Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 339 (2008) 996–1002
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Strong convergence of the composite Halpern iteration

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Received 16 November 2005
Available online 1 August 2007
Submitted by D. Khavinson

Abstract

Let C be a closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping with a nonempty fixed points set. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $(0, 1)$, the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\alpha_n \to 0$, $\beta_n \to 0$ and $0 < a \leqslant \gamma_n$, for some $a \in (0, 1)$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a composite iteration process defined by

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n) T x_n, \\ y_n = \beta_n x_n + (1 - \beta_n) T z_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases}$$

then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Keywords: Iteration scheme; Sunny and nonexpansive retraction; Nonexpansive mapping; Banach space

1. Introduction and preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \text{for all } x, y \in C.$$

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doi:10.1016/j.jmaa.2007.07.062

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is $F(T) = \{x \in C\colon\ Tx = x\}$. It is assumed throughout that T is a nonexpansive mapping such that $F(T) \neq \emptyset$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [3,11]. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C,$$

where $u \in C$ is a arbitrary (but fixed) point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of $\{x_t\}$ as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [3] proved that, if E is a uniformly smooth Banach space, then $\{x_t\}$ converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

In 1967, Halpern [6] first introduced the following iteration scheme:

$$\begin{cases} x_0 = x \in C, & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \end{cases} \tag{1.1}$$

see also Browder [2]. He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that, if the iteration scheme (1.1) converges to a fixed point of T, then these conditions must be satisfied. Ten years later, Lions [8] investigated the general case in Hilbert space under the conditions

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{(\alpha_n - \alpha_{n+1})^2}{\alpha_{n+1}} = 0$$

on the parameters. However, Lions' conditions on the parameters were more restrictive and did not include the natural candidate $\{\alpha_n\} = \frac{1}{n}$. In 1980, Reich [11] gave the iteration scheme (1.1) in the case when E is uniformly smooth and $\{\alpha_n\} = n^{-\delta}$ with $0 < \delta < 1$.

In 1992, Wittmann [12] studied the iteration scheme (1.1) in the case when E is a Hilbert space and $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

In 1994, Reich [10] obtained a strong convergence of the iterates (1.1) with two necessary and decreasing conditions on parameters for convergence in the case when E is uniformly smooth with a weakly continuous duality mapping.

Recently Chang [4] studied the iteration scheme (1.1) in the case when E is a uniformly smooth Banach space and $\{\alpha_n\}$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \|Tx_n - x_n\| \to 0,$$

then $\{x_n\}$ converges strongly to a fixed point of T.

This paper introduces the composite iteration scheme as follows:

$$\begin{cases} z_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.2}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, that $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T.

If $\{\beta_n\} = 0$ and $\{\gamma_n\} = 1$ in (1.2) then we have the usual Halpern iterative sequence $\{x_n\}$ defined by (1.1).

On the other hand, the composite iterations this paper introduced is a modified Ishikawa iteration. If $\{\gamma_n\} = 1$ in (1.2) then (1.2) can be viewed as a modified Mann iteration

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.3}$$

which was considered by Kim and Xu [7].

It is our purpose in this paper to introduce this composite iteration scheme for approximating a fixed point of nonexpansive mappings in the framework of uniformly smooth Banach spaces. We establish strong convergence of the composite iteration scheme $\{x_n\}$ defined by (1.2). The results improve and extend results of Chang [4], Wittmann [12], Kim and Xu [7] and many others.

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{ f \in E^*\colon \langle x, f\rangle = \|x\|^2 = \|f\|^2 \}, \quad x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. The norm of E is said to be Gâteaux differentiable (and E is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.4}$$

exists for each x, y in its unit sphere $U = \{x \in E\colon \|x\| = 1\}$. A Banach space E whose norm is uniformly Gâteaux differentiable; then the duality map J is single-valued and norm-to-weak* uniformly continuous on bounded sets of E. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth) if the limit in (1.4) is attained uniformly for $(x, y) \in U \times U$.

We need the following lemmas for the proof of our main results.

