ is continuous}\}$ and $C_b^1(J, V) = \{\phi : J \to V \text{ is continuously differentiable bounded}\}$. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $[x]^+ = (|x_1|, |x_2|, \ldots, |x_n|)^{\mathrm{T}}$ and for any $Q = (q_{ij})_{n\times n} \in \mathbb{Z}^{n\times n}$, $[Q]^+ = (|q_{ij}|)_{n\times n}$. In addition, the notation $\star$ always denotes the symmetric block in one symmetric matrix.

Consider the following neural networks model:

$$\begin{cases} \dot{x}(t) = -Cx(t-\sigma) + Af(x(t-\tau(t))) \\ \quad + B\displaystyle\int_{-\infty}^{t} K(t-s)f(x(s))\,\mathrm{d}s + D\dot{x}(t-h(t)) + J, \qquad t > 0, \\ x(s) = \varphi(s),\ s \in (-\infty, 0], \end{cases} \tag{1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ is the neuron state vector of the neural networks and $\dot{x}$ denotes the time derivative of the neuron state; $C = \mathrm{diag}(c_1, \ldots, c_n)$ is a diagonal matrix with $c_i > 0, i \in \Lambda$; A, B and D are the interconnection matrices representing the weight coefficients of the neurons; J is an external input; $f(x(\cdot)) = (f_1(x_1(\cdot)), \ldots, f_n(x_n(\cdot)))^{\mathrm{T}}$ is the neuron activation function; $K(\cdot) = \mathrm{diag}(k_1(\cdot), \ldots, k_n(\cdot))$ is the delay kernel; $\sigma \geqslant 0$ is a constant which denotes the leakage delay; $\tau(t)$ and $h(t)$ are time-varying transmission delays and satisfy $0 \leqslant \tau(t) \leqslant \tau$, $0 \leqslant h(t) \leqslant h$, $\dot{h}(t) \leqslant h^* < 1$; initial condition $\varphi(\cdot) \in C_b^1((-\infty, 0], \mathbb{R}^n)$, the norm is defined by

$$\|\varphi\|_h = \max\left\{\sup_{s\leqslant 0}\|\varphi(s)\|,\ \sup_{-h\leqslant s\leqslant 0}\|\dot{\varphi}(s)\|\right\}.$$

In this paper, we give the following assumptions:

(H_1) The neurons activation functions f_j, $j \in \Lambda$, are continuous on $\mathbb{R}$ and satisfy

$$l_j^- \leqslant \frac{f_j(u) - f_j(v)}{u - v} \leqslant l_j^+, \qquad \text{for any } u, v \in \mathbb{R}, \quad u \neq v, \quad j \in \Lambda,$$

where l_j^- and l_j^+ are some real constants and they may be positive, zero or negative.

(H_2) The delay kernels $k_j, j \in \Lambda$, are some real value non-negative continuous functions defined in $[0, \infty)$ and satisfy

$$\int_0^\infty k_j(s)\,\mathrm{d}s = \kappa_j, \qquad \int_0^\infty s k_j(s)\,\mathrm{d}s < \infty, \qquad j \in \Lambda,$$

where κ_j, $j \in \Lambda$ are some positive constants.

Definition 2.1 ([32]). *Assume that $\Omega \in \mathbb{R}^n$ is a bounded and open set, $\mathscr{F}(u) : \Omega \to \mathbb{R}^n$ is a continuous and differentiable function. If $p \overline{\in} \mathscr{F}(\partial\Omega)$ and $J_{\mathscr{F}}(u) \neq 0$ for any $u \in \mathscr{F}^{-1}(p)$, where $J_{\mathscr{F}}$ denotes the Jacobian determinant relative to $\mathscr{F}$, then the topological degree relative to Ω and p is defined by*

$$\deg(\mathscr{F}, \Omega, p) = \begin{cases} \displaystyle\sum_{u \in \mathscr{F}^{-1}(p)} \operatorname{sgn} J_{\mathscr{F}}(u), & \mathscr{F}^{-1}(p) \neq \emptyset, \\ 0, & \mathscr{F}^{-1}(p) = \emptyset. \end{cases}$$

Remark 2.1. Generally speaking, the topological degree of $\mathscr{F}(u)$ relative to Ω and p can be regarded as the algebraic number of solution of $\mathscr{F}(u) = p$ in Ω if $\mathscr{F}(\partial\Omega) \neq 0$. For instance, $\deg(\mathscr{F}, \Omega, 0) = \pm 1$ implies that $\mathscr{F}(u) = 0$ has at least one solution in Ω.

Lemma 2.1 ([33]). *Given any real matrix $M = M^{\mathrm{T}} > 0$ of appropriate dimension, and a vector function $\omega(\cdot) : [a, b] \to \mathbb{R}^n$, such that the integrations concerned are well defined, then*

$$\left[\int_a^b \omega(s)\,\mathrm{d}s\right]^{\mathrm{T}} M \left[\int_a^b \omega(s)\,\mathrm{d}s\right] \leqslant (b - a) \int_a^b \omega^{\mathrm{T}}(s) M \omega(s)\,\mathrm{d}s.$$

3. Main results

First, we present a sufficient condition to guarantee the existence of equilibrium point for model (1).

Theorem 3.1. *Assume that assumptions (H_1) and (H_2) hold. Then model (1) has at least one equilibrium point if $C - [A + BK]^{+} L$ is an M-matrix, where $L = \operatorname{diag}(l_1, \ldots, l_n)$, $l_j = \max\{|l_j^-|, |l_j^+|\}$, $K = \operatorname{diag}(k_1, \ldots, k_n)$.*

Proof. The equilibrium point of model (1) is defined by the constant vector $x^* \in \mathbb{R}^n$, where x^* satisfies

$$-Cx^* + Af(x^*) + BKf(x^*) + J = 0,$$

which is equal to

$$x^* - C^{-1}(A + BK)f(x^*) - C^{-1}J = 0, \tag{2}$$

in view of $C > 0$. Clearly, now we only need to show that system (2) has at least one solution. For convenience, it can be rewritten as

$$h(x) = x - Wf(x) - J' = 0,$$

where $W = C^{-1}(A + BK)$, $J' = C^{-1}J$.

