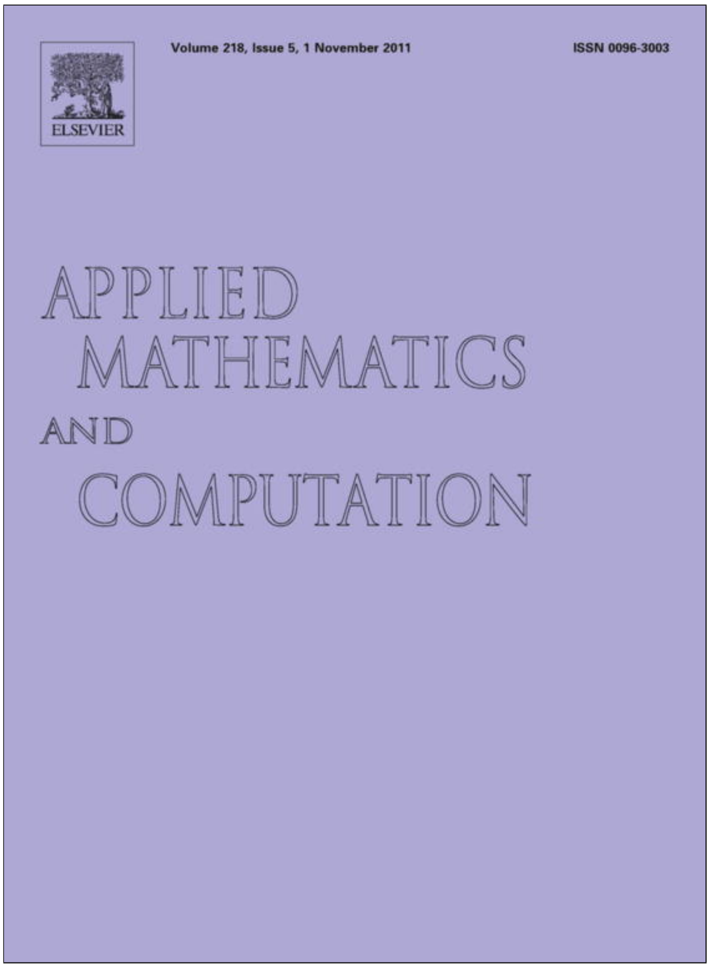
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Multiplicity results of periodic solutions for systems of first order delay differential equation ☆

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ABSTRACT

This paper is concerned with the periodic solutions of the following delay non-autonomous systems

$$u'(t) = -f(t, u(t-r)), \tag{1}$$

where $r > 0$, $f \in C(\mathbf{R^1} \times \mathbf{R^n}, \mathbf{R^n})$ satisfies $f(t+r, z) = f(t, z)$ for all $z \in \mathbf{R^n}$. Some multiplicity results of periodic solutions of (1) are obtained via variational methods.

1. Introduction and preliminaries

In this paper, we consider the multiplicity problems of periodic solutions for the following non-autonomous delay systems

$$u'(t) = -f(t, u(t-r)) \tag{1.1}$$

via variational methods, where $r > 0$, $f \in C(\mathbf{R^1} \times \mathbf{R^n}, \mathbf{R^n})$ satisfies $f(t+r, z) = f(t, z)$ for all $z \in \mathbf{R^n}$.

For autonomous delay differential equations dealing with scalar, the existence of the periodic solutions has been extensively studied in the past years via fixed point theory and some other techniques, for example, see [1–7]. It is not our purpose to give a survey in this paper. We only mention some related work here. In 2005, Guo and Yu [8] took the lead in using the variational approaches to study the existence of multiple periodic solutions for (1.1), and a multiplicity result was given by using a pseudo-index theory. Up to now, to the authors' knowledge, there is not any other existence and multiplicity results of periodic solutions for (1.1) dealing with variational approaches. In the present paper, our main purpose is to study the multiplicity of periodic orbits for the systems (1.1) via some recent critical point theorems for strongly indefinite functionals.

Now, we give some preliminaries. Let X and Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. Let $\mathcal{S} \subset X^*$ be a dense subset. For each $s \in \mathcal{S}$, there is a semi-norm on E defined by

$$p_s : E \to \mathbf{R^1}, \quad p_s(u) = |s(x)| + \|y\| \quad \text{for } u = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_{\mathcal{S}}$ the topology on E induced by semi-norm family $\{p_s\}$, and let w and w^* denote the weak-topology and weak*-topology, respectively.

For a functional $\Phi \in C^1(E, \mathbf{R^1})$ we write $\Phi_a = \{u \in E : \Phi(u) \geqslant a\}$. Recall that Φ' is said to be weak sequentially continuous if for any $u_k \rightharpoonup u$ in E, one has $\lim_{k\to\infty} \Phi'(u_k)v \to \Phi'(u)v$ for each $v \in E$, i.e. $\Phi' : (E, w) \to (E^*, w^*)$ is sequentially continuous. For

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$c \in \mathbf{R}^1$ we say that Φ satisfies the $(C)_c$ condition if any sequence $\{u_k\} \subset E$ such that $\Phi(u_k) \to c$ and $(1 + \|u_k\|)\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence. Similarly, we say that Φ satisfies the $(PS)_c$ condition if any sequence $\{u_k\} \subset E$ such that $\Phi(u_k) \to c$ and $\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.

Suppose that

(Φ_0) for any $c \in \mathbf{R}^1$, Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed, and $\Phi' : (\Phi_c, \mathcal{T}_{\mathcal{S}}) \to (E^*, w^*)$ is continuous,

(Φ_1) there exists a $\rho > 0$ such that $\kappa := \inf \Phi(\partial B_\rho \bigcap Y) > 0$, where

$$B_\rho = \{u \in E : \|u\| < \rho\},$$

(Φ_2) there exists a finite dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \backslash S_0) < \inf \Phi(B_\rho \bigcap Y)$, where

$$E_0 := X \oplus Y_0, \quad \text{and} \quad S_0 = \{u \in E_0 : \|u\| \leqslant R\}.$$

(Φ_3) there exists an increasing sequence of finite dimensional subspaces $Y_n \subset Y$ and there exist $R_n > \rho$ such that $\sup \Phi(E_n) < \infty$ and $\sup \Phi(E_n \backslash S_n) < \inf \Phi(B_\rho \bigcap Y)$, where $E_n := X \oplus Y_n$, $S_n = \{u \in E_n : \|u\| \leqslant R_n\}$.