Lemma 1.1. *A Banach space E is uniformly smooth if and only if the duality map J is single-valued and norm-to-norm uniformly continuous on bounded sets of E.*

In our convergence results in the next sections, we need to estimate the square-norm $\|x_{n+1} - p\|^2$ in terms of the square-norm $\|x_n - p\|^2$, where x_i is the ith iterate for $i \geqslant 1$, and p is a fixed point of the mapping T. To do this, we need the following well-known (subdifferential) inequality:

Lemma 1.2. *In a Banach space E, there holds the inequality*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad x, y \in E,$$

where $j(x+y) \in J(x+y)$.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty, closed, convex and $D \subset C$, then a map $Q : C \to D$ is sunny [1,9] provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geqslant 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [1,5,9]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leqslant 0, \quad \text{for all } x \in C,\ y \in D.$$

Reich [11] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C onto D and it can be constructed as follows:

Lemma 1.3. *(See Reich [11].) Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property*

$$\langle u - Qu, J(z - Qu)\rangle \leqslant 0, \quad u \in C,\ z \in F(T).$$

Lemma 1.4. *(See Xu [13,14].) Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leqslant (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geqslant 0,$$

where $\{\gamma\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma\}_{n=0}^{\infty}$ are such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,
(ii) *either* $\limsup_{n\to\infty} \sigma_n \leqslant 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n \sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ *converges to zero.*

2. Main results

Theorem 2.1. *Let C be a closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0, 1]$, the following conditions are satisfied*:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\alpha_n \to 0$, $\beta_n \to 0$ *and* $0 < a \leqslant \gamma_n$, *for some* $a \in (0, 1)$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be composite process defined by* (1.2), *then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Proof. First we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, if we take a fixed point p of T, noting that

$$\|z_n - p\| \leqslant \gamma_n \|x_n - p\| + (1 - \gamma_n)\|Tx_n - p\| \leqslant \|x_n - p\|. \tag{2.1}$$

It follows from (1.2) and (2.1) that

$$\begin{aligned} \|y_n - p\| &\leqslant \beta_n \|x_n - p\| + (1 - \beta_n)\|Tz_n - p\| \\ &\leqslant \beta_n \|x_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leqslant \|x_n - p\|, \end{aligned}$$

which yields that

$$\begin{aligned} \|x_{n+1} - p\| &\leqslant \alpha_n \|u - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\leqslant \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leqslant \max\bigl\{\|u - p\|, \|x_n - p\|\bigr\}. \end{aligned}$$

Now, by simple induction yields

$$\|x_n - p\| \leqslant \max\bigl\{\|u - p\|, \|x_0 - p\|\bigr\}, \quad n \geqslant 0. \tag{2.2}$$

This implies that $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\|x_{n+1} - x_n\| \to 0. \tag{2.3}$$

In order to prove (2.3) from (1.2), after some manipulations we have

$$\begin{aligned} x_{n+1} - x_n = {} & (1 - \alpha_n)(1 - \beta_n)(Tz_n - Tz_{n-1}) + (1 - \alpha_n)\beta_n(x_n - x_{n-1}) \\ & + \bigl[(\beta_n - \beta_{n-1})(1 - \alpha_n) - (\alpha_n - \alpha_{n-1})\beta_{n-1}\bigr](x_{n-1} - Tz_{n-1}) \\ & + (\alpha_n - \alpha_{n-1})(u - Tz_{n-1}). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - x_n\| \leqslant {} & (1 - \beta_n)(1 - \alpha_n)\|Tz_n - Tz_{n-1}\| + (1 - \alpha_n)\beta_n \|x_n - x_{n-1}\| \\ & + \bigl|(\beta_n - \beta_{n-1})(1 - \alpha_n) - (\alpha_n - \alpha_{n-1})\beta_{n-1}\bigr|\|x_{n-1} - Tz_{n-1}\| \\ & + |\alpha_n - \alpha_{n-1}|\|u - Tz_{n-1}\|. \end{aligned} \tag{2.4}$$

That is

$$\begin{aligned}\|x_{n+1}-x_n\| \leqslant{}& (1-\beta_n)(1-\alpha_n)\|z_n-z_{n-1}\|+(1-\alpha_n)\beta_n\|x_n-x_{n-1}\| \\ &+\big|(\beta_n-\beta_{n-1})(1-\alpha_n)-(\alpha_n-\alpha_{n-1})\beta_{n-1}\big|\|x_{n-1}-Tz_{n-1}\| \\ &+|\alpha_n-\alpha_{n-1}|\|u-Tz_{n-1}\|. \end{aligned} \tag{2.5}$$

