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A Lie algebraic condition for exponential stability of discrete hybrid systems and application to hybrid synchronization

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A Lie algebraic condition for global exponential stability of linear discrete switched impulsive systems is presented in this paper. By considering a Lie algebra generated by all subsystem matrices and impulsive matrices, when not all of these matrices are Schur stable, we derive new criteria for global exponential stability of linear discrete switched impulsive systems. Moreover, simple sufficient conditions in terms of Lie algebra are established for the synchronization of nonlinear discrete systems using a hybrid switching and impulsive control. As an application, discrete chaotic system's synchronization is investigated by the proposed method. © *2011 American Institute of Physics*. [doi:10.1063/1.3594046]

Nowadays, the control of discrete system especially, the discrete hybrid system, has received a great deal of attention, which provides a natural framework for mathematical modeling of many real-world phenomena such as evolutionary process, biological systems, flying object motions, and so on. However, until recently, the general theory of discrete systems remained much weaker than that in continuous-time case. This motivates us to investigate the stability and its applications to discrete hybrid systems. In this paper, we propose a new approach for the stability of discrete switched impulsive systems and synchronization of discrete nonlinear systems using a hybrid switching and impulsive control. The Lie algebraic approach ensures the global exponential stability of discrete switched impulsive systems under predesigned dwell time conditions. Moreover, similar technique is developed to accomplish the synchronization of discrete nonlinear systems using a hybrid switching and impulsive control. The main difficulty lies in the existence of both stable and unstable subsystems as well as impulsive behaviors. The conditions are easy to check and the hybrid control demonstrates good performance.

I. INTRODUCTION

In the past decades, the study of hybrid systems has been a hot research topic in the control loop, see Refs. 1–5 and the references therein. Switched systems are a class of hybrid dynamical systems consisting of a family of continuous and/or discrete-time subsystems, and a rule that orchestrates the switching between them. Most recent efforts have focused on the stability, stabilization, and controllability of switched systems for both theoretical and practical reasons. Among the methods contributed to the stability of switched systems, the Lie algebraic approach is recognized to be effective and interesting. Based on the assumption on system matrices, Lie algebraic conditions can ensure the existence of common quadratic Lyapunov function for the stability analysis and system design of switched systems.[3,4]

On the other hand, discrete dynamic systems, which arise from modeling processes with successive changes in variables at discrete time, can be found in many applications such as finance, economics, and so on. However, the control theory of discrete systems is not fully developed compared with the continuous-time counterpart.[6–10] Different techniques were utilized to investigate the analysis and control of discrete hybrid systems, such as reachability,[5] global stabilization, stability, and bifurcation of nonlinear discrete systems[11,12] and generating chaos for discrete delay systems.[13]

In practice, one kind of hybrid systems characterized by switches of states and abrupt changes at switching instants with certain logic rules is called switched impulsive system. The instantaneous changes at certain moments come from unexpected internal or external perturbations. Moreover, switched systems with impulses have been used extensively to describe systems in various applications, including information science, electronics, and automatic control systems. Because of technical difficulties, the dynamical behaviors for such hybrid systems with switching and impulsive effects are still under investigation from different viewpoints.[14–16] From the control point of view, the hybrid switching and impulsive control is effective in achieving stability and accomplishing synchronization.[17–19] Nowadays, synchronization of chaotic dynamics has been an active research area due to its role in understanding the basic features of coupled nonlinear systems and potential applications in communication, time series analysis, and modeling.[20,21] There have been several results on the synchronization of hybrid systems using impulsive control[22–25] or hybrid switching and impulsive control.[16,18]

However, to the best of our knowledge, there is no result about the stability analysis of discrete switched impulsive systems by the Lie algebraic approach. We present simple sufficient conditions for global exponential stability of linear

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discrete switched impulsive systems based on Lie algebra generated by coefficient matrices. It is worth mentioning that all these matrices are not necessarily Schur stable. Lie algebraic feature of system matrices combining with dwell time design achieves the global exponential stability. Moreover, the advantage of hybrid control motivates us to study the synchronization of discrete nonlinear systems by hybrid switching and impulsive control. The main concern is how to co-design the switching and impulsive control gain matrices and dwell time for accomplishing the synchronization. The advantage of this method is that conditions are easy to check and the hybrid control can be designed flexibly according to practical requirements.

The rest of this paper is organized as follows: Sec. II gives some basic concepts and a lemma. In Sec. III, a Lie algebraic condition is established for global exponential stability of discrete switched impulsive systems. Section IV develops a new hybrid switching and impulsive control strategy for the synchronization of nonlinear systems. In Sec. V, a typical chaotic system synchronization is presented to show the effectiveness of proposed results and comparisons with switching control and impulsive control are discussed. Some concluding remarks are drawn in Sec. VI.

II. PRELIMINARIES

Consider the following linear discrete switched impulsive system

$$\begin{aligned} x(k+1) &= A_{i_k}x(k), \quad k \neq \tau_k, \\ x(\tau_k+1) &= B_{j_k}x(\tau_k), \quad k = 1,2,..., \\ x(k_0^+) &= x_0, \end{aligned} \tag{1}$$

where the variable $k \in \mathbb{Z}^+$ denotes the discrete time, the state variable $x(k) := x_k \in \mathbb{R}^n$, and A_{i_k} and B_{j_k} are $n \times n$ matrices with $i_k \in \{1,2,\ldots,l\}$ and $j_k \in \{1,2,\cdots m\}$. The switching sequence $\{(1,i_1),(2,i_2),\cdots,(k,i_k),\cdots\}$ specifies which subsystem is activated at certain discrete-time instant k. If $i_{k+1} \neq i_k$, the system switches from the dynamics governed by A_{i_k} to that by $A_{i_{k+1}}$ due to changes in a modeling's operating condition or a control action at discrete instants. $\{\tau_k : \tau_k \in \mathbb{Z}^+\}$ is the sequence of impulsive instants, $\tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ with $\lim_{k\to\infty} \tau_k = \infty$ and $\tau_{k+1} - \tau_k > 1$. When the system is switched from the ith subsystem to the jth subsystem at the instant τ_k, a sudden change of the state happens due to external or internal effect, which is described by $x(\tau_k+1) = B_{j_k}x(\tau_k)$, $j_k \in \{1,2,\ldots,m\}$. This kind of systems can be regarded as impulsive controlled discrete switched systems which has much more complex dynamic behavior than that of discrete systems. Considering the initial condition $x(k_0) = x_{k_0}$ and a switching sequence, the solution sequence denoted by $x(k)$ can be determined for $k \geq k_0$. Let $\|x\|$ denotes the Euclidean vector norm, i.e., $\|x\| = \sqrt{x^\top x}$. The definition of exponential stability of system (1) is presented as follows.