Theorem 1.1 [9]. *Assume that Φ is even and (Φ_0)–(Φ_2) are satisfied. Then Φ has at least $m = \dim Y_0$ pairs of critical points with critical values less than or equal to $\bar{c}$ provided Φ satisfies the $(C)_c$ condition for all $c \in [\kappa, \bar{c}]$.*

Theorem 1.2 [10]. *Assume that Φ is even and (Φ_0), (Φ_1) and (Φ_3) are satisfied. Then Φ has an unbounded sequence of critical values provided Φ satisfies the $(PS)_c$ condition for every $c \in (0, \infty)$.*

In our applications we take $\mathcal{S} = X^*$, so that $\mathcal{T}_{\mathcal{S}}$ is the product topology on $E = X \oplus Y$ given by the weak topology on X and the strong topology on Y. Moreover, we need the following lemma which can be found in [10,11].

Lemma 1.1. *Suppose $\Phi \in C^1(E, \mathbf{R}^1)$ be the form*

$$\Phi(u) = \frac{1}{2}(\|y\|^2 - \|x\|^2) - \Psi(u) \quad \text{for } u = x + y \in E = X \oplus Y$$

such that

(i) *$\Psi \in C^1(E, \mathbf{R}^1)$ is bounded from below,*
(ii) *$\Psi : (E, w) \to \mathbf{R}^1$ is sequentially lower semicontinuous, that is, $u_k \to u$ in (E, w) implies*

$$\Psi(u) \leqslant \liminf_k \Psi(u_k),$$

(iii) *Ψ': $(E, w) \to (E^*, w^*)$ is sequentially continuous,*
(iv) *v:$E \to \mathbf{R}^1$, $v(u) = \|u\|^2$ is C^1 and v': $(E, w) \to (E^*, w^*)$ is sequentially continuous.Then Φ satisfies Φ_0.*

2. The variational set

First of all, one can easily find that (1.1) can be transformed to the equation

$$u'(t) = -\lambda f\left(\lambda t, u\left(t - \frac{\pi}{2}\right)\right) \tag{2.1}$$

by making the change of variable $t \mapsto \frac{\pi}{2r} t = \lambda^{-1} t$. This implies that a $4r$-periodic solution of (1.1) corresponds to a 2π-periodic solution of (2.1). Hence we will only seek for the 2π-periodic orbits of (2.1) in the sequel.

Throughout this paper, we always assume that

(f_1) $f(t,z)$ is odd in z, i.e. $f(t, -z) = -f(t,z)$ for all $t \in [0, r]$,
(f_2) there exists a continuously differentiable function $F(t,z) \in C^1(\mathbf{R}^1 \times \mathbf{R}^n, \mathbf{R}^1)$ such that $\nabla_z F(t,z) = f(t,z)$ for all $(t,z) \in \mathbf{R}^1 \times \mathbf{R}^n$,
(f_3) one of the following conditions holds:
(I) there exists a symmetric matrix $B = (b_{ij})_{n \times n}$ such that $\lim_{|z| \to \infty} \frac{|f(t,z) - Bz|}{|z|} = 0$ uniformly for all $t \in [0, r]$.
(II) there exist constants $a > 0$ and $p > 2$ such that

$$|f(t,z)| \leqslant a\left(1 + |z|^{p-1}\right)$$

for all $(t,z) \in [0, r] \times \mathbf{R}^n$.

Similar to the treatment in [8], we introduce the following variational set. Let $L^2(S^1,\mathbf{R^n})$ be the space of square integrable 2π periodic vector-valued functions with dimension n. Let $C^\infty(S^1,\mathbf{R^n})$ be the space of 2π-periodic C^∞ vector-valued functions with dimension n. For any $u\in C^\infty(S^1,\mathbf{R^n})$, it has the following Fourier expansion in the sense that it is convergent in the space $L^2(S^1,\mathbf{R^n})$

$$u(t)=\frac{a_0^u}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{j=1}^{+\infty}(a_j^u\cos jt+b_j^u\sin jt),$$

where $a_0^u,a_j^u,b_j^u\in\mathbf{R^n}$. Let $H=H^{\frac12}(S^1,\mathbf{R^n})$ be the closure of $C^\infty(S^1,\mathbf{R^n})$ with respect to the Hilbert norm

$$\|u\|_{H^{\frac12}}=\left[|a_0^u|^2+\sum_{j=1}^{+\infty}(1+j)\left(|a_j^u|^2+|b_j^u|^2\right)\right]^{\frac12}.$$

More specifically, $H^{\frac12}(S^1,\mathbf{R^n})=\left\{u\in L^2(S^1,\mathbf{R^n}):\|u\|_{H^{\frac12}}<+\infty\right\}$ with the inner product

$$\langle u,v\rangle_H=(a_0^u,a_0^v)+\sum_{j=1}^{+\infty}(1+j)\left[(a_j^u,a_j^v)+(b_j^u,b_j^v)\right]$$

for any $u,\ v\in H^{\frac12}(S^1,\mathbf{R^n})$, where $(\cdot,\cdot)$ denotes the usual inner product in $\mathbf{R^n}$. The norm on H is defined by

$$\|u\|_H=\left[|a_0^u|^2+\sum_{j=1}^{+\infty}(1+j)\left(|a_j^u|^2+|b_j^u|^2\right)\right]^{\frac12}.$$

By Proposition 6.6 in [12] we know that H is compactly embedded in $L^s(S^1,\mathbf{R^n})$, where $s\in[1,\infty)$.