Consider the following homotopic mapping:

$$H(x, \lambda) = \lambda h(x) + (1 - \lambda)x, \qquad \lambda \in [0, 1].$$

Note that $C - [A + BK]^{+}L$ is an M-matrix; it can be deduced that $I - [W]^{+}L$ is also an M-matrix. This implies that $(I - [W]^{+}L)^{-1} \geqslant 0$ and there exists a positive vector $X_0 \in \mathbb{R}^n$ such that $(I - [W]^{+}L)X_0 > 0$. It then follows that

$$\begin{aligned}
[H(x,\lambda)]^{+} &= [\lambda h(x) + (1-\lambda)x]^{+} = [x - \lambda W f(x) - \lambda J']^{+} \\
&\geqslant [x]^{+} - \lambda[Wf(x)]^{+} - \lambda[J']^{+} \\
&\geqslant [x]^{+} - \lambda[W]^{+}[f(x)]^{+} - \lambda[J']^{+} \\
&\geqslant [x]^{+} - \lambda[W]^{+}L[x]^{+} - \lambda[J']^{+} \\
&\geqslant (1-\lambda)[x]^{+} + \lambda\{(I - [W]^{+}L)[x]^{+} - [J']^{+}\} \\
&\geqslant (1-\lambda)[x]^{+} + \lambda(I - [W]^{+}L)\{[x]^{+} - (I - [W]^{+}L)^{-1}[J']^{+}\}.
\end{aligned}$$

Let

$$\Omega = \{x \mid [x]^{+} \leqslant (I - [W]^{+}L)^{-1}[J']^{+} + X_0\},$$

then the set Ω is not empty and for any $x \in \partial\Omega$, we have

$$[H(x,\lambda)]^{+} \geqslant (1-\lambda)[x]^{+} + \lambda(I - [W]^{+}L)X_0 > 0, \qquad \lambda \in [0,1],$$

which implies that $H(x,\lambda) \neq 0$ for all $x \in \partial\Omega$ and $\lambda \in [0,1]$. By topological degree invariance theory, we obtain

$$\deg(h(x),\Omega,0) = \deg(H(x,\lambda),\Omega,0) = \deg(H(x,0),\Omega,0) = 1.$$

Therefore, from remark 2.1, we know that system (2) has at least one solution in Ω. This completes the proof. □

Remark 3.1. It should be noted that theorem 3.1 can guarantee the existence of the equilibrium point but not the uniqueness. In the following work, we shall derive some sufficient conditions to guarantee not only the global asymptotic stability but also the uniqueness of the equilibrium point.

Assume that x^* is an equilibrium point of model (1), then we can shift the equilibrium point x^* to the origin by a simple transformation $y(t) = x(t) - x^*$. Thus, model (1) can be rewritten as

$$\begin{cases}
\dot{y}(t) = -Cy(t-\sigma) + Ag(y(t-\tau(t))) & \\
\quad + B\displaystyle\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s + D\dot{y}(t-h(t)), & t > 0, \\
y(s) = \varphi(s) - x^*, & s \in [-\rho, 0],
\end{cases} \tag{3}$$

where $g(y(\cdot)) = f(y(\cdot) + x^*) - f(x^*)$. Moreover, it has an equivalent form as follows:

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}t}\left[y(t) - C\displaystyle\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right] = -Cy(t) + Ag(y(t-\tau(t))) & \\
\quad + B\displaystyle\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s + D\dot{y}(t-h(t)), & t > 0, \\
y(s) = \varphi(s) - x^*, & s \in [-\rho, 0].
\end{cases} \tag{4}$$

Then we have the following global asymptotical stability result.

Theorem 3.2. *Under the conditions in theorem 3.1, model (1) has a unique equilibrium point which is globally asymptotically stable if there exist two $n \times n$ inverse matrices*

Q_1, Q_2, *three* $n \times n$ *matrices* $P > 0, Q_3 > 0, Q_4 > 0$, *four* $n \times n$ *diagonal matrices* $U_1 > 0, U_2 > 0, Q_5 > 0, Q_6 > 0$ *and a* $2n \times 2n$ *matrix* $\begin{pmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{pmatrix} > 0$ *such that*

$$\begin{pmatrix}
\Pi_{11} & 0 & PD & T_{12}^{\mathrm{T}} & 0 & CPC & U_1\Sigma_2 & PA & PB \\
\star & \Pi_{22} & Q_1D + Q_2^{\mathrm{T}}D & 0 & -Q_1C & 0 & 0 & Q_1A & Q_1B \\
\star & \star & \Pi_{33} & 0 & D^{\mathrm{T}}Q_2C & -D^{\mathrm{T}}PC & 0 & -D^{\mathrm{T}}Q_2A & -D^{\mathrm{T}}Q_2B \\
\star & \star & \star & \Pi_{44} & 0 & 0 & 0 & U_2\Sigma_2 & 0 \\
\star & \star & \star & \star & -Q_3 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -Q_4 & 0 & -CPA & -CPB \\
\star & \star & \star & \star & \star & \star & Q_5\mathcal{K} - U_1 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -U_2 & 0 \\
\star & \star & \star & \star & \star & \star & \star & 0 & -Q_5
\end{pmatrix} < 0, \tag{5}$$

where

$$\Pi_{11} = -PC - CP + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1,$$
$$\Pi_{22} = \tau T_{22} + Q_6 - Q_1 - Q_1^{\mathrm{T}},$$
$$\Pi_{33} = -Q_6(1 - h^*) - D^{\mathrm{T}}Q_2D - D^{\mathrm{T}}Q_2^{\mathrm{T}}D,$$
$$\Pi_{44} = \tau T_{11} - T_{12} - T_{12}^{\mathrm{T}} - U_2\Sigma_1,$$
$$\mathcal{K} = \mathrm{diag}(\kappa_1^2, \kappa_2^2, \ldots, \kappa_n^2),$$
$$\Sigma_1 = \mathrm{diag}(l_1^- l_1^+, \ldots, l_n^- l_n^+),$$
$$\Sigma_2 = \mathrm{diag}\left(\frac{l_1^- + l_1^+}{2}, \ldots, \frac{l_n^- + l_n^+}{2}\right).$$