Definition 1. For discrete switched impulsive system (1), the trivial equilibrium point is exponentially stable if there exist positive constants $c > 0$ and $\gamma \in (0,1)$ *such that*

$$\|x(k)\| \leq c\gamma^k\|x(0)\|, \quad k \geq 0, \tag{2}$$

where γ is called the exponential convergence rate. If Eq. (2) is satisfied for any initial condition $x(0) \in \mathbb{R}^n$, the trivial equilibrium point is globally exponentially stable for discrete-time switched impulsive system (1).

Next, some preliminaries of Lie algebra for integrity are introduced. A Lie algebra $\mathcal{L}$, is a vector space over a field equipped with a Lie bracket $[\cdot,\cdot]$. In the case of matrix Lie algebra, the standard Lie bracket is defined as $[A,B] \overset{\Delta}{=} AB - BA$. The descending sequence of ideals $\mathcal{L}^{(k)}$ is defined inductively as follows: $\mathcal{L}^{(1)} := \mathcal{L}, \mathcal{L}^{(k+1)} := [\mathcal{L}^{(k)},\mathcal{L}^{(k)}] \subset \mathcal{L}^{(k)}$. If $\mathcal{L}^{(k)} = 0$, for k sufficiently large, then $\mathcal{L}$ is called *solvable*. For example, if $\mathcal{L}$ is a Lie algebra generated by two matrices A and B, i.e., $\mathcal{L} = \{A,B\}_{LA}$, then we have: $\mathcal{L}^{(1)} = \mathcal{L} = \text{span}\{A,B,[A,B],[A,[A,B]],...\}$, $\mathcal{L}^{(2)} = \text{span}\{[A,B],[A,[A,B]],...\}$, $\mathcal{L}^{(3)} = \text{span}\{[[A,B],[A,[A,B]]],...\}$, and so on.

For convenience, we confuse the set $\{A_{i_k}, i_k \in \{1,2,\cdots l\}\}$ with $\{A_i, i \in \{1,2,\cdots l\}\}$ and $\{B_{j_k}, j_k \in \{1,2,\cdots m\}\}$ with $\{B_j, j \in \{1,2,\cdots m\}\}$, respectively. If all the eigenvalues of a matrix A lie in the unit plane, then A is called Schur stable. Without loss of generality, it is assumed that matrices $A_1,...,A_{s_1}$ $(0 \leq s_1 \leq l)$ and $B_1,...,B_{s_2}$ $(0 \leq s_2 \leq m,\ s_1+s_2 \geq 1)$ are Schur stable while the others (if existing) are not stable. During $[0, k]$, we define t_s and t_u as *total number* dwelling on Schur stable and unstable subsystems, i.e., switched gain matrices A_i are Schur stable and unstable, respectively. Similarly, $d_s(k)$ and $d_u(k)$ are the *total number* of Schur stable and unstable impulsive behavior during $[0, k]$. Let $m_s = t_s + d_s$, $m_u = t_u + d_u$, $t = t_s + t_u$, and $d = d_s + d_u$.

Definition 2. If not all the subsystems matrices and impulsive coefficient matrices are Schur stable and there exists a common positive definite matrix P satisfying

$$\begin{aligned} &\left(\frac{A_i}{\beta_s}\right)^T P\left(\frac{A_i}{\beta_s}\right) - P < 0, \quad i = 1,\cdots,s_1, \\ &\left(\frac{B_j}{\beta_s}\right)^T P\left(\frac{B_j}{\beta_s}\right) - P < 0, \quad j = 1,\cdots,s_2, \end{aligned} \tag{3}$$

$$\begin{aligned} &\left(\frac{A_i}{\beta_u}\right)^T P\left(\frac{A_i}{\beta_u}\right) - P < 0, \quad s_1 < i \leq l, \\ &\left(\frac{B_j}{\beta_u}\right)^T P\left(\frac{B_j}{\beta_u}\right) - P < 0, \quad s_2 < j \leq m, \end{aligned} \tag{4}$$

with scalars $0 < \beta_s < 1$, $\beta_u > 1$, then $V(x) = x^T Px$ is called common quadratic Lyapunov-like funtion (CQLLF) for all the subsystems.

We first present the following lemma which plays a key role in the subsequent discussion.

Lemma 1. If not all the subsystems and impulsive coefficient matrices are Schur stable, and the Lie algebra

$$\{A_i, \quad i = 1,\ldots,l; \quad B_j, \quad j = 1,\ldots,m\}_{LA} \tag{5}$$

is solvable, then there exists a CQLLF for all the subsystems satisfying Eqs. (3) and (4).

Proof. As the Lie algebra (5) is solvable, there exists a nonsingular complex matrix U such that for all i, j,

$$A_i = U^{-1}\tilde{A}_i U, \;\; B_j = U^{-1}\tilde{B}_j U, \;\; i = 1, 2, \ldots, l, \;\; j = 1, 2, \ldots, m, \tag{6}$$

where $\tilde{A}_i$ and $\tilde{B}_j$ are upper-triangular matrices.

For Schur stable matrices A_i and B_j, there exists a positive scalar $0 < \beta_s < 1$ such that $\frac{A_i}{\beta_s}$ and $\frac{B_j}{\beta_s}$ remain Schur stable. For unstable matrices A_i and B_j, there exists a constant $\beta_u > 1$ such that $\frac{A_i}{\beta_u}$ and $\frac{B_j}{\beta_u}$ become Schur stable. From Eq. (6), it yields that

$$\frac{A_i}{\beta_s} = U^{-1}\frac{\tilde{A}_i}{\beta_s}U, \;\; 1 \le i \le s_1, \quad \frac{B_j}{\beta_s} = U^{-1}\frac{\tilde{B}_j}{\beta_s}U, \;\; 1 \le j < s_2,$$
$$\frac{A_i}{\beta_u} = U^{-1}\frac{\tilde{A}_i}{\beta_u}U, \;\; s_1 < i \le l, \quad \frac{B_j}{\beta_u} = U^{-1}\frac{\tilde{B}_j}{\beta_u}U, \;\; s_2 < j \le m.$$

Note that all the matrices $\frac{\tilde{A}_i}{\beta_s}(1 \le i \le s_1)$, $\frac{\tilde{A}_i}{\beta_u}$ $(s_1 < i \le l)$, $\frac{\tilde{B}_j}{\beta_s}$ $(1 \le j \le s_2)$, and $\frac{\tilde{B}_j}{\beta_u}$ $(s_2 < j \le m)$ are Schur stale and still upper-triangular. Then using the similar technique as in the proof of Theorem 3 in Ref. 4, we can construct a common symmetric positive definite matrix P satisfying Eqs. (3) and (4). The existence of CQLLF for all subsystems is then guaranteed. This completes the proof.