Now consider a functional I defined on H, given by

$$I(u)=\int_0^{2\pi}\left[\frac12\left(\dot u\left(t+\frac{\pi}{2}\right),u(t)\right)+\lambda F(\lambda t,u(t))\right]dt \tag{2.2}$$

for any $u\in H$, where $\dot u(t)$ denotes the weak derivative of u. We define an operator $L:H\to H^*$ as follows: for any $u\in H$, which is given by

$$Lu(v)=\int_0^{2\pi}\left(\dot u\left(t+\frac{\pi}{2}\right),v(t)\right)dt$$

for all $v\in H$, where H^* denotes the dual space of H. By the Riesz representation theorem, we can identify H^* with H. Thus, Lu can also be viewed as a function belonging to H such that $\langle Lu,v\rangle_H=Lu(v)$ for any $u,v\in H$.

For any $u\in H$, define a bounded linear operator $\zeta:H\to H$ as follows: $\zeta u(\cdot)=u\left(\cdot+\frac{\pi}{2}\right)$. Set $E=\{u\in H:\zeta^2u=-u\}$. Then E is a closed subspace of H and invariant with respect to L. It is easy to check that L is a bounded linear operator on H. Moreover, $L|_E$ is self-adjoint.

Let $e_1,e_2,\ldots,e_n$ denote the usual normal orthogonal bases in $\mathbf{R^n}$. Define the subspaces E^+ and E^- of E as follows:

$$E^+=\overline{\mathrm{span}\{e_k\cos(2j-1)t,e_k\sin(2j-1)t:j\in Z^+,j\text{ is even},\ k=1,2,\ldots,n\}},$$
$$E^-=\overline{\mathrm{span}\{e_k\cos(2j-1)t,e_k\sin(2j-1)t:j\in Z^+,j\text{ is odd},\ k=1,2,\ldots,n\}},$$

where Z^+ is the set of all positive integers. By using the definition of E and a Fourier series argument, we see that $E=E^+\oplus E^-$. Moreover, for any $u\in E^+$, it has a Fourier expansion as follows:

$$u(t)=\frac{1}{\sqrt{\pi}}\sum_{j=1}^{+\infty}\left[a_{4j-1}^u\cos(4j-1)t+b_{4j-1}^u\sin(4j-1)t\right].$$

Thus,

$$\langle Lu,u\rangle_H=\int_0^{2\pi}(\dot u(t+\frac{\pi}{2}),u(t))dt=\sum_{j=1}^{+\infty}(4j-1)\left(|a_{4j-1}^u|^2+|b_{4j-1}^u|^2\right)\geqslant\frac12\sum_{j=1}^{+\infty}4j\left(|a_{4j-1}^u|^2+|b_{4j-1}^u|^2\right)=\frac12\|u\|_H^2.$$

Similarly, $\langle Lu,u\rangle_H\leqslant-\frac12\|u\|_H^2$ for any $u\in E^-$. Then we can define an equivalent norm $\|\cdot\|$ on E given by

$$\|u\|^2=\langle Lu^+,u^+\rangle_H-\langle Lu^-,u^-\rangle_H$$

for $u=u^++u^-\in E^+\oplus E^-$. Denote by $\langle\cdot,\cdot\rangle$ the inner product corresponding to $\|\cdot\|$ on E. Clearly, the spaces E^+ and E^- are mutually orthogonal with respect to the inner products $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{L^2}$ by the orthogonality of trigonometric functions, where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the usual inner product on $L^2(S^1,\mathbf{R^n})$.

Let

$$G(u)=\int_0^{2\pi}\lambda F(\lambda t,u(t))dt$$

for any $u \in H$. Then $I(u)$ can be rewritten as

$$I(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + G(u) \tag{2.3}$$

for $u = u^+ + u^- \in E$.

Lemma 2.1. *G is weakly sequentially continuous on H under the assumption (f_3).*

Proof. Since $f(t,z)$ is r-periodic in t, by (f_3), there are constants $c_1, c_2 > 0$ such that

$$|f(t,z)| \leqslant c_1 + c_2|z|^{p-1} \tag{2.4}$$

for all $(t,z) \in \mathbf{R^1} \times \mathbf{R^n}$. Let $\{u_k\}$ be any sequence converging to some u weakly in H. By the compactness of embedding, one has $u_k \to u$ in $L^p(S^1, \mathbf{R^n})$. By Hölder inequality we have

$$\begin{aligned}
|G(u_k) - G(u)| &= \left|\lambda \int_0^{2\pi} F(\lambda t, u_k) - F(\lambda t, u)dt\right| = \left|\lambda \int_0^{2\pi}\int_0^1 (f(\lambda t, u + s(u_k - u)), u_k - u)dsdt\right| \\
&\leqslant \lambda \int_0^{2\pi}\int_0^1 \left(c_1 + c_2|u + s(u_k - u)|^{p-1}\right)|u_k - u|dsdt \\
&\leqslant \lambda \int_0^{2\pi} c_1|u_k - u|dt + \lambda \int_0^{2\pi} c_2(|u| + |u_k - u|)^{p-1}|u_k - u|dt \\
&\leqslant \lambda \int_0^{2\pi} c_1|u_k - u|dt + \lambda \int_0^{2\pi} 2^{p-1}c_2\left(|u|^{p-1} + |u_k - u|^{p-1}\right)|u_k - u|dt \\
&\leqslant c_3\left(1 + \|u\|_{L^p}^{p-1} + \|u_k - u\|_{L^p}^{p-1}\right)\|u_k - u\|_{L^p},
\end{aligned}$$

where $c_3 > 0$ is a constant. This implies $G(u_k) \to G(u)$. The proof is completed. □

By Proposition B.37 in [12], we have the following lemma.