Proof. Construct a Lyapunov–Krasovskii functional in the form of

$$V(t, y(t)) = V_1(t, y(t)) + V_2(t, y(t)) + V_3(t, y(t)) + V_4(t, y(t)) + V_5(t, y(t)) + V_6(t, y(t)), \tag{6}$$

where

$$V_1(t, y(t)) = \left[y(t) - C\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} P \left[y(t) - C\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right],$$
$$V_2(t, y(t)) = \int_0^t \int_{u-\tau(u)}^{u} \begin{bmatrix} y(u - \tau(u)) \\ \dot{y}(s) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{bmatrix} \begin{bmatrix} y(u - \tau(u)) \\ \dot{y}(s) \end{bmatrix} \mathrm{d}s\,\mathrm{d}u,$$
$$V_3(t, y(t)) = \int_{-\tau}^{0}\int_{t+u}^{t} \dot{y}^{\mathrm{T}}(s)T_{22}\dot{y}(s)\,\mathrm{d}s\,\mathrm{d}u,$$
$$V_4(t, y(t)) = \int_{t-\sigma}^{t} y^{\mathrm{T}}(s)Q_3y(s)\,\mathrm{d}s + \int_{t-h(t)}^{t} \dot{y}^{\mathrm{T}}(s)Q_6\dot{y}(s)\,\mathrm{d}s,$$
$$V_5(t, y(t)) = \sigma\int_{t-\sigma}^{t}\int_{s}^{t} y^{\mathrm{T}}(u)Q_4y(u)\,\mathrm{d}u\,\mathrm{d}s,$$
$$V_6(t, y(t)) = \sum_{j=1}^{n} q_j^{(5)}\kappa_j \int_0^{\infty} k_j(u)\int_{t-u}^{t} g_j^2(y_j(s))\,\mathrm{d}s\,\mathrm{d}u,$$

where $Q_5 = \mathrm{diag}(q_1^{(5)}, q_2^{(5)}, \ldots, q_n^{(5)}) > 0$.

For the sake of brevity, we denote $V(t) = V(t, y(t))$ and $V_i(t) = V_i(t, y(t)), i = 1, \ldots, 6$. Then calculating the time derivative of V along the solution of (3) or (4), we have

$$
\begin{aligned}
\dot{V}_1(t) = 2&\left[y(t) - C\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} P\left[-Cy(t) + Ag(y(t-\tau(t)))\right. \\
&\left. + B\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s + D\dot{y}(t-h(t))\right] \\
= {}&-2y^{\mathrm{T}}(t)PCy(t) + 2y^{\mathrm{T}}(t)PAg(y(t-\tau(t))) \\
&+ 2y^{\mathrm{T}}(t)PB\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s \\
&+ 2y^{\mathrm{T}}(t)PD\dot{y}(t-h(t)) + 2y^{\mathrm{T}}(t)CPC\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u \\
&- 2\left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} CPAg(y(t-\tau(t))) \\
&- 2\left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} CPB\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s \\
&- 2\left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} CPD\dot{y}(t-h(t)),
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\dot{V}_2(t) = {}&\int_{t-\tau(t)}^{t} \begin{bmatrix} y(t-\tau(t)) \\ \dot{y}(s) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{bmatrix} \begin{bmatrix} y(t-\tau(t)) \\ \dot{y}(s) \end{bmatrix} \mathrm{d}s \\
= {}&\tau(t)y^{\mathrm{T}}(t-\tau(t))T_{11}y(t-\tau(t)) + 2y^{\mathrm{T}}(t)T_{12}^{\mathrm{T}}y(t-\tau(t)) \\
&- 2y^{\mathrm{T}}(t-\tau(t))T_{12}^{\mathrm{T}}y(t-\tau(t)) + \int_{t-\tau(t)}^{t} \dot{y}^{\mathrm{T}}(s)T_{22}\dot{y}(s)\,\mathrm{d}s \\
\leqslant {}& y^{\mathrm{T}}(t-\tau(t))[\tau T_{11} - 2T_{12}^{\mathrm{T}}]y(t-\tau(t)) + 2y^{\mathrm{T}}(t)T_{12}^{\mathrm{T}}y(t-\tau(t)) \\
&+ \int_{t-\tau}^{t} \dot{y}^{\mathrm{T}}(s)T_{22}\dot{y}(s)\,\mathrm{d}s,
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\dot{V}_3(t) &= \tau\dot{y}^{\mathrm{T}}(t)T_{22}\dot{y}(t) - \int_{-\tau}^{0} \dot{y}^{\mathrm{T}}(t+u)T_{22}\dot{y}(t+u)\,\mathrm{d}u \\
&= \tau\dot{y}^{\mathrm{T}}(t)T_{22}\dot{y}(t) - \int_{t-\tau}^{t} \dot{y}^{\mathrm{T}}(s)T_{22}\dot{y}(s)\,\mathrm{d}s,
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\dot{V}_4(t) = {}& y^{\mathrm{T}}(t)Q_3y(t) - y^{\mathrm{T}}(t-\sigma)Q_3y(t-\sigma) + \dot{y}^{\mathrm{T}}(t)Q_6\dot{y}(t) \\
&- \dot{y}^{\mathrm{T}}(t-h(t))Q_6\dot{y}(t-h(t))(1-\dot{h}(t)) \\
\leqslant {}& y^{\mathrm{T}}(t)Q_3y(t) - y^{\mathrm{T}}(t-\sigma)Q_3y(t-\sigma) + \dot{y}^{\mathrm{T}}(t)Q_6\dot{y}(t) \\
&- \dot{y}^{\mathrm{T}}(t-h(t))Q_6\dot{y}(t-h(t))(1-h^*).
\end{aligned}
\tag{10}
$$

It follows from lemma 2.1 that

$$\dot{V}_5(t) = \sigma^2 y^{\mathrm{T}}(t) Q_4 y(t) - \sigma \int_{t-\sigma}^{t} y^{\mathrm{T}}(u) Q_4 y(u)\,\mathrm{d}u$$
$$\leqslant \sigma^2 y^{\mathrm{T}}(t) Q_4 y(t) - \left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} Q_4 \left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]. \tag{11}$$

By the well-known Cauchy–Schwarz inequality, we know

$$\dot{V}_6(t) = \sum_{j=1}^{n} q_j^{(5)} \kappa_j \int_0^{\infty} k_j(u) g_j^2(y_j(t))\,\mathrm{d}u - \sum_{j=1}^{n} q_j^{(5)} \kappa_j \int_0^{\infty} k_j(u) g_j^2(y_j(t-u))\,\mathrm{d}u$$
$$\leqslant g^{\mathrm{T}}(y(t)) Q_5 \mathcal{K} g(y(t)) - \sum_{j=1}^{n} q_j^{(5)} \int_0^{\infty} k_j(u)\,\mathrm{d}u \int_0^{\infty} k_j(u) g_j^2(y_j(t-u))\,\mathrm{d}u$$
$$\leqslant g^{\mathrm{T}}(y(t)) Q_5 \mathcal{K} g(y(t)) - \sum_{j=1}^{n} q_j^{(5)} \left(\int_0^{\infty} k_j(u) g_j(y_j(t-u))\,\mathrm{d}u\right)^2$$
$$= g^{\mathrm{T}}(y(t)) Q_5 \mathcal{K} g(y(t))$$
$$- \left(\int_{-\infty}^{t} K(t-s) g(y(s))\,\mathrm{d}s\right)^{\mathrm{T}} Q_5 \left(\int_{-\infty}^{t} K(t-s) g(y(s))\,\mathrm{d}s\right). \tag{12}$$