Remark 1. Although it has been shown in Ref. 4 that CQLLF can be constructed explicitly, the computation depends on the transformation matrix U. As obtaining U *may need some efforts when using standard numerical methods mentioned in Ref. 4, it may be more effective to solve linear matrix inequalities (LMIs) (3) and (4) with respect to $P > 0$ directly using the existing* LMI *software. In addition, β_s can be chosen as a positive constant such that $1 > \beta_s > \lambda$, where $\lambda = \max|\lambda(A_{i_k})|$ with $\lambda(A_{i_k})$ being eigenvalues of A_{i_k}.*

III. GLOBAL EXPONENTIAL STABILITY

The main purpose of this section is to derive the global exponential stability criteria for discrete switched impulsive systems by the Lie algebraic condition and dwell time design.

Theorem 1. *When not all the matrices $A_i(i = 1, 2, \ldots l)$ and $B_j(j = 1, 2, \ldots m)$ of system (1) are Schur stable, if the Lie algebra (5) is solvable and any of the following conditions holds,*

1. *for any given scalar β satisfying $\beta_s < \beta < 1$,*

$$\frac{m_u}{m_s} \le \frac{\ln(\beta) - \ln(\beta_s)}{\ln(\beta_u) - \ln(\beta)}, \tag{7}$$

2. *for any given scalar β satisfying $\beta_s^2 < \beta \le \beta_s$,*

$$\frac{m_u}{m_s} \le \frac{\ln(\beta) - 2\ln(\beta_s)}{2\ln(\beta_u) - \ln(\beta)}, \tag{8}$$

then system (1) is globally exponentially stable.

Proof. According to Lemma 1, we can obtain a positive definite matrix P for $(\frac{A_i}{\beta_s})$ $(1 \le i \le s_1)$, $\frac{A_i}{\beta_u}$ $(s_1 < i \le l)$, $(\frac{B_j}{\beta_s})$ $(1 \le j \le s_2)$, and $(\frac{B_j}{\beta_u})$ $(s_2 < j \le m)$ such that Eqs. (3) and (4) hold. Then let a Lyapunov function in the form of $V(k) := V(x(k)) = x^\top(k)Px(k)$.

When $k \ne \tau_k$, from Eq. (1), we have

$$\begin{aligned} V(k+1) &= x(k+1)^\top P x(k+1) \\ &= (A_{i_k}x(k))^\top P(A_{i_k}x(k)), \quad i_k \in \{1, 2, \ldots, l\}. \end{aligned} \tag{9}$$

Without loss of generality, when $k = k_0 \ne \tau_k$, the Schur stable subsystem $x(k+1) = A_{i_1}x(k)$ is assumed to be activated, from Eq. (3), we have

$$\begin{aligned} V(k_0+1) &= x(k_0+1)^\top P x(k_0+1) \\ &= (A_{i_1}x(k_0))^\top P(A_{i_1}x(k_0)) < \beta_s^2 V(k_0). \end{aligned}$$

At the first impulsive instant τ_1, the Schur stable impulsive behavior is supposed to happen. Using Eq. (3), it follows that

$$\begin{aligned} V(\tau_1+1) &= x(\tau_1+1)^\top P x(\tau_1+1) \\ &= x^\top(\tau_1)B_{j_1}^\top P B_{j_1}x(\tau_1) < \beta_s^2 V(\tau_1) \end{aligned}$$

Therefore, in general, according to Eqs. (3) and (4), we obtain that

$$V(k+1) < \begin{cases} \beta_s^2 V(k), & \text{when } A_{i_k}, B_{j_k} \text{ are Schur stable,} \\ \beta_u^2 V(k), & \text{when } A_{i_k}, B_{j_k} \text{ are unstable.} \end{cases}$$

It is easy to get that no matter what activation order is

$$V(k) < \beta_s^{2m_s}\beta_u^{2m_u}V(k_0), \tag{10}$$

where m_u and m_s are defined in Sec. II. In the following, two cases should be considered.

1. In order that $V(k) < \beta^{2(m_s+m_u)}V(k_0)$ for a given scalar β with $\beta_s < \beta < 1$, we need the inequality as follows:

$$\beta_s^{2m_s}\beta_u^{2m_u} \le \beta^{2(m_s+m_u)},$$

which implies that

$$\left(\frac{\beta_u}{\beta}\right)^{2m_u} \le \left(\frac{\beta}{\beta_s}\right)^{2m_s}.$$

Taking the logarithm on both sides of the above inequality follows that

$$m_u \ln\left(\frac{\beta_u}{\beta}\right) \le m_s \ln\left(\frac{\beta}{\beta_s}\right),$$

equivalently,

$$\frac{m_u}{m_s} \le \frac{\ln(\beta) - \ln(\beta_s)}{\ln(\beta_u) - \ln(\beta)}.$$

2. To obtain that $V(k) < \beta^{(m_s+m_u)}V(k_0)$ for a given scalar β with $\beta_s^2 < \beta \leq \beta_s$, the following inequality needs to hold

$$\beta_s^{2m_s}\beta_u^{2m_u} \leq \beta^{(m_s+m_u)},$$

which means that

$$\left(\frac{\beta_u^2}{\beta}\right)^{m_u} \leq \left(\frac{\beta}{\beta_s^2}\right)^{m_s}.$$

Similarly, taking the logarithm on both sides of the above inequality gives

$$\frac{m_u}{m_s} \leq \frac{\ln(\beta) - 2\ln(\beta_s)}{2\ln(\beta_u) - \ln(\beta)}.$$

Hence, from the above analysis, under the dwell time design (7) or (8), system (1) is globally exponentially stable. This completes the proof.