Lemma 2.2. *Assume that f satisfies (f_2) and (f_3). Then the functional I is continuously differentiable on H and $I'(u)$ is defined by*

$$I'(u)v = \int_0^{2\pi}\left[\frac{1}{2}\left(\dot{u}\left(t + \frac{\pi}{2}\right) - \dot{u}(t - \frac{\pi}{2}), v(t)\right) + \lambda(f(\lambda t, u(t)), v(t))\right]dt$$

for all $v \in H$. In particular,

$$I'(u)v = \int_0^{2\pi}\left[\left(\dot{u}\left(t + \frac{\pi}{2}\right), v(t)\right) + \lambda(f(\lambda t, u(t)), v(t))\right]dt$$

for all $u, v \in E$.

Moreover, $G' : H \to H^*$ is a compact mapping and

$$G'(u)v = \int_0^{2\pi} \lambda(f(\lambda t, u(t)), v(t))dt,$$

for any $v \in H$.

By the Riesz theorem, we can view $G'(u)$ as an element of H for any $u \in H$. In addition, one can easily prove that E is invariant with respect to G' under condition (f_1)(see [8]). As usual, we identify $u \in H$ with its continuous representant.

Since E is invariant with respect to L and G', an argument as in [8] yields.

Lemma 2.3. *Assume that f satisfies (f_1), (f_2) and (f_3). Then a critical point of functional I restricted to E is a 2π-periodic solution of system (2.1).*

Remark 2.1. It is pointed in [8] that a critical point u of I in H is a weak solution of (2.1). However, a simple regularity argument shows that $u \in C^1(S^1, \mathbf{R^n})$ (see the proof of Theorem 6.10 in [12]).

Remark 2.2. As usual, we should deal with (2.2) in the space H. But, according to Lemma 2.3, we only need to treat the functional I in the subspace E of H. From now on we will view I as $I|_E$.

3. Main results

In this section we denote by Z^+ the set of all positive integers; c_i stand for different positive constants for $i \in Z^+$.

The following hypotheses will be used in our main results.

(f_4) $\lim_{|z|\to 0}\frac{|F(t,z)|}{|z|^2}=0$ uniformly for $t\in[0,r]$,
(f_5) $(Bz,z)>\frac{\pi}{2r}|z|^2$ for all $z\in\mathbf{R^n}\backslash\{0\}$,
(f_6) for any positive integer j, $(-1)^{j+1}\frac{(2j-1)\pi}{2r}\notin\sigma(B)$, where $\sigma(B)$ is the set of all eigenvalues of B; B is the $n\times n$ symmetric matrix appearing in (f_3)(I).

Define $m=\max\left\{j\in Z^+ : (4j-3)|z|^2<\frac{2r}{\pi}(Bz,z) \text{ for } z\neq 0\right\}$. Then we have the following main result.

Theorem 3.1. *Assume that f satisfies (f_1), (f_2), (f_3)(I) and (f_4)–(f_6). Then (1.1) possesses at least 2mn pairs of 4r-periodic classical solutions.*

Proof. We will show that $\Phi(u)=-I(u)$ satisfies all hypotheses of Theorem 1.1. The proof of this theorem will be divided into several parts.

Step 1. We prove that Φ satisfies (Φ_0). Let $X=E^+$, $Y=E^-$ and $\Psi(u)=G(u)$. Then

$$\Phi(u)=\frac{1}{2}(\|y\|^2-\|x\|^2)-\Psi(u)\quad \text{for } u=x+y\in X\oplus Y,$$

and $\Psi(u)\in C^1(E,\mathbf{R^1})$ satisfies (ii) of Lemma 1.1 by Lemma 2.1.

Let $\{u_k\}$ be any sequence converging to u weakly in E. For $1\leqslant r<\infty$, since the injection of E into $L^r(S^1,\mathbf{R^n})$ is continuous, the sequence $\{u_k\}$ converges to u weakly in $L^r(S^1,\mathbf{R^n})$. Hence, in $L^r(S^1,\mathbf{R^n})$, any convergent subsequence of $\{u_k\}$ converges to u, and hence

$$\int_0^{2\pi}|u|^r dt=\liminf_{k\to\infty}\int_0^{2\pi}|u_k|^r dt\leqslant\limsup_{k\to\infty}\int_0^{2\pi}|u_k|^r dt=\int_0^{2\pi}|u|^r dt.$$

It follows that $u_k\to u$ in $L^r(S^1,\mathbf{R^n})$ and $u_k\to u$ a.e. on $[0,2\pi]$. Thus, for every $v\in E$ we get that $(f(\lambda t,u_k(t)),v(t))\to(f(\lambda t,u(t)),v(t))$ a.e. for $t\in[0,2\pi]$. Moreover, by (2.4), one has

$$\left|\int_0^{2\pi}(f(\lambda t,u_k(t)),v(t))dt\right|\leqslant\int_0^{2\pi}\left[c_1|v|+c_2|u_k|^{p-1}|v|\right]dt\leqslant c_1\|v\|_{L^1}+c_2\|u_k\|_{L^p}^{(p-1)}\|v\|_{L^p}\to c_1\|v\|_{L^1}+c_2\|u\|_{L^p}^{(p-1)}\|v\|_{L^p}.$$

Thus, the Vitali theorem is applicable

$$\int_0^{2\pi}(f(\lambda t,u_k(t)),v(t))dt\to\int_0^{2\pi}(f(\lambda t,u(t)),v(t))dt,$$

that is, $\Psi'(u_k)v\to\Psi'(u)v$ for any $v\in E$. Hence Ψ satisfies (iii) of Lemma 1.1. Moreover, note that E is a Hilbert space. (iv) of Lemma 1.1 holds, obviously.