In addition, we note that

$$0 = 2\dot{y}^{\mathrm{T}}(t) Q_1 \{-\dot{y}^{\mathrm{T}}(t) + \dot{y}^{\mathrm{T}}(t)\}$$
$$= 2\dot{y}^{\mathrm{T}}(t) Q_1 \bigg\{ - \dot{y}^{\mathrm{T}}(t) - C y(t-\sigma) + A g(y(t-\tau(t))) + B \int_{-\infty}^{t} K(t-s) g(y(s))\,\mathrm{d}s$$
$$+ D\dot{y}(t-h(t)) \bigg\}$$
$$= -2\dot{y}^{\mathrm{T}}(t) Q_1 \dot{y}^{\mathrm{T}}(t) - 2\dot{y}^{\mathrm{T}}(t) Q_1 C y(t-\sigma) + 2\dot{y}^{\mathrm{T}}(t) Q_1 A g(y(t-\tau(t)))$$
$$+ 2\dot{y}^{\mathrm{T}}(t) Q_1 B \int_{-\infty}^{t} K(t-s) g(y(s))\,\mathrm{d}s + 2\dot{y}^{\mathrm{T}}(t) Q_1 D \dot{y}(t-h(t)) \tag{13}$$

and

$$0 = 2\dot{y}^{\mathrm{T}}(t-h(t)) D^{\mathrm{T}} Q_2 \{-D\dot{y}^{\mathrm{T}}(t-h(t)) + D\dot{y}^{\mathrm{T}}(t-h(t))\}$$
$$= 2\dot{y}^{\mathrm{T}}(t-h(t)) D^{\mathrm{T}} Q_2 \bigg\{ - D\dot{y}^{\mathrm{T}}(t-h(t)) + \dot{y}(t) + C y(t-\sigma) - A g(y(t-\tau(t)))$$
$$- B \int_{-\infty}^{t} K(t-s) g(y(s))\,\mathrm{d}s \bigg\}$$
$$= -2\dot{y}^{\mathrm{T}}(t-h(t)) D^{\mathrm{T}} Q_2 D \dot{y}^{\mathrm{T}}(t-h(t)) + 2\dot{y}^{\mathrm{T}}(t) Q_2^{\mathrm{T}} D \dot{y}(t-h(t))$$
$$+ 2\dot{y}^{\mathrm{T}}(t-h(t)) D^{\mathrm{T}} Q_2 C y(t-\sigma) - 2\dot{y}^{\mathrm{T}}(t-h(t)) D^{\mathrm{T}} Q_2 A g(y(t-\tau(t)))$$
$$- 2\dot{y}^{\mathrm{T}}(t-h(t)) D^{\mathrm{T}} Q_2 B \int_{-\infty}^{t} K(t-s) g(y(s))\,\mathrm{d}s. \tag{14}$$

Moreover, for any $n \times n$ diagonal matrices $U_1 > 0$, $U_2 > 0$, the following inequality holds by the methods in [31]:

$$0 \leqslant \left\{ \begin{bmatrix} y(t) \\ g(y(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -U_1\Sigma_1 & U_1\Sigma_2 \\ \star & -U_1 \end{bmatrix} \begin{bmatrix} y(t) \\ g(y(t)) \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} y(t-\tau(t)) \\ g(y(t-\tau(t))) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -U_2\Sigma_1 & U_2\Sigma_2 \\ \star & -U_2 \end{bmatrix} \cdot \begin{bmatrix} y(t-\tau(t)) \\ g(y(t-\tau(t))) \end{bmatrix} \right\}. \tag{15}$$

Utilizing relations (7)–(15), we get

$$\begin{aligned}
\dot{V}(t) \leqslant\ & y^{\mathrm{T}}(t)[-2PC + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1]y(t) + 2y^{\mathrm{T}}(t)PD\dot{y}(t-h(t)) \\
& + 2y^{\mathrm{T}}(t)T_{12}^{\mathrm{T}}y(t-\tau(t)) \\
& + 2y^{\mathrm{T}}(t)U_1\Sigma_2 g(y(t)) + 2y^{\mathrm{T}}(t)PAg(y(t-\tau(t))) + 2y^{\mathrm{T}}(t)CPC\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u \\
& + 2y^{\mathrm{T}}(t)PB\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s + \dot{y}^{\mathrm{T}}(t)[\tau T_{22} + Q_6 - 2Q_1]\dot{y}(t) \\
& + 2\dot{y}^{\mathrm{T}}(t)[Q_1 D + Q_2^{\mathrm{T}} D]\dot{y}(t-h(t)) - 2\dot{y}^{\mathrm{T}}(t)Q_1 Cy(t-\sigma) \\
& + 2\dot{y}^{\mathrm{T}}(t)Q_1 Ag(y(t-\tau(t))) + 2\dot{y}^{\mathrm{T}}(t)Q_1 B\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s \\
& \times\ \dot{y}^{\mathrm{T}}(t-h(t))[-Q_6(1-h^*) - 2D^{\mathrm{T}}Q_2 D]\dot{y}(t-h(t)) \\
& + 2\dot{y}^{\mathrm{T}}(t-h(t))D^{\mathrm{T}}Q_2 Cy(t-\sigma) - 2\dot{y}^{\mathrm{T}}(t-h(t))D^{\mathrm{T}}PC\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u \\
& + 2\dot{y}^{\mathrm{T}}(t-h(t))D^{\mathrm{T}}Q_2 Cy(t-\sigma) - 2\dot{y}^{\mathrm{T}}(t-h(t))D^{\mathrm{T}}Q_2 Ag(y(t-\tau(t))) \\
& - 2\dot{y}^{\mathrm{T}}(t-h(t))D^{\mathrm{T}}Q_2 B\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s \\
& + y^{\mathrm{T}}(t-\tau(t))[\tau T_{11} - 2T_{12}^{\mathrm{T}} - U_2\Sigma_1]y(t-\tau(t)) + y^{\mathrm{T}}(t-\tau(t))U_2\Sigma_2 g(y(t-\tau(t))) \\
& - y^{\mathrm{T}}(t-\sigma)Q_3 y(t-\sigma) - \left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} Q_4 \left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right] \\
& - 2\left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} CPAg(y(t-\tau(t))) \\
& - 2\left[\int_{t-\sigma}^{t} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} CPB\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s \\
& + g^{\mathrm{T}}(y(t))[Q_5\mathcal{K} - U_1]g(y(t)) - g^{\mathrm{T}}(y(t-\tau(t)))U_2 g(y(t-\tau(t))) \\
& - \left(\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s\right)^{\mathrm{T}} Q_5 \left(\int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s\right) \\
=\ & \xi^{\mathrm{T}}(t)\Xi\xi(t),
\end{aligned}$$