Remark 2. According to Theorem 1, for a given positive scalar $\beta \in (\beta_s^2, 1)$, we can design the dwell time scheme in the form of Eq. (7) or (8) to obtain different exponential convergence rates according to practical requirements. From Eqs. (7) and (8), we can find that the scalar β affects the dwell time design. The expression on the right hand in Eq. (7) becomes greater with the increase of the value of β, which implies that the choice of dwell time becomes more flexible. Moreover, due to the existence of both Schur stable and unstable coefficient matrices, a dwell time scheme is necessary to guarantee the stability which is different from known results by Lie algebraic approach, where exponential stability can be achieved under arbitrary switching law.

Corollary 1. If we choose the Lyapunov function in a more general form $w(k) = M(k)\, V(k)$, where $M(k)$ is a discrete-time nonincreasing function satisfying $M(k) \geq m$, $m > 0, k \in \mathbb{Z}^+$. Then system (1) is globally exponentially stable under condition (7) or (8).

Corollary 2. When the system (1) is reduced to the discrete switched system, the inequality (10) becomes inequality (20) in Ref. 4 if the continuous-time subsystems vanish. Hence, the dwell time design is the same as that in Ref. 4. The Lie algebraic approach for the stability of discrete systems is extended to the case of discrete switched impulsive systems.

IV. GLOBAL EXPONENTIAL SYNCHRONIZATION

In this section, the issue of synchronization of nonlinear discrete system is investigated using Lie algebraic approach. The chaotic system is synchronized by hybrid controlled response system based on the Lie algebraic condition and the dwell time design. It is easy to observe that many discrete chaotic systems can be written in the following form:

$$x(k+1) = Ax(k) + f(k, x_k), \tag{11}$$

where $k \in \mathbb{Z}^+$, $x \in \mathbb{R}^n$, A is a known $n \times n$ matrix, and $f(k, x_k) : \mathbb{Z}^+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a discrete vector-value function guaranteeing the existence and uniqueness of solutions of Eq. (11) for the initial value problem. For convenience, denote $x(k)$ as x without leading to confusion. Regarding Eq. (11) as a drive system, the response system can be described as

$$y(k+1) = Ay(k) + f(k, y) + u(k, x, y), \tag{12}$$

where $u(k, x, y)$ is the control input. Construct a hybrid switching and impulsive control $u = u_1 + u_2$ for the response system (12) as follows:

$$\begin{cases} u_1(k) = B_{1k}[y(k) - x(k)], k \neq \tau_k, \\ u_2(\tau_k) = B_{2k}[y(\tau_k) - x(\tau_k)] - [f(\tau_k, y) - f(\tau_k, x)], \end{cases} \tag{13}$$

where B_{1k} and B_{2k} are $n \times n$ constant matrices to be determined. It is clear from Eq. (13) that

$$y(k+1) = Ay(k) + f(k, y) + B_{1k}(y(k) - x(k)), k \neq \tau_k,$$

which implies that $u_1(k)$ is a switching control and switches its value at every instant $k \neq \tau_k$. When $k = \tau_k$, Eqs. (12) and (13) yield that

$$y(\tau_k + 1) = Ay(\tau_k) + B_{2k}[y(\tau_k) - x(\tau_k)] + f(\tau_k, x), \tau_k \in \mathbb{Z}^+,$$

which means that the controller $u_2(\tau_k)$ is an impulsive control with the effect of changing the state of response system (12) instantaneously at the point τ_k. In the subsequent, the control gain matrices B_{1k} and B_{2k} will be chosen from a finite matrix set.

Accordingly, under control (13), the closed-loop response system of Eq. (12) becomes

$$\begin{cases} y(k+1) = Ay(k) + f(k,y) + B_{1k}(y(k) - x(k)), \ k \neq \tau_k, \\ y(k+1) = Ay(k) + B_{2k}[y(k) - x(k)] + f(k,x), k = \tau_k, k \in \mathbb{Z}^+, \\ y(k_0) \quad = y_0. \end{cases} \tag{14}$$

Let the synchronization error be $e(k) = y(k) - x(k)$. Systems (11) and (14) can be reformulated as

$$\begin{cases} e(k+1) = (A + B_{1k})e(k) + f(k,y) - f(k,x), \quad k \neq \tau_k, \\ e(k+1) = (A + B_{2k})e(k), \quad k = \tau_k, \\ e(k_0) \quad = e_0, \end{cases} \tag{15}$$

where B_{1k} and B_{2k} are to be determined. Now, we study the stability of the synchronization error system (15). Let the switching control gain matrices $B_{1k} \in \{B_{11}, B_{12}, \ldots, B_{1l}\}$, and let the impulsive control gain matrices $B_{2k} \in \{B_{21}, B_{22}, \ldots, B_{2m}\}$. Then system (15) becomes a discrete switched impulsive system

$$\begin{cases} e(k+1) = A_{i_k}e(k) + f(k,y) - f(k,x), \quad k \neq \tau_k, \\ e(k+1) = C_{j_k}e(k), \quad k = \tau_k, \\ e(k_0) \quad = e_0, \quad k = 1, 2, \ldots, \end{cases} \tag{16}$$

where $A_{i_k} = A + B_{1i_k}$ and $C_{j_k} = A + B_{2j_k}$, $i_k \in \{1, 2, \ldots, l\}$, $j_k \in \{1, 2, \ldots, m\}$.

Assume that $\{A_1, A_2, \cdots, A_l, C_1, C_2, \cdots, C_m\}$ is a compact (with respect to the usual topology in $\mathbb{R}^{n\times n}$) set of real $n\times n$ matrices, and there exists a positive definite symmetric matrix P satisfying inequalities (3) and (4) and a function $L(k)\geq 0$ such that

$$\parallel f(k,y)-f(k,x)\parallel \leq L(k)\parallel x-y\parallel . \tag{17}$$

Assume that there exists a constant L such that $L=\sup_{k\in\mathbb{Z}^+} L(k)$. Define $\alpha_s(k)$, $\alpha_u(k)$, and ρ such that

$$\begin{aligned}&\beta_s^2+2\lambda\rho L(k)+\rho L^2(k)=\alpha_s(k),\\ &\beta_u^2+2\lambda\rho L(k)+\rho L^2(k)=\alpha_u(k),\quad \rho=\frac{\lambda_{max}(P)}{\lambda_{min}(P)},\end{aligned}\tag{18}$$

where $\lambda_{max}(P)$ and $\lambda_{min}(P)$ denote the maximum and minimum eigenvalues of P.

Based on the discussion in Theorem 1, we obtain the synchronization criteria for systems (11) and (12).