It remains to prove that Ψ is bounded from below. Notice that $f\in C(\mathbf{R^1}\times\mathbf{R^n},\mathbf{R^n})$ and $f(t,z)$ is r-periodic in t. Hence (f_3)(I) implies that there exists a constant $c>0$ such that

$$|f(t,z)-Bz|\leqslant\frac{1}{2\lambda}(|z|+c)$$

for all $(t,z)\in\mathbf{R^1}\times\mathbf{R^n}$, where $\lambda=\frac{2r}{\pi}$. Clearly, it can be deduced from (f_4) that $F(t,0)=0$. Consequently, by (f_5) and the above inequality, one has

$$\begin{aligned}\Psi(u)&=\int_0^{2\pi}\lambda F(\lambda t,u)dt=\int_0^{2\pi}\int_0^1\lambda(f(\lambda t,su),u)dsdt=\frac{1}{2}\int_0^{2\pi}\lambda(Bu,u)dt+\int_0^{2\pi}\int_0^1\lambda(f(\lambda t,su)-sBu,u)dsdt\\&\geqslant\frac{1}{2}\int_0^{2\pi}|u|^2dt-\frac{1}{2}\int_0^{2\pi}\int_0^1(s|u|+c)|u|dsdt\geqslant\frac{1}{4}\|u\|_{L^2}^2-c_3\|u\|_{L^2},\end{aligned}$$

which implies that Ψ is bounded from below. By virtue of Lemma 1.1, Φ satisfies (Φ_0).

Step 2. Φ satisfies (Φ_1). Indeed, for any $\varepsilon>0$, by (2.4) and (f_4), there is a $c=c(\varepsilon)>0$ such that

$$|F(t,z)|\leqslant\varepsilon|z|^2+c|z|^p$$

for all $(t,z)\in\mathbf{R^1}\times\mathbf{R^n}$. Hence, for $u\in Y$ and small ε, we have

$$\Phi(u)=\frac{1}{2}\|u\|^2-\int_0^{2\pi}\lambda F(\lambda t,u)dt\geqslant\frac{1}{2}\|u\|^2-\lambda\varepsilon\|u\|_{L^2}^2-\lambda c\|u\|_{L^p}^p\geqslant\frac{1}{4}\|u\|^2-c_4\|u\|^p.$$

Since $p>2$, there is a small $\rho>0$ such that $\frac{1}{8}\rho^2\geqslant c_4\rho^p$. Therefore,

$$\kappa:=\inf\Phi\left(\partial B_\rho\bigcap Y\right)\geqslant\frac{1}{8}\rho^2>0 \tag{3.1}$$

and hence (Φ_1) holds.

Step 3. (Φ_2) is satisfied under the hypotheses of Theorem 3.1. Let

$$Y_0 = \operatorname{span}\{e_k\cos(4j-3)t, e_k\sin(4j-3)t : j \in Z^+, j \leqslant m, k = 1,2,\ldots,n\}.$$

Obviously, $Y_0 \subset Y$ and $\dim Y_0 = 2mn$. In order to obtain the desired conclusion, it is sufficient to prove that $\Phi(u) \to -\infty$ as $u \in E_0$ and $\|u\| \to \infty$. By the definition of m, there exists a $\delta (0 < \delta \leqslant 1)$ such that

$$(4m-3+\delta)|z|^2 \leqslant \lambda(Bz,z) \tag{3.2}$$

for all $z \in \mathbf{R^n}\setminus\{0\}$. Notice that

$$(4m-3)\|y\|_{L^2}^2 \geqslant \|y\|^2 \tag{3.3}$$

for any $y \in Y_0$. Let $\widetilde{F}(t,z) = F(\lambda t, z) - \frac{1}{2}(Bz,z)$. Then for $u = x + y \in E_0$, by (3.2) and (3.3), one has

$$\begin{aligned}\Phi(u) &= \frac{1}{2}(\|y\|^2 - \|x\|^2) - \int_0^{2\pi}\lambda F(\lambda t,u)dt \leqslant \frac{1}{2}(4m-3)\|y\|_{L^2}^2 - \frac{1}{2}\|x\|^2 - \frac{1}{2}\int_0^{2\pi}\lambda(Bu,u)dt - \int_0^{2\pi}\lambda\widetilde{F}(t,u)dt \\ &\leqslant \frac{1}{2}(4m-3)\|y\|_{L^2}^2 - \frac{1}{2}\|x\|^2 - \frac{1}{2}(4m-3+\delta)\|u\|_{L^2}^2 - \int_0^{2\pi}\lambda\widetilde{F}(t,u)dt \\ &\leqslant \frac{1}{2}(4m-3)\|y\|_{L^2}^2 - \frac{1}{2}\|x\|^2 - \frac{1}{2}(4m-3+\delta)\|y\|_{L^2}^2 - \int_0^{2\pi}\lambda\widetilde{F}(t,u)dt \leqslant -\frac{\delta}{8m-6}\|y\|^2 - \frac{1}{2}\|x\|^2 - \int_0^{2\pi}\lambda\widetilde{F}(t,u)dt \\ &\leqslant -\frac{\delta}{8m-6}\|u\|^2 - \int_0^{2\pi}\lambda\widetilde{F}(t,u)dt.\end{aligned}$$

It remains to show that

$$\frac{1}{\|u\|^2}\int_0^{2\pi}\widetilde{F}(t,u)dt \to 0 \tag{3.4}$$

as $\|u\| \to \infty$. For any $\varepsilon > 0$, by the continuity of f, (f_3)(I) and the periodicity of $f(\cdot,z)$ we know that there exists a positive constant $c = c(\varepsilon)$ such that

$$|f(t,z) - Bz| \leqslant \varepsilon|z| + c \tag{3.5}$$

for $(t,z) \in \mathbf{R^1} \times \mathbf{R^n}$. Thus, for $u \in E_0$ with $\|u\| \neq 0$ we have

$$\begin{aligned}\left|\frac{1}{\|u\|^2}\int_0^{2\pi}\widetilde{F}(t,u)dt\right| &= \frac{1}{\|u\|^2}\left|\int_0^{2\pi}\int_0^1 (f(\lambda t,su) - B(su),u)dsdt\right| \leqslant \frac{1}{\|u\|^2}\int_0^{2\pi}\int_0^1(\varepsilon|su|+c)|u|dsdt \\ &\leqslant \frac{1}{\|u\|^2}\left(\varepsilon\|u\|_{L^2}^2 + c\|u\|_{L^1}\right) \leqslant c_5\left(\varepsilon + \frac{c}{\|u\|}\right),\end{aligned}$$

which implies that (3.4) is true by the arbitrariness of ε. Hence (Φ_2) holds.