Since

$$z_n-z_{n-1}=(1-\gamma_n)(Tx_n-Tx_{n-1})+\gamma_n(x_n-x_{n-1})+(\gamma_{n-1}-\gamma_n)(Tx_{n-1}-x_{n-1}),$$

we have

$$\|z_n-z_{n-1}\| \leqslant \|x_n-x_{n-1}\|+|\gamma_{n-1}-\gamma_n|\|Tx_{n-1}-x_{n-1}\|. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$\begin{aligned}\|x_{n+1}-x_n\| \leqslant{}& (1-\alpha_n)\|x_n-x_{n-1}\|+(1-\alpha_n)(1-\beta_n)|\gamma_n-\gamma_{n-1}|\|Tx_{n-1}-x_{n-1}\| \\ &+\big|(\beta_n-\beta_{n-1})(1-\alpha_n)-(\alpha_n-\alpha_{n-1})\beta_{n-1}\big|\|x_{n-1}-Tz_{n-1}\| \\ &+|\alpha_n-\alpha_{n-1}|\|u-Tz_{n-1}\|, \end{aligned} \tag{2.7}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x_n\| \leqslant{}& (1-\alpha_n)\|x_n-x_{n-1}\| \\ &+M_1\big(|\gamma_n-\gamma_{n-1}|+|\beta_n-\beta_{n-1}|+2|\alpha_n-\alpha_{n-1}|\big), \end{aligned} \tag{2.8}$$

where M_1 is a constant such that

$$M_1 \geqslant \max\big\{\|u-Tz_{n-1}\|, \|x_{n-1}-Tx_{n-1}\|, \|x_{n-1}-Tz_{n-1}\|\big\}$$

for all n. By assumptions (i)–(iii), we have that

$$\lim_{n\to\infty}\alpha_n=0, \qquad \sum_{n=1}^{\infty}\alpha_n=\infty,$$

and

$$\sum_{n=1}^{\infty}\big(|\beta_n-\beta_{n-1}|+2|\alpha_n-\alpha_{n-1}|+|\gamma_n-\gamma_{n-1}|\big)<\infty.$$

Hence, Lemma 1.4 is applicable to (2.8) and we obtain

$$\|x_{n+1}-x_n\| \to 0 \quad \text{as } n\to\infty. \tag{2.9}$$

On the other hand, from condition (ii), we have

$$\|x_{n+1}-y_n\|=\alpha_n\|u-y\| \to 0 \quad \text{as } n\to\infty, \tag{2.10}$$

and

$$\|y_n-Tz_n\|=\beta_n\|x_n-Tz_n\| \to 0 \quad \text{as } n\to\infty. \tag{2.11}$$

Again, it follows from (1.2) and the fact that T is nonexpansive that

$$\begin{aligned}\|Tx_n-x_n\| &\leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-Tz_n\|+\|Tz_n-Tx_n\| \\ &\leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-Tz_n\|+(1-\gamma_n)\|Tx_n-x_n\|. \end{aligned} \tag{2.12}$$

It follows that

$$\gamma_n\|Tx_n-x_n\| \leqslant \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\|y_n-Tz_n\|.$$

From condition (ii) and (2.9)–(2.11) we have

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \tag{2.13}$$

Next, we claim that

$$\limsup_{n\to\infty}\big\langle u-q, J(x_n-q)\big\rangle \leqslant 0, \tag{2.14}$$

where $q=Qu=\lim_{t\to 0} z_t$ with z_t being the fixed point of the contraction $z\mapsto tu+(1-t)Tz$. First, z_t solves the fixed point equation

$$z_t = tu+(1-t)Tz_t.$$

Therefore, we have

$$\|z_t-x_n\| = \big\|(1-t)(Tz_t-x_n)+t(u-x_n)\big\|.$$

It follows from Lemma 1.2, the nonexpansive property of T, and the definition of J that

$$\begin{aligned}\|z_t-x_n\|^2 &\leqslant (1-t)^2\|Tz_t-x_n\|^2+2t\big\langle u-x_n, J(z_t-x_n)\big\rangle\\ &\leqslant \big(1-2t+t^2\big)\|z_t-x_n\|^2+f_n(t)+2t\big\langle u-z_t, J(z_t-x_n)\big\rangle+2t\|z_t-x_n\|^2,\end{aligned} \tag{2.15}$$

where

$$f_n(t)=\big(2\|z_t-x_n\|+\|x_n-Tx_n\|\big)\|x_n-Tx_n\|\to 0 \quad \text{as } n\to 0. \tag{2.16}$$