Theorem 2. *Assume that not all the matrices $A_i, C_j, i=1,2,\cdots l, j=1,2,\cdots m$ of system (16) are Schur stable and the Lie algebra $\{A_i, C_j, i=1,2,\cdots l, j=1,2,\cdots m\}_{LA}$ is solvable, $\alpha_s(k)\leq\alpha_1$, $\alpha_u(k)\leq\alpha_2$, where α_1 and α_2 are positive constants with $\alpha_2>1$. If one of the following conditions is satisfied, then the trivial solution of system (16) is globally exponentially stable which implies that the drive system (11) and the response system (12) are globally exponentially synchronized by the hybrid switching and impulsive control.*

1. If $0<\alpha_1<1$, and for any given scalars α,β satisfying $0<\alpha_1<\alpha<1, \beta_s<\beta<1$,

$$\frac{t_u}{t_s}\leq\frac{\ln\alpha-\ln\alpha_1}{\ln\alpha_2-\ln\alpha},\ \frac{d_u}{d_s}\leq\frac{\ln\beta-\ln\beta_s}{\ln\beta_u-\ln\beta}. \tag{19}$$

2. If $0<\alpha_1<1$, and for any given scalars α,β satisfying $0<\alpha_1<\alpha<1, \beta_s^2<\beta\leq\beta_s$,

$$\frac{t_u}{t_s}\leq\frac{\ln\alpha-\ln\alpha_1}{\ln\alpha_2-\ln\alpha},\ \frac{d_u}{d_s}\leq\frac{\ln\beta-2\ln\beta_s}{2\ln\beta_u-\ln\beta}. \tag{20}$$

3. If $\alpha_1\geq 1$, and for any given scalar β satisfying $\beta_s<\beta<1$

$$\frac{t+2d_u}{2d_s}\leq\frac{\ln\beta-\ln\beta_s}{\max\{\ln\alpha_2-\ln\beta,\ln\beta_u-\ln\beta\}}. \tag{21}$$

4. If $\alpha_1\geq 1$, and for any given scalar β satisfying $\beta_s^2<\beta\leq\beta_s$,

$$\frac{t+d_u}{d_s}\leq\frac{\ln\beta-2\ln\beta_s}{\max\{\ln\alpha_2-\ln\beta,2\ln\beta_u-\ln\beta\}}. \tag{22}$$

Proof. Noting that the Lie algebra $\{A_{i_k}, C_{j_k}: i_k\in\{1,2,\ldots,l\}, j_k\in\{1,2,\ldots,m\}\}_{LA}$ is solvable, we can explicitly construct a symmetric positive definite $n\times n$ matrix P satisfying (3) and (4) and let a Lyapunov function $V(k):=V(e(k))=e^\top Pe$.

According to matrix inequality theory, for $k\neq\tau_k$, we have

$$\begin{aligned}[f(k,y)-f(k,x)]^\top PA_{i_k}e&\leq L(k)(e^\top e)^{\frac{1}{2}}(e^\top A_{i_k}^\top PPA_{i_k}e)^{\frac{1}{2}}\\ &\leq\rho L(k)\sqrt{\lambda_{\max}(A_{i_k}^\top A_{i_k})}e^\top Pe.\end{aligned}$$

Also it can be derived that

$$\begin{aligned}&[f(k,y)-f(k,x)]^\top P[f(k,y)-f(k,x)]\\ &\quad\leq\lambda_{\max}(P)[f(k,y)-f(k,x)]^\top[f(k,y)-f(k,x)]\\ &\quad\leq\rho L^2(k)e^\top Pe, k\neq\tau_k.\end{aligned}$$

Let $\lambda=\max_{i_k=1,\ldots,l}\sqrt{\lambda_{\max}(A_{i_k}^\top A_{i_k})}$. For $k\neq\tau_k$, if the Schur stable subsystem is activated, we have

$$\begin{aligned}V(k+1)&=e(k+1)^\top Pe(k+1)\\ &=[A_{i_k}e(k)+f(k,y)-f(k,x)]^\top P[A_{i_k}e(k)+f(k,y)\\ &\quad-f(k,x)]\\ &=e(k)^\top A_{i_k}^\top PA_{i_k}e(k)+2[f(k,y)-f(k,x)]^\top PA_{i_k}e(k)\\ &\quad+[f(k,y)-f(k,x)]^\top P[f(k,y)-f(k,x)]\\ &\leq[\beta_s^2+2L(k)\lambda\rho+\rho L^2(k)]V(k)\end{aligned}$$

Similarly, when the Schur unstable subsystem is activated, it follows that

$$V(k+1)\leq[\beta_u^2+2L(k)\lambda\rho+\rho L^2(k)]V(k)$$

At impulsive instants $k=\tau_k$, the Schur stable impulsive behavior gives

$$\begin{aligned}V(\tau_k+1)&=e(\tau_k+1)^\top Pe(\tau_k+1)\\ &=[C_{j_k}e(\tau_k)]^\top P[C_{j_k}e(\tau_k)]\leq\beta_s^2V(\tau_k),\end{aligned}$$

and the unstable impulsive effect follows that $V(\tau_k+1)\leq\beta_u^2V(\tau_k)$.

The above inequalities imply that, on time interval $[k_0,k]$,

$$\begin{aligned}V(k)&\leq\Pi_{t_s}\alpha_s(k)\Pi_{t_u}\alpha_u(k)\beta_s^{2d_s}\beta_u^{2d_u}V(k_0)\\ &\leq\alpha_1^{t_s}\alpha_2^{t_u}\beta_s^{2d_s}\beta_u^{2d_u}V(k_0),\end{aligned}\tag{23}$$

where the symbol "Π" represents the successively multiplying $\alpha_s(k)$ and $\alpha_u(k)$, in the time order k. Next, four cases will be considered based on different choices of α_1 and β.