Step 4. Φ satisfies the $(C)_c$ condition for any $c \in \mathbf{R^1}$. Let $\{u_k\} \subset E$ be any sequence such that

$$\Phi(u_k) \to c, \ \ (1+\|u_k\|)\Phi'(u_k) \to 0 \tag{3.6}$$

as $k \to \infty$. We claim that $\{u_k\}$ is bounded in E. Assume by contradiction that $\|u_k\| \to \infty$ as $k \to \infty$. Let $\varphi_k = \frac{u_k}{\|u_k\|}$, then $\|\varphi_k\| = 1$. Without loss of generality, we can assume that $\varphi_k \rightharpoonup \varphi$ in E and $\varphi_k \to \varphi$ in $L^2(S^1,\mathbf{R^n})$. Hence for each $v \in E$, by (3.5) and Hölder inequality, one has

$$\begin{aligned}\left|\frac{1}{\|u_k\|}\int_0^{2\pi}(f(\lambda t,u_k),v)dt - \int_0^{2\pi}(B\varphi,v)dt\right| &\leqslant \left|\frac{1}{\|u_k\|}\int_0^{2\pi}(f(\lambda t,u_k) - Bu_k,v)dt\right| + \left|\int_0^{2\pi}(B\varphi_k - B\varphi,v)dt\right| \\ &\leqslant \frac{1}{\|u_k\|}\int_0^{2\pi}(\varepsilon|u_k|+c)|v|dt + \left(\max_{1\leqslant i,j\leqslant n}|b_{ij}|\right)\int_0^{2\pi}|\varphi_k - \varphi||v|dt \\ &\leqslant \frac{1}{\|u_k\|}\left(\varepsilon\|u_k\|_{L^2}\|v\|_{L^2} + c\|v\|_{L^1}\right) + c_6\|\varphi_k - \varphi\|_{L^2}\|v\|_{L^2} \\ &\leqslant c_7\left(\varepsilon + \frac{c}{\|u_k\|} + \|\varphi_k - \varphi\|_{L^2}\right)\|v\|.\end{aligned}$$

By $\|u_k\| \to \infty$, $\|\varphi_k - \varphi\|_{L^2} \to 0$ and the arbitrariness of ε, we get that

$$\frac{1}{\|u_k\|}\int_0^{2\pi}(f(\lambda t,u_k),v)dt \to \int_0^{2\pi}(B\varphi,v)dt \tag{3.7}$$

as $k \to \infty$. This yields

$$\begin{aligned}\frac{\Phi'(u_k)v}{\|u_k\|} &= \langle\varphi_k^-, v^-\rangle - \langle\varphi_k^+, v^+\rangle - \frac{1}{\|u_k\|}\int_0^{2\pi}\lambda(f(\lambda t, u_k), v)dt = \langle\varphi^-, v^-\rangle - \langle\varphi^+, v^+\rangle - \int_0^{2\pi}\lambda(B\varphi, v)dt + o(1)\\ &= -\langle L\varphi, v\rangle_H - \int_0^{2\pi}\lambda(B\varphi, v)dt + o(1).\end{aligned}$$

It can be deduced from the above equality that

$$\langle L\varphi, v\rangle_H + \int_0^{2\pi}\lambda(B\varphi, v)dt = 0, \quad \forall v \in E. \tag{3.8}$$

Using the definition of E we can set

$$\varphi(t) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{+\infty}[a_{2j-1}^{\varphi}\cos(2j-1)t + b_{2j-1}^{\varphi}\sin(2j-1)t]$$

and

$$v(t) = \frac{1}{\sqrt{\pi}}\sum_{j=1}^{+\infty}\left[a_{2j-1}^{v}\cos(2j-1)t + b_{2j-1}^{v}\sin(2j-1)t\right].$$

Then, by (3.8) one can obtain

$$\sum_{j=1}^{+\infty}\left[\left((\lambda B + (-1)^j(2j-1)I)a_{2j-1}^{\varphi}, a_{2j-1}^{v}\right) + \left((\lambda B + (-1)^j(2j-1)I)b_{2j-1}^{\varphi}, b_{2j-1}^{v}\right)\right] = 0,$$

where I is the $n \times n$ unit matrix. For any j, take $v(t) = \frac{1}{\sqrt{\pi}}e_i\cos(2j-1)t$ and $v(t) = \frac{1}{\sqrt{\pi}}e_i\sin(2j-1)t$, where $i = 1,2,\ldots,n$. An easy computation shows that

$$(\lambda B + (-1)^j(2j-1)I)a_{2j-1}^{\varphi} = 0$$

and

$$(\lambda B + (-1)^j(2j-1)I)b_{2j-1}^{\varphi} = 0.$$

Hence, by $\lambda^{-1}(-1)^{j+1}(2j-1) \notin \sigma(B)$ we know $\varphi \equiv 0$ and

$$\frac{1}{\|u_k\|}\int_0^{2\pi}(f(\lambda t, u_k), \varphi_k^-)dt = \frac{1}{\|u_k\|}\int_0^{2\pi}(f(\lambda t, u_k), \varphi_k^-)dt - \int_0^{2\pi}(B\varphi, \varphi_k^-)dt.$$

This shows by replacing v with φ_k^- in the proof of (3.7) that

$$\frac{1}{\|u_k\|}\int_0^{2\pi}(f(\lambda t, u_k), \varphi_k^-)dt \to 0. \tag{3.9}$$