It follows that

$$\big\langle z_t-u, J(z_t-x_n)\big\rangle \leqslant \frac{t}{2}\|z_t-x_n\|^2+\frac{1}{2t}f_n(t). \tag{2.17}$$

Letting $n\to\infty$ in (2.17) and noting (2.16) yields

$$\limsup_{n\to\infty}\big\langle z_t-u, J(z_t-x_n)\big\rangle \leqslant \frac{t}{2}M_2, \tag{2.18}$$

where $M_2>0$ is a constant such that $M_2\geqslant \|z_t-x_n\|^2$ for all $t\in(0,1)$ and $n\geqslant 1$. Letting $t\to 0$ from (2.18), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\big\langle z_t-u, J(z_t-x_n)\big\rangle \leqslant 0.$$

So, for any $\epsilon>0$, there exists a positive number δ_1, when $t\in(0,\delta_1)$ we get

$$\limsup_{n\to\infty}\big\langle z_t-u, J(z_t-x_n)\big\rangle \leqslant \frac{\epsilon}{2}. \tag{2.19}$$

On the other hand, $z_t\to q$ as $t\to 0$, from Lemma 1.1, $\exists\delta_2>0$, such that when $t\in(0,\delta_2)$ we have

$$\begin{aligned}&\big|\big\langle u-q, J(x_n-q)\big\rangle-\big\langle z_t-u, J(z_t-x_n)\big\rangle\big|\\ &\quad\leqslant \big|\big\langle u-q, J(x_n-q)\big\rangle-\big\langle u-q, J(x_n-z_t)\big\rangle\big|+\big|\big\langle u-q, J(x_n-z_t)\big\rangle-\big\langle z_t-u, J(z_t-x_n)\big\rangle\big|\\ &\quad\leqslant \big|\big\langle u-q, J(x_n-q)-J(x_n-z_t)\big\rangle\big|+\big\langle z_t-q, J(x_n-z_t)\big\rangle \leqslant \frac{\epsilon}{2}.\end{aligned}$$

Choosing $\delta=\min\{\delta_1,\delta_2\}$; $\forall t\in(0,\delta)$, we have

$$\big\langle u-q, J(x_n-q)\big\rangle \leqslant \big\langle z_t-u, J(z_t-x_n)\big\rangle+\frac{\epsilon}{2},$$

which yields that

$$\limsup_{n\to\infty}\big\langle u-q, J(x_n-q)\big\rangle \leqslant \lim_{n\to\infty}\big\langle z_t-u, J(z_t-x_n)\big\rangle+\frac{\epsilon}{2}.$$

It follows from (2.19) that

$$\limsup_{n\to\infty}\big\langle u-q, J(x_n-q)\big\rangle \leqslant \epsilon.$$

Since ϵ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\big\langle u-q, J(x_n-q)\big\rangle \leqslant 0. \tag{2.20}$$

Finally, we show that $x_n \to q$ strongly and this concludes the proof. Indeed, using Lemma 1.2 again we obtain

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \big\|(1-\alpha_n)(y_n - q) + \alpha_n(u - q)\big\|^2 \\ &\leqslant (1-\alpha_n)^2\|y_n - q\|^2 + 2\alpha_n\big\langle u - q, J(x_{n+1} - q)\big\rangle \\ &\leqslant (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\big\langle u - q, J(x_{n+1} - q)\big\rangle.\end{aligned}$$

Now we use (2.20) and apply Lemma 4 to see that $\|x_n - q\| \to 0$. □

As a corollary of Theorem 2.1, we have the following immediately:

Corollary 2.2. *(See Kim and Xu [7].) Let C be a closed convex subset of a uniformly smooth Banach space E and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0, 1]$, the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $\alpha_n \to 0$, $\beta_n \to 0$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Let $\{x_n\}_{n=1}^{\infty}$ be a composite process defined by (1.3)*, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.*

Remark. Kim and Xu [7] proved the sequence defined by iteration scheme (1.3) converges to fixed point of T, under the conditions

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $\alpha_n \to 0$, $\beta_n \to 0$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ on the parameters.

Actually, the condition $\sum_{n=0}^{\infty} \beta_n = \infty$ can be removed, and the condition $\beta_n \to 0$ also can be relaxed to $0 < \beta_n \leqslant a < 1$, for some $a \in (0, 1)$.

Acknowledgments

The authors are extremely grateful to the referees for useful suggestions that improved the content of the paper.

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