1. If $0<\alpha_1<1$, we want prove that $V(k)\leq\gamma^{(k-k_0)}V(k_0)$, where $\gamma=\max\{\alpha,\beta\}$ with predesigned positive constants α,β satisfying $\alpha_1<\alpha<1, \beta_s<\beta<1$. Then the following two conditions need to hold.

$$\alpha_1^{t_s}\alpha_2^{t_u}\leq\alpha^{t_s+t_u}, \beta_s^{2d_s}\beta_u^{2d_u}\leq\beta^{2(d_s+d_u)}.$$

Simple deduction follows that $(\frac{\alpha_2}{\alpha})^{t_u} \le (\frac{\alpha}{\alpha_1})^{t_s}$, $(\frac{\beta_u}{\beta})^{d_u} \le (\frac{\beta}{\beta_s})^{d_s}$, which implies that

$$\frac{t_u}{t_s} \le \frac{\ln\alpha - \ln\alpha_1}{\ln\alpha_2 - \ln\alpha}, \frac{d_u}{d_s} \le \frac{\ln\beta - \ln\beta_s}{\ln\beta_u - \ln\beta}.$$

2. If $0 < \alpha_1 < 1$, then proceed with similar proof to case (1) for any given scalar β satisfying $\beta_s^2 < \beta \le \beta_s$. We can obtain that $V(k) \le \gamma^{(k-k_0)}V(k_0)$.
3. If $\alpha_1 \ge 1$, then $V(k) \le \alpha_2^{t_s+t_u}\beta_s^{2d_s}\beta_u^{2d_u}V(k_0) \le \alpha_2^t\beta_s^{2d_s}\beta_u^{2d_u}V(k_0)$. For any given positive scalar β satisfying $\beta_s < \beta < 1$, to prove that $V(k) \le \beta^{(k-k_0)}V(k_0)$ needs to prove the following inequality

$$\alpha_2^t\beta_s^{2d_s}\beta_u^{2d_u} \le \beta^{t+2(d_s+d_u)},$$

If $(\max\{\frac{\alpha_2}{\beta}, \frac{\beta_u}{\beta}\})^{t+2d_u} \le (\frac{\beta}{\beta_s})^{2d_s}$, then $V(k) \le \beta^{(k-k_0)}V(k_0)$. This inequality implies

$$\frac{t+2d_u}{2d_s} \le \frac{\ln\beta - \ln\beta_s}{\max\{\ln\alpha_2 - \ln\beta, \ln\beta_u - \ln\beta\}},$$

which coincides with the condition (21).
4. Similar argument to case (3), we can easily verify that condition (22) leads to $V(k) \le \beta^{(k-k_0)}V(k_0)$.

In conclusion, under the above dwell time design, the drive system (11) and the response system (12) are globally exponentially synchronized by the hybrid switching and impulsive control. This completes the proof.

Remark 3. By the hybrid switching and impulsive control technique, we can flexibly design the control gain matrices to satisfy practical requirements such as convergence rate and control cost. The control gain matrices can be chosen such that all matrices A_{i_k}, C_{j_k} are Schur stable at the expense of high control cost, which will be presented in Corollary 3 below. In view of this, the proposed control design has the advantage that not all matrices A_{i_k}, C_{j_k} are necessarily Schur stable.

Corollary 3. When B_{1k} and B_{2k} are chosen such that all of the matrices A_{i_k} and C_{j_k} are Schur stable and $\alpha_1 > 1$, if the Lie algebra $\{A_{i_k}, C_{j_k} : i_k \in \{1,2,\ldots,l\}, j_k \in \{1,2,\ldots,m\}\}_{LA}$ is solvable, then Eq. (23) is reduced to the following concise form

$$V(k+1) \le \alpha_1^t \cdot \beta_s^{2d}V(k_0).$$

For any give scalar β satisfying $\beta_s < \beta < 1$ ($\beta_s^2 < \beta \le \beta_s$), if $\frac{t}{d} \le 2\cdot\frac{\ln\beta - \ln\beta_s}{\ln\alpha_1 - \ln\beta}$ ($\frac{t}{d} \le \frac{\ln\beta - 2\ln\beta_s}{\ln\alpha_1 - \ln\beta}$), systems (11) and (12) are exponentially synchronized by the hybrid switching and impulsive control.

V. NUMERICAL SIMULATION AND DISCUSSION

In this section, a numerical example on the synchronization of nonlinear chaotic systems by the hybrid control is presented. Compared with the switching control and impulsive control, the hybrid control achieves good performance.

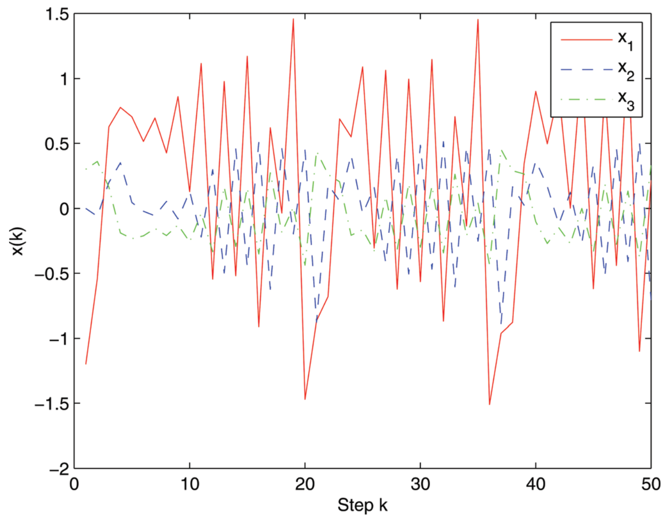

FIG. 1. (Color online) System trajectory of Hénon discrete chaotic system (24).

Consider 3-dimensional generalized discrete chaotic system with Hénon map[9]

$$\begin{cases} x_1(k+1) = 1 - ax_1^2(k) + x_2(k) \\ x_2(k+1) = bx_1(k) + x_3(k) \\ x_3(k+1) = -bx_1(k). \end{cases} \quad (24)$$

When a = 1.07 and b = 0.3, the chaotic behavior of system (24) is presented in Fig. 1.

Rewrite the system (24) as

$$x(k+1) = Ax(k) + f(x(k)), \quad (25)$$

where $x = (x_1, x_2, x_3)^\top$, $f(x) = [1 - ax_1^2(k), 0, 0]^\top$, and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0.3 & 0 & 1 \\ -0.3 & 0 & 0 \end{bmatrix}$.

Regarding Eq. (25) as a drive system, under hybrid switching and impulsive control (13), the corresponding response system becomes

$$\begin{cases} y(k+1) = Ay(k) + f(k,y) + B_{1i_k}[y(k) - x(k)], \quad k \ne \tau_k, \\ y(k+1) = Ay(k) + B_{2j_k}[y(k) - x(k)] + f(k,x), k = \tau_k, k \in \mathbb{Z}, \\ y(k_0) \quad = y_0, \end{cases} \quad (26)$$

with B_{1i_k} and B_{2j_k} being 3×3 matrices, $B_{1i_k} \in \{B_{11}, B_{12}, \ldots, B_{1l}\}$, $B_{2j_k} \in \{B_{21}, B_{22}, \ldots, B_{2m}\}$, and $\tau_k \to \infty$. Thus, from Eqs. (25) and (26), the synchronization error system is a discrete switched impulsive system

$$\begin{cases} e(k+1) = (A + B_{1i_k})e(k) + f(k,y) - f(k,x), \quad k \ne \tau_k, \\ e(k+1) = (A + B_{2j_k})e(k), \quad k = \tau_k, \\ e(k_0) \quad = e_0, \end{cases} \quad (27)$$

where $e(k) = y(k) - x(k)$ is the synchronization error and we denote $C_{j_k} = (A + B_{2j_k})$.