It follows from (3.9) that

$$o(1) = \frac{\Phi'(u_k)\varphi_k^-}{\|u_k\|} = \langle\varphi_k^-, \varphi_k^-\rangle - \frac{1}{\|u_k\|}\int_0^{2\pi}\lambda(f(\lambda t, u_k), \varphi_k^-)dt = \|\varphi_k^-\|^2 + o(1),$$

which implies $\|\varphi_k^-\| \to 0$. Similarly, $\|\varphi_k^+\| \to 0$. It is impossible since $\|\varphi_k\| = 1$ for any k. Consequently, $\{u_k\}$ is bounded in E. Moreover, by the compactness of Ψ', going if necessary to a subsequence, we can assume that $u_k \rightharpoonup u$ and $\Psi'(u_k) \to \Psi'(u)$ in E. Then

$$\|u_k^- - u^-\|^2 = \langle u_k^- - u^-, u_k^- - u^-\rangle = (\Phi'(u_k) - \Phi'(u))(u_k^- - u^-) + (\Psi'(u_k) - \Psi'(u))(u_k^- - u^-) \to 0.$$

Similarly, $\|u_k^+ - u^+\|^2 \to 0$. Hence $u_k \to u$ in E and the $(C)_c$ condition is satisfied. Finally, Φ is even since $f(t,z)$ is odd in z and $F(t,0) = 0$. Hence Theorem 3.1 follows from Theorem 1.1. The proof is completed. □

Theorem 3.2. *Assume that f satisfies (f_1), (f_2), $(f_3)(I)$, (f_4), (f_6) and the following condition*

(f_7) $(Bz, z) < -\frac{3\pi}{2r}|z|^2$ for all $z \in R^n \backslash \{0\}$.

Then (1.1) possesses at least $2mn$ pairs of $4r$-periodic classical solutions, where $m = \max\left\{j \in Z^+ : -(4j-1)|z|^2 > \frac{2r}{\pi}(Bz, z) \text{ for } z \neq 0\right\}$.

Proof. Let $X = E^+$, $Y = E^-$, $\Phi(u) = I(u)$, $\Psi(u) = -G(u)$ and

$$Y_0 = \mathrm{span}\left\{e_k\cos(4j-1)t,\ e_k\sin(4j-1)t : j \in Z^+,\ j \leqslant m,\ k = 1,2,\ldots,n\right\}.$$

Then the conclusion will be obtained by the same argument as Theorem 3.1. The proof is completed. □

Theorem 3.3. *Assume that f satisfies* (f_1), (f_2), (f_3)(I), (f_4), (f_5) *and the following condition*

(f_8) $\widehat{F}(t,z) \geqslant 0$ *and* $\widehat{F}(t,z) \to +\infty$ *as* $|z| \to \infty$ *uniformly for* $t \in [0,r]$, *where* $\widehat{F}(t,z) = \frac{1}{2}(f(t,z),z) - F(t,z)$.

Let m be given by Theorem 3.1. *Then* (1.1) *possesses at least 2mn pairs of 4r-periodic classical solutions.*

Proof. Let Φ and Ψ be that in Theorem 3.1. From the proof of Theorem 3.1 we see that the condition (f_6) was only used to prove the $(C)_c$ condition. Hence, it is sufficient to prove that Φ satisfies the $(C)_c$ condition.

Let $\{u_k\} \subset E$ be any sequence such that $\Phi(u_k) \to c$, $(1 + \|u_k\|)\Phi'(u_k) \to 0$ as $k \to \infty$. We claim that $\{u_k\}$ is bounded in E. Assume by contradiction that $\|u_k\| \to \infty$ as $k \to \infty$. Let $\varphi_k = \frac{u_k}{\|u_k\|}$, then $\|\varphi_k\| = 1$. Without loss of generality, we can assume that $\varphi_k \rightharpoonup \varphi$ in E, $\varphi_k \to \varphi$ in $L^2(S^1,\mathbf{R^n})$ and $\varphi_k(t) \to \varphi(t)$ for almost all $t \in [0,2\pi]$. If $\varphi \equiv 0$, the argument of Theorem 3.1 shows $\|\varphi_k\| \to 0$. This contradicts $\|\varphi_k\| = 1$. Hence the case $\varphi \equiv 0$ will not occur, and hence $\varphi \not\equiv 0$. Set $\Omega = \{t \in [0,2\pi] : \varphi_k(t) \to \varphi(t) \neq 0\}$. Then Ω has a positive measure and $u_k(t) \to \infty$ for all $t \in \Omega$. It follows from (f_8) that

$$c = \lim_{k\to\infty}\left[\Phi(u_k) - \frac{1}{2}\Phi'(u_k)u_k\right] = \lim_{k\to\infty}\int_0^{2\pi}\lambda\widehat{F}(t,u_k)dt \geqslant \int_\Omega \liminf_{k\to\infty}\lambda\widehat{F}(t,u_k)dt \to +\infty.$$

This is a contradiction. Therefore, $\{u_k\}$ is bounded. Moreover, by arguing as in Theorem 3.1 we know that $\{u_k\}$ has a convergent subsequence. The proof is completed. □

At the end of this paper, we discuss the infinitely many solutions for system (1.1).

Theorem 3.4. *Assume that f satisfies* (f_1), (f_2), (f_3)(II), (f_4) *and the following condition*

(f_9) *there exists an* $\bar{r} > 0$ *such that*

$$(f(t,z),z) \geqslant pF(t,z) > 0$$

for $t \in [0,r]$ *and* $|z| \geqslant \bar{r}$, *where p appears in* (f_3)(II). *Then* (1.1) *possesses an unbounded sequence of 4r-periodic classical solutions.*

Proof. Let Φ, Ψ and X, Y be given by Theorem 3.1. The proof of this theorem will be completed with the aid of Theorem 1.2.