where

$$\Xi = \begin{pmatrix} \Pi_{11} & 0 & PD & T_{12}^{\mathrm{T}} & 0 & CPC & U_1\Sigma_2 & PA & PB \\ \star & \Pi_{22} & Q_1D + Q_2^{\mathrm{T}}D & 0 & -Q_1C & 0 & 0 & Q_1A & Q_1B \\ \star & \star & \Pi_{33} & 0 & D^{\mathrm{T}}Q_2C & -D^{\mathrm{T}}PC & 0 & -D^{\mathrm{T}}Q_2A & -D^{\mathrm{T}}Q_2B \\ \star & \star & \star & \Pi_{44} & 0 & 0 & 0 & U_2\Sigma_2 & 0 \\ \star & \star & \star & \star & -Q_3 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -Q_4 & 0 & -CPA & -CPB \\ \star & \star & \star & \star & \star & \star & Q_5\mathcal{K} - U_1 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & 0 & -Q_5 \end{pmatrix},$$

$$\xi(t) = \bigg(y(t),\ \dot{y}(t),\ \dot{y}(t-h(t)),\ y(t-\tau(t)),\ y(t-\sigma),\ \int_{t-\sigma}^{t} y(s)\,\mathrm{d}s,\ g(y(t)),$$
$$g(y(t-\tau(t))), \int_{-\infty}^{t} K(t-s)g(y(s))\,\mathrm{d}s\bigg)^{\mathrm{T}}. \tag{16}$$

By (5), it yields

$$\dot{V}(t) \leqslant -\xi^{\mathrm{T}}(t)\Xi^*\xi(t), \qquad t > 0,$$

where $\Xi^* = -\Xi > 0$.

Thus, it can be deduced that

$$V(t) + \int_0^t \xi^{\mathrm{T}}(u)\Xi^*\xi(u)\,\mathrm{d}u \leqslant V(0) < \infty, \qquad t \geqslant 0, \tag{17}$$

where

$$\begin{aligned} V(0) \leqslant{}& \left[y(0) - C\int_{-\sigma}^{0} y(u)\,\mathrm{d}u\right]^{\mathrm{T}} P \left[y(0) - C\int_{-\sigma}^{0} y(u)\,\mathrm{d}u\right] \\ &+ \int_{-\tau}^{0}\int_{u}^{0} \dot{y}^{\mathrm{T}}(s)T_{22}\dot{y}(s)\,\mathrm{d}s\,\mathrm{d}u + \int_{-\sigma}^{0} y^{\mathrm{T}}(s)Q_3y(s)\,\mathrm{d}s + \int_{-h}^{0} \dot{y}^{\mathrm{T}}(s)Q_6\dot{y}(s)\,\mathrm{d}s \\ &+ \sigma\int_{-\sigma}^{0}\int_{s}^{0} y^{\mathrm{T}}(u)Q_4y(u)\,\mathrm{d}u\,\mathrm{d}s + \sum_{j=1}^{n} q_j^{(5)}\kappa_j \int_0^{\infty} k_j(u)\int_{-u}^{0} g_j^2(y_j(s))\,\mathrm{d}s\,\mathrm{d}u \\ \leqslant{}& \Big\{2\lambda_{\max}(P)(1 + \sigma^2 \max_{i\in\Lambda} c_i) + \tau^2\lambda_{\max}(T_{22}) + \sigma\lambda_{\max}(Q_3) \\ &+ h\lambda_{\max}(Q_6) + \sigma^3\lambda_{\max}(Q_4) + \sum_{j=1}^{n} q_j^{(5)}\kappa_j \max_{j\in\Lambda} l_j^2 \int_0^{\infty} uk_j(u)\,\mathrm{d}u\Big\}\|\varphi\|_h^2 < \infty. \end{aligned}$$

From the definition of $V_4(t)$ and lemma 2.1, we know

$$\begin{aligned} \left\|\int_{t-\sigma}^{t} y(s)\,\mathrm{d}s\right\|^2 &= \left[\int_{t-\sigma}^{t} y(s)\,\mathrm{d}s\right]^{\mathrm{T}} \left[\int_{t-\sigma}^{t} y(s)\,\mathrm{d}s\right] \leqslant \sigma\int_{t-\sigma}^{t} y^{\mathrm{T}}(s)y(s)\,\mathrm{d}s \\ &\leqslant \frac{\sigma}{\lambda_{\min}(Q_3)}\int_{t-\sigma}^{t} y^{\mathrm{T}}(s)Q_3y(s)\,\mathrm{d}s \\ &\leqslant \frac{\sigma}{\lambda_{\min}(Q_3)}V(t) \leqslant \frac{\sigma}{\lambda_{\min}(Q_3)}V(0), \end{aligned}$$

which together with the definition of $V_1(t)$ yields

$$\|y(t)\| \leqslant \left\|C\int_{t-\sigma}^{t} y(s)\,\mathrm{d}s\right\| + \sqrt{\frac{V_1(t)}{\lambda_{\min}(P)}} \leqslant \left\|C\int_{t-\sigma}^{t} y(s)\,\mathrm{d}s\right\| + \sqrt{\frac{V(0)}{\lambda_{\min}(P)}}$$
$$\leqslant \left\{\sqrt{\sum_{i=1}^{n} c_i \frac{\sigma}{\lambda_{\min}(Q_3)}} + \sqrt{\frac{1}{\lambda_{\min}(P)}}\right\}\sqrt{V(0)}.$$

This implies that the equilibrium point of model (1) is locally stable. Next we shall prove that $\|y(t)\| \to 0$ as $t \to \infty$.