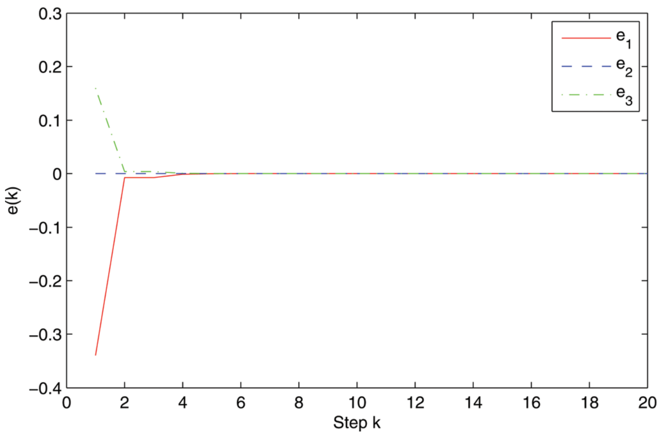

FIG. 2. (Color online) Synchronization error under the hybrid control.

From Ref. 25, we know that $\|f(y)-f(x)\| \le L\|y-x\|$, with $L=4.2$. If we choose $l=m=2$, B_{1i}, B_{2i}, $A_i=A+B_{1i}$, and $C_i=A+B_{2i}$, $i=1,2$ as follows:

$$B_{11}=\begin{bmatrix}0.12 & -1.18 & 0\\ -0.3 & 0.15 & -1.18\\ 0.3 & 0 & 0.18\end{bmatrix},\quad B_{12}=\begin{bmatrix}1.17 & -2.04 & 0\\ -0.3 & 1.04 & -2.17\\ 0.3 & 0 & 1.105\end{bmatrix},$$

$$B_{21}=\begin{bmatrix}0.15 & -1 & 0\\ -0.3 & 0.12 & -1\\ 0.3 & 0 & 0.15\end{bmatrix},\quad B_{22}=\begin{bmatrix}1.04 & -1 & 0\\ -0.3 & 1.17 & -1\\ 0.3 & 0 & 1.04\end{bmatrix},$$

$$A_1=\begin{bmatrix}0.12 & -0.18 & 0\\ 0 & 0.15 & -0.18\\ 0 & 0 & 0.18\end{bmatrix},\quad A_2=\begin{bmatrix}1.17 & -1.04 & 0\\ 0 & 1.04 & -1.17\\ 0 & 0 & 1.105\end{bmatrix},$$

$$C_1=\begin{bmatrix}0.15 & 0 & 0\\ 0 & 0.12 & 0\\ 0 & 0 & 0.15\end{bmatrix},\quad C_2=\begin{bmatrix}1.04 & 0 & 0\\ 0 & 1.17 & 0\\ 0 & 0 & 1.04\end{bmatrix}.$$

It is clear that matrices A_1 and C_1 are Schur stable, and A_2 and C_2 are unstable.

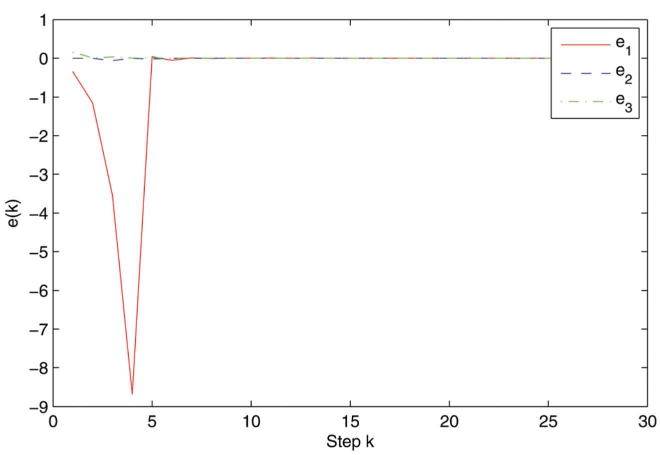

FIG. 3. (Color online) Synchronization error under the switch control.

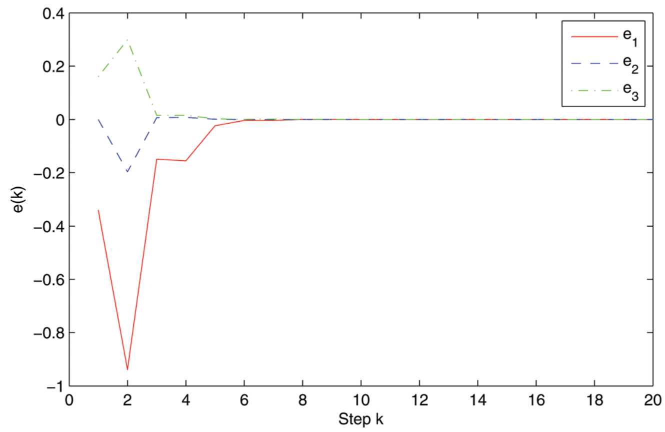

FIG. 4. (Color online) Synchronization error under the impulsive control.