First, since the assumption (f_9) is the Ambrosetti–Rabinowitz condition, it is well known that there exist constants $c_1, c_2 > 0$ such that

$$F(t,z) \geqslant c_1|z|^p - c_2 \tag{3.10}$$

for $(t,z) \in \mathbf{R^1} \times \mathbf{R^n}$. This implies that Ψ is bounded from below. Moreover, by the argument of Theorem 3.1 we see that Φ satisfies (Φ_0) and (Φ_1).

Next, we check that Φ satisfies (Φ_3). To do this, let $\widetilde{Y}$ be any finite dimensional subspace of Y. It is sufficient to prove that $\Phi(u) \to -\infty$ as $u \in \widetilde{E} := X \oplus \widetilde{Y}$ and $\|u\| \to \infty$.

Since $\widetilde{Y}$ is finite dimensional, there is a $\tilde{\delta} = \tilde{\delta}(\widetilde{Y}) > 0$ such that

$$\|y\|_{L^2} \geqslant \tilde{\delta}\|y\| \tag{3.11}$$

for any $y \in \widetilde{Y}$. Hence for $u = u^+ + u^- \in X \oplus \widetilde{Y}$, by (3.10) and (3.11) and $\frac{2}{p} < 1$, one has

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - \int_0^{2\pi}\lambda F(\lambda t,u)dt \leqslant \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - \int_0^{2\pi}\lambda(c_1|u|^p - c_2)dt \\
&\leqslant \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - c_3\left(\int_0^{2\pi}|u^+ + u^-|^2dt\right)^{\frac{p}{2}} + 2\pi\lambda c_2 \leqslant \frac{1}{2}\left(\|u^-\|^2 - \|u^+\|^2\right) - c_3\|u^-\|_{L^2}^p + 2\pi\lambda c_2 \\
&\leqslant \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2 - c_3\tilde{\delta}^p\|u^-\|^p + 2\pi\lambda c_2 \leqslant -\frac{1}{2}\|u^-\|^2 - \frac{1}{2}\|u^+\|^2 + c_4 = -\frac{1}{2}\|u\|^2 + c_4.
\end{aligned}$$

This yields $\Phi(u) \to -\infty$ as $u \in \widetilde{E}$ and $\|u\| \to \infty$. Hence (Φ_3) holds.

Finally, we prove that the $(PS)_c$ condition holds for any $c \in (0,\infty)$. Let $\{u_k\}$ be any sequence such that $\Phi(u_k) \to c > 0$ and $\Phi'(u_k) \to 0$ as $k \to \infty$. We can assume $\|\Phi'(u_k)\| \leqslant 1$. By (f_9) and (3.10) we have

$$\begin{aligned}
2c + \|u_k\| &\geqslant 2\Phi(u_k) - \Phi'(u_k)u_k = \int_0^{2\pi}\lambda[(f(\lambda t,u_k),u_k) - 2F(\lambda t,u_k)]dt \\
&\geqslant \int_0^{2\pi}\lambda(p-2)F(\lambda t,u_k)dt + \int_0^{2\pi}\lambda[(f(\lambda t,u_k),u_k) - pF(\lambda t,u_k)]dt \geqslant \int_0^{2\pi}\lambda(p-2)(c_1|u_k|^p - c_2)dt - c_5 \\
&\geqslant c_6\|u_k\|_{L^p}^p - c_7
\end{aligned}$$

which implies

$$\|u_k\|_{L^p} \leqslant c_8\left(1+\|u_k\|^{\frac{1}{p}}\right) \tag{3.12}$$

Write $u_k = u_k^+ + u_k^- \in X \oplus Y$. Then for large k, by (f_3)(II) and Hölder inequality we get that

$$\begin{aligned}\|u_k^-\| &\geqslant |\Phi'(u_k)u_k^-| = |\|u_k^-\|^2 - \int_0^{2\pi} \lambda(f(\lambda t, u_k), u_k^-)dt| \geqslant \|u_k^-\|^2 - \int_0^{2\pi} \lambda a(1+|u_k|^{p-1})|u_k^-|dt \\ &\geqslant \|u_k^-\|^2 - \lambda a\|u_k^-\|_{L^1} - \lambda a\|u_k\|_{L^p}^{p-1}\|u_k^-\|_{L^p} \geqslant \|u_k^-\|^2 - c_9\|u_k^-\| - c_{10}\|u_k\|_{L^p}^{p-1}\|u_k^-\|.\end{aligned}$$

This yields

$$\|u_k^-\| \leqslant c_{11}\left(1+\|u_k\|_{L^p}^{p-1}\right). \tag{3.13}$$

Similarly, one can easily get that

$$\|u_k^+\| \leqslant c_{11}\left(1+\|u_k\|_{L^p}^{p-1}\right). \tag{3.14}$$

The combination of (3.12)–(3.14) shows that

$$\|u_k\| \leqslant \|u_k^-\| + \|u_k^+\| \leqslant c_{12}\left(1+\|u_k\|^{\frac{p-1}{p}}\right).$$

It implies that $\{u_k\}$ is bounded in E. Moreover, $\{u_k\}$ has a convergent subsequence according to the argument in Theorem 3.1. Hence $(PS)_c$ condition holds for any $c \in (0,\infty)$.

We have pointed out the fact that Φ is even in Theorem 3.1. By virtue of Theorem 1.2, Φ has a sequence of critical points $\{u_n\} \subset E$ such that $|\Phi(u_n)| \to \infty$. If $\{u_n\}$ is bounded in E, then by the assumption (f_3)(II) and the definition of Φ, one know that $\{|\Phi(u_n)|\}$ is also bounded, a contradiction. Hence $\{u_n\}$ is unbounded in E. The proof is completed. $\square$

Remark 3.1. Similar to the treatment of Theorem 3.2, we can get the same conclusion as Theorem 3.4 by replacing (f_9) with the following condition

$(-f_9)$ there exists an $\bar{r} > 0$ such that

$$(-f(t,z),z) \geqslant -pF(t,z) > 0$$

for $t \in [0,r]$ and $|z| \geqslant \bar{r}$.

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