First, for any constant $\theta \in [0, 1]$, it follows from (16), (17) and lemma 2.1 that

$$\begin{aligned}\|y(t+\theta) - y(t)\|^2 &= \left[\int_t^{t+\theta} \dot{y}(s)\,\mathrm{d}s\right]^{\mathrm{T}}\left[\int_t^{t+\theta} \dot{y}(s)\,\mathrm{d}s\right] \\ &\leqslant \theta\int_t^{t+\theta} \dot{y}^{\mathrm{T}}(s)\dot{y}(s)\,\mathrm{d}s \\ &\leqslant \int_t^{t+1} \dot{y}^{\mathrm{T}}(s)\dot{y}(s)\,\mathrm{d}s \\ &\leqslant \frac{1}{\lambda_{\min}(\Xi^*)}\int_t^{t+1} \xi^{\mathrm{T}}(s)\Xi^*\xi(s)\,\mathrm{d}s \to 0 \qquad \text{as } t \to \infty,\end{aligned}$$

which implies that for any $\varepsilon > 0$, there exists a $T_1 = T_1(\varepsilon) > 0$ such that

$$\|y(t+\theta) - y(t)\| < \frac{\varepsilon}{2}, \qquad t > T_1, \quad \theta \in [0, 1]. \tag{18}$$

On the other hand, from (17) we get

$$\begin{aligned}\left\|\int_t^{t+1} y(s)\,\mathrm{d}s\right\|^2 &= \left[\int_t^{t+1} y(s)\,\mathrm{d}s\right]^{\mathrm{T}}\left[\int_t^{t+1} y(s)\,\mathrm{d}s\right] \\ &\leqslant \int_t^{t+1} y^{\mathrm{T}}(s)y(s)\,\mathrm{d}s \\ &\leqslant \frac{1}{\lambda_{\min}(\Xi^*)}\int_t^{t+1} \xi^{\mathrm{T}}(s)\Xi^*\xi(s)\,\mathrm{d}s \to 0 \qquad \text{as } t \to \infty,\end{aligned}$$

which implies that for any $\varepsilon > 0$, there exists a $T_2 = T_2(\varepsilon) > 0$ such that

$$\left\|\int_t^{t+1} y(s)\,\mathrm{d}s\right\| < \frac{\varepsilon}{2}, \qquad t > T_2.$$

Note that $y(s)$ is continuous on $[t, t+1]$, $t > 0$. Applying the integral mean value theorem, we know there exists a vector $\zeta_t = (\zeta_{t1}, \ldots, \zeta_{tn})^{\mathrm{T}} \in \mathbb{R}^n$, $\zeta_{tj} \in [t, t+1]$, such that

$$\|y(\zeta_t)\| = \left\|\int_t^{t+1} y(s)\,\mathrm{d}s\right\| < \frac{\varepsilon}{2}, \qquad t > T_2. \tag{19}$$

By (18) and (19), we obtain that for any $\varepsilon > 0$, there exists a $T = \max\{T_1, T_2\} > 0$ such that $t > T$ implies

$$\|y(t)\| \leqslant \|y(t) - y(\zeta_t)\| + \|y(\zeta_t)\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, we can conclude that model (1) has a unique equilibrium point which is globally asymptotically stable. This completes the proof. □

Remark 3.2. The LMI criterion given in theorem 3.2 is dependent on the leakage delay, time-varying delays and continuously distributed delays. It is well known that the delay-dependent criterion is less conservative than the delay-independent criterion when the delay is small. In particular when $\sigma = 0$, model (1) becomes the well-known case which has been directly or indirectly investigated by many authors, for instance, see [22–30, 34, 35]. However, it always assumes that the activation functions are bounded, monotonic or differentiable and the time delays are constant delays or time-varying delays which are differentiable and their derivatives are bounded. By theorem 3.2, we have the following result which removes those undesirable restrictions.

Corollary 3.1. *Under the conditions in theorem 3.1, model* (1) *with* $\sigma = 0$ *has a unique equilibrium point which is globally asymptotically stable if there exist two* $n\times n$ *inverse matrices* Q_1, Q_2, *an* $n\times n$ *matrix* $P > 0$, *four* $n\times n$ *diagonal matrices* $U_1 > 0, U_2 > 0, Q_5 > 0, Q_6 > 0$ *and a* $2n \times 2n$ *matrix* $\begin{pmatrix} T_{11} & T_{12} \\ \star & T_{22}\end{pmatrix} > 0$ *such that*

$$\begin{pmatrix}
\Pi_{11} & 0 & PD & T_{12}^{\mathrm{T}} & U_1\Sigma_2 & PA & PB \\
\star & \Pi_{22} & Q_1D + Q_2^{\mathrm{T}}D & 0 & 0 & Q_1A & Q_1B \\
\star & \star & \Pi_{33} & 0 & 0 & -D^{\mathrm{T}}Q_2A & -D^{\mathrm{T}}Q_2B \\
\star & \star & \star & \Pi_{44} & 0 & U_2\Sigma_2 & 0 \\
\star & \star & \star & \star & Q_5\mathcal{K} - U_1 & 0 & 0 \\
\star & \star & \star & \star & \star & -U_2 & 0 \\
\star & \star & \star & \star & \star & 0 & -Q_5
\end{pmatrix} < 0,$$

where

$$\begin{aligned}
&\Pi_{11} = -PC - CP + Q_3 - U_1\Sigma_1, \\
&\Pi_{22} = \tau T_{22} + Q_6 - Q_1 - Q_1^{\mathrm{T}}, \\
&\Pi_{33} = -Q_6(1-h^*) - D^{\mathrm{T}}Q_2D - D^{\mathrm{T}}Q_2^{\mathrm{T}}D, \\
&\Pi_{44} = \tau T_{11} - T_{12} - T_{12}^{\mathrm{T}} - U_2\Sigma_1, \\
&\mathcal{K} = \mathrm{diag}(\kappa_1^2, \kappa_2^2, \ldots, \kappa_n^2), \\
&\Sigma_1 = \mathrm{diag}(l_1^- l_1^+, \ldots, l_n^- l_n^+), \\
&\Sigma_2 = \mathrm{diag}\left(\frac{l_1^- + l_1^+}{2}, \ldots, \frac{l_n^- + l_n^+}{2}\right).
\end{aligned}$$

When $D = 0$, *model (1) becomes*

$$\begin{cases}
\dot{x}(t) = -Cx(t-\sigma) + Af(x(t-\tau(t))) & \\
+ B\displaystyle\int_{-\infty}^{t} K(t-s)f(x(s))\,\mathrm{d}s + J, & t > 0, \\
x(s) = \varphi(s), & s \in (-\infty, 0],
\end{cases} \tag{20}$$

Then we have the following result.

Corollary 3.2. *Under the conditions in theorem 3.1, model* (20) *has a unique equilibrium point which is globally asymptotically stable if there exist an* $n \times n$ *inverse matrix* Q_1, *three* $n \times n$

matrices $P > 0$, $Q_3 > 0$, $Q_4 > 0$, *three* $n \times n$ *diagonal matrices* $U_1 > 0$, $U_2 > 0$, $Q_5 > 0$ *and a* $2n \times 2n$ *matrix* $\begin{pmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{pmatrix} > 0$ *such that*

$$\begin{pmatrix}
\Pi_{11} & 0 & T_{12}^{\mathrm{T}} & 0 & CPC & U_1\Sigma_2 & PA & PB \\
\star & \Pi_{22} & 0 & -Q_1C & 0 & 0 & Q_1A & Q_1B \\
\star & \star & \Pi_{44} & 0 & 0 & 0 & U_2\Sigma_2 & 0 \\
\star & \star & \star & -Q_3 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -Q_4 & 0 & -CPA & -CPB \\
\star & \star & \star & \star & \star & Q_5\mathcal{K} - U_1 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -U_2 & 0 \\
\star & \star & \star & \star & \star & \star & 0 & -Q_5
\end{pmatrix} < 0,$$

where

$$\begin{aligned}
\Pi_{11} &= -PC - CP + Q_3 + \sigma^2 Q_4 - U_1\Sigma_1, \\
\Pi_{22} &= \tau T_{22} - Q_1 - Q_1^{\mathrm{T}}, \\
\Pi_{44} &= \tau T_{11} - T_{12} - T_{12}^{\mathrm{T}} - U_2\Sigma_1, \\
\mathcal{K} &= \mathrm{diag}(\kappa_1^2, \kappa_2^2, \ldots, \kappa_n^2), \\
\Sigma_1 &= \mathrm{diag}(l_1^- l_1^+, \ldots, l_n^- l_n^+), \\
\Sigma_2 &= \mathrm{diag}\left(\frac{l_1^- + l_1^+}{2}, \ldots, \frac{l_n^- + l_n^+}{2}\right).
\end{aligned}$$

Remark 3.3. In this paper, the differentiability of time-varying delay $\tau(t)$ is removed successfully, which improves the results in [22, 25–30, 34, 35]. Unfortunately, it still requires the time-varying delay $h(t)$ in the neutral term is differentiable and its derivative is bounded. In the future, we will carry out some studies to overcome the shortage.

4. Numerical examples

In this section, we present two numerical examples to demonstrate the effectiveness of the proposed method.

Example 4.1. Consider a two-neuron neural networks model of neutral type with leakage delay:

$$\begin{cases}
\dot{x}(t) = -Cx(t-\sigma) + Af(x(t-\tau(t))) \\
\quad + B\displaystyle\int_{-\infty}^{t} K(t-s)f(x(s))\,\mathrm{d}s + D\dot{x}(t-h(t)) + J, & t > 0, \\
x(s) = \varphi(s), & s \in (-\infty, 0],
\end{cases} \tag{21}$$

where $f_1(s) = f_2(s) = \tanh(s)$, $k_1(s) = k_2(s) = \mathrm{e}^{-s}$, $\tau(t) = 0.2 - 0.1\cos t$, $\sigma = h(t) = 0.1$, $J = (1, 2)^{\mathrm{T}}$ and parameters C, A, B and D are given as follows:

$$C = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 0.6 & 0.3 \\ -0.5 & -0.8 \end{pmatrix}, \quad B = \begin{pmatrix} 0.7 & -0.2 \\ -0.2 & 0.5 \end{pmatrix},$$

$$D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

Table 1. The maximal allowable upper bounds of τ with different values of h^* and σ.

	σ				
h^*	0	0.05	0.1	0.15	0.2
0	0.3527	0.3422	0.3075	0.2370	0.0894
0.5	0.2850	0.2744	0.2399	0.1692	0.0203
0.8	0.1509	0.1404	0.1057	0.0343	—

It is easy to see that $\tau = 0.3, h^* = 0, l_j^- = 0, l_j^+ = 1, \kappa_j = 1, j = 1, 2$. Note that

$$C - [A + BK]^+ L = \begin{pmatrix} 2.7 & -0.1 \\ -0.7 & 3.7 \end{pmatrix} \text{ is an } M\text{-matrix.}$$

By theorem 3.1, we know that system (21) has a equilibrium point. Then via the MATLAB LMI toolbox, one can see that the LMI given in theorem 3.2 is feasible with the following solutions:

$$P = \begin{pmatrix} 548.7279 & 122.1073 \\ 122.1073 & 339.5536 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 44.7214 & 18.9438 \\ 18.9844 & 32.0665 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 75.8819 & 16.7414 \\ 13.6274 & 38.1192 \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 998.1184 & 370.3787 \\ 370.3787 & 668.6616 \end{pmatrix},$$

$$Q_4 = 10^4 \times \begin{pmatrix} 3.2432 & 0.8399 \\ 0.8399 & 2.7603 \end{pmatrix}, \qquad Q_5 = \begin{pmatrix} 391.8827 & 0 \\ 0 & 391.8827 \end{pmatrix},$$

$$Q_6 = \begin{pmatrix} 10.4651 & 3.3773 \\ 3.3773 & 7.3038 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 812.8138 & 0 \\ 0 & 812.8138 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 261.4367 & 0 \\ 0 & 261.4367 \end{pmatrix}, \qquad T_{11} = 10^3 \times \begin{pmatrix} 486.8372 & 299.6064 \\ 299.6064 & 503.9215 \end{pmatrix},$$

$$T_{12} = \begin{pmatrix} 197.5417 & 116.8963 \\ 95.0797 & 172.4107 \end{pmatrix}, \qquad T_{22} = \begin{pmatrix} 87.1718 & 44.8965 \\ 44.8965 & 67.8071 \end{pmatrix}.$$

Hence, from theorem 3.2, system (21) has a unique equilibrium point which is globally asymptotically stable.

Remark 4.1. In fact, when $\sigma = h = 0.1$ (i.e. $h^* = 0$), by the MATLAB LMI toolbox, the maximum allowable upper bounds of τ in example 4.1 satisfying the LMI in theorem 3.2 can be calculated as $\tau < 0.3075$. Moreover, one can obtain the maximum allowable upper bounds of τ with different values of h^* and σ, which are shown in table 1.

Example 4.2. Consider a three-neuron neural networks model of neutral type with leakage delay:

$$\begin{cases} \dot{x}(t) = -Cx(t-\sigma) + Af(x(t-\tau(t))) & \\ + B\displaystyle\int_{-\infty}^{t} K(t-s)f(x(s))\,\mathrm{d}s + D\dot{x}(t-h(t)) + J, & t > 0, \\ x(s) = \varphi(s), & s \in (-\infty, 0], \end{cases} \tag{22}$$

where $f_j(s) = |s|, k_j(s) = \mathrm{e}^{-s}, j = 1, 2, 3, \tau(t) = 0.09 - 0.01[\cos t]^*, \sigma = 0.05, h(t) = 0.2 - 0.1 \sin t, J = (-1, 1, 3)^{\mathrm{T}}$ and parameters C, A, B and D are given as follows:

$$C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} 0.3 & 0.2 & -0.1 \\ -0.2 & 0.1 & 0 \\ 0.4 & 0.7 & 0.2 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.1 & -0.32 & 0.4 \\ 0.1 & 0.1 & 0.1 \\ 0.55 & 0.8 & 0.34 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.08 & 0.1 & -0.02 \\ -0.1 & 0.02 & 0.1 \\ 0.2 & 0.08 & 0.02 \end{pmatrix}.$$

It is easy to see that $\tau = 0.1, h^* = 0.1, l_j^- = -1, l_j^+ = 1, \kappa_j = 1, j = 1, 2$. Note that

$$C - [A + BK]^+ L = \begin{pmatrix} 4.8 & -0.12 & -0.3 \\ -0.1 & 3.8 & -0.1 \\ -0.95 & -1.5 & 2.46 \end{pmatrix} \text{ is an } M\text{-matrix.}$$

By theorem 3.1, we know that system (22) has a equilibrium point. Then via the MATLAB LMI toolbox, one can see that the LMI given in theorem 3.2 is feasible with the following solutions:

$$P = \begin{pmatrix} 30.1311 & 3.7443 & 0.1663 \\ 3.7443 & 32.1185 & -0.0306 \\ 0.1663 & -0.0306 & 22.8015 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 3.0422 & 0.7996 & -0.0821 \\ 0.7847 & 3.8467 & -0.0986 \\ -0.0887 & -0.0992 & 1.0720 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 3.3722 & 0.8465 & 0.1215 \\ 0.7695 & 3.9364 & 0.0723 \\ 0.4983 & 0.3989 & 3.0136 \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 82.2154 & 16.9686 & 0.6060 \\ 16.9686 & 63.0709 & -0.1288 \\ 0.6060 & -0.1288 & 13.0611 \end{pmatrix},$$

$$Q_4 = 10^3 \times \begin{pmatrix} 3.8018 & 0.3561 & 0.0152 \\ 0.3561 & 2.1439 & -0.0040 \\ 0.0152 & -0.0040 & 0.7286 \end{pmatrix}, \qquad Q_5 = \begin{pmatrix} 26.5064 & 0 & 0 \\ 0 & 26.5064 & 0 \\ 0 & 0 & 26.5064 \end{pmatrix},$$

$$Q_6 = \begin{pmatrix} 2.1599 & 0.8117 & -0.0411 \\ 0.8117 & 1.5563 & 0.0128 \\ -0.0411 & 0.0128 & 0.2637 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 29.0933 & 0 & 0 \\ 0 & 29.0933 & 0 \\ 0 & 0 & 29.0933 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 15.4283 & 0 & 0 \\ 0 & 15.4283 & 0 \\ 0 & 0 & 15.4283 \end{pmatrix}, \qquad T_{11} = \begin{pmatrix} 100.0042 & -0.6855 & -1.9765 \\ -0.6855 & 124.2743 & -4.7513 \\ -1.9765 & -4.7513 & 62.8956 \end{pmatrix},$$

$$T_{12} = \begin{pmatrix} 22.3245 & -0.1488 & -1.3364 \\ -0.2034 & 27.4341 & -1.3801 \\ -1.2899 & -1.1650 & 16.7021 \end{pmatrix}, \qquad T_{22} = \begin{pmatrix} 7.7892 & -0.0598 & -1.3016 \\ -0.0598 & 12.4886 & -1.1067 \\ -1.3016 & -1.1067 & 5.1436 \end{pmatrix}.$$

Hence, from theorem 3.2, system (22) has a unique equilibrium point which is globally asymptotically stable.

Table 2. The maximal allowable upper bounds of τ with different values of h^* and σ.

	σ			
h^*	0	0.05	0.1	0.15
0	0.1821	0.1731	0.1474	0.0939
0.3	0.1601	0.1505	0.1229	0.0649
0.6	0.1058	0.0947	0.0630	—

Remark 4.2. Note that $\sigma = 0.05$, $h^* = 0.1$ in example 4.2, by the MATLAB LMI toolbox, the maximum allowable upper bounds of τ satisfying the LMI in theorem 3.2 can be calculated as $\tau < 0.1672$. Moreover, the maximum allowable upper bounds of τ with different values of h^* and σ can also be obtained, which are shown in table 2.

Remark 4.3. It is obvious that the results in [22–30, 34, 35] are ineffective for system (22) even with $\sigma = 0$, since the activation functions are unbounded and the time-varying delay $\tau(t)$ is a piecewise function, which shows the advantage of the proposed method in this paper. However, from tables 1 and 2, one may observe that the maximum allowable upper bounds of τ are small with different values of h^* and σ, which shows the limited adaptive ranges of the development results in this paper. So there is still some room for us to develop and explore. In the future, we will do some further studies on this problem.

5. Conclusions

In this paper, we have dealt with the global asymptotic stability of neural networks of neutral type with mixed delays. The mixed delays include leakage constant delay, time-varying delays and continuously distributed delays. Based on the topological degree theory, Lyapunov method and LMI approach, some LMI-based conditions ensuring the existence, uniqueness and global asymptotic stability of the equilibrium point have been presented, which are dependent on both the discrete and distributed time delays and can be easily checked by MATLAB LMI toolbox. Even there is no leakage delay, the obtained results are also less restrictive than some recent works since the assumptions on boundedness, monotonicity or differentiability of the activation functions are removed. Two numerical examples have been illustrated to demonstrate the usefulness of the proposed method.

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