Some standard Lie brackets are computed as follows:

$$[A_1,A_2]=\begin{bmatrix}0 & 0.0546 & 0.0234\\ 0 & 0 & 0.0234\\ 0 & 0 & 0\end{bmatrix},$$

$$[A_1,C_1]=\begin{bmatrix}0 & 0.0054 & 0\\ 0 & 0 & -0.0054\\ 0 & 0 & 0\end{bmatrix},$$

$$[A_1,C_2]=\begin{bmatrix}0 & -0.0234 & 0\\ 0 & 0 & 0.0234\\ 0 & 0 & 0\end{bmatrix},$$

$$[A_2,C_2]=\begin{bmatrix}0 & -0.1352 & 0\\ 0 & 0 & 0.1521\\ 0 & 0 & 0\end{bmatrix},$$

$$[[A_1,A_2],[A_1,C_1]]=\begin{bmatrix}0 & 0 & -0.4212\times 10^{-3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$[[A_1,A_2],[A_2,C_1]]=\begin{bmatrix}0 & 0 & -0.0026\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

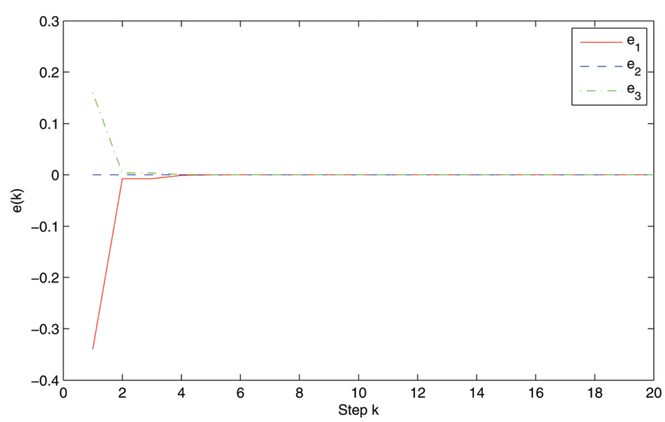

FIG. 5. (Color online) Synchronization error systems under the hybrid control.

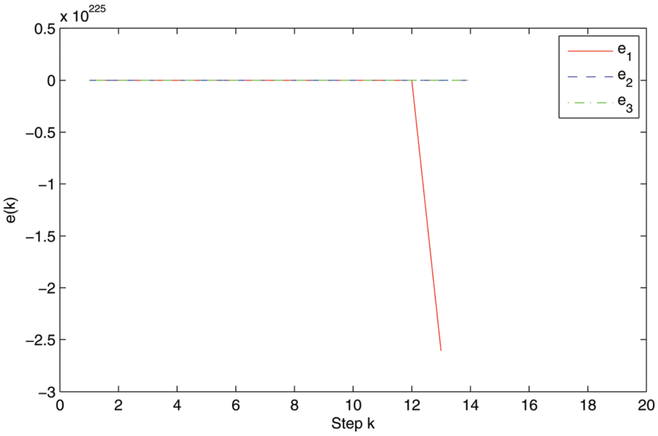

FIG. 6. (Color online) Synchronization fails under the switching control.

Further computation shows that the Lie algebra $\{A_1, A_2, C_1, C_2\}_{LA}$ is solvable with k = 4.

Choose $\beta_s = 0.3$ and $\beta_u = 1.3$, then $\frac{A_1}{\beta_s}$, $\frac{A_2}{\beta_u}$, $\frac{C_1}{\beta_s}$, and $\frac{C_2}{\beta_u}$ are Schur stable. By solving LMIs, we can obtain a positive definite matrix P satisfying Eqs. (3) and (4) as follows:

$$P = \begin{bmatrix} 0.2891 & 0.0416 & -0.0137 \\ 0.0416 & 0.2898 & 0.0509 \\ -0.0137 & 0.0509 & 0.3357 \end{bmatrix}.$$

It is easy to get $\rho = 1.6288$, $\lambda = 1.9722$, and $\beta_u^2 + 2L\lambda\rho + \rho L^2 < 57.4060$. Given $\beta = 0.9$, according to Theorem 2, under dwell time scheme $\frac{t+2d_u}{2d_s} \leq 0.2644$, the chaotic drive system (24) and the controlled response system (26) are globally exponentially synchronized using hybrid switching and impulsive control. The synchronization performance by hybrid control under the dwell time $\frac{t+2d_u}{2d_s} = 0.2$ is illustrated in Fig. 2 for initial states $x(0) = [-1.2, 0, 0.3]^\top$ and $y(0) = [-1.54, 0, 0.46]^\top$. Figs. 3 and 4 show the synchronization performances by switching control and impulsive control, respectively. It can be found that the hybrid control performs better than switching control and impulsive control.

Moreover, if we choose another set of B_{11}, B_{12}, B_{21}, and B_{22} as follows:

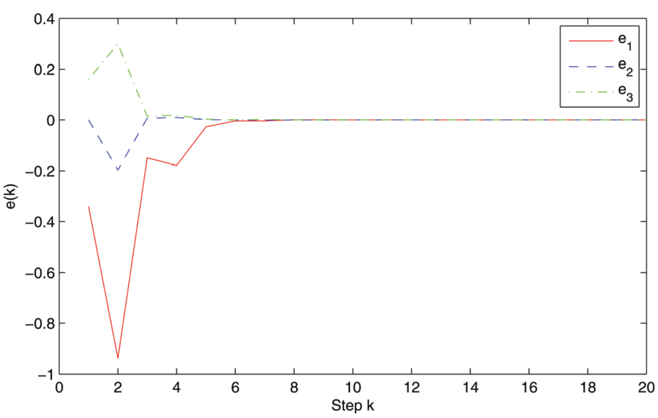

FIG. 7. (Color online) Synchronization error under the impulsive control.

$$B_{11} = \begin{bmatrix} 0.12 & -1.18 & 0 \\ -0.3 & 0.15 & -1.18 \\ 0.3 & 0 & 0.18 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 1.35 & -2.2 & 0 \\ -0.3 & 1.2 & -2.35 \\ 0.3 & 0 & 1.2 \end{bmatrix},$$

$$B_{21} = \begin{bmatrix} 0.15 & -1 & 0 \\ -0.3 & 0.12 & -1 \\ 0.3 & 0 & 0.15 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 1.2 & -1 & 0 \\ -0.3 & 1.35 & -1 \\ 0.3 & 0 & 1.2 \end{bmatrix}.$$

Denote the parameter matrices of the error system (27) $A_i = A + B_{1i}$, $C_i = A + B_{2i}$, and $i = 1, 2$. It is clear that A_1 and C_1 are Schur stable, and A_2 and C_2 are unstable. Similar computation on Lie bracket shows that the Lie algebra $\{A_1, A_2, C_1, C_2\}_{LA}$ is solvable with k = 4. Choose $\beta_s = 0.3$ and $\beta_u = 1.5$. Given $\beta = 0.9$, it follows from similar process that under the dwell time $\frac{t+2d_u}{2d_s} \leq 0.2641$, the hybrid control can accomplish the synchronization for $x(0) = [-1.2, 0, 0.3]^\top$ and $y(0) = [-1.54, 0, 0.46]^\top$, which is shown in Fig. 5. However, Fig. 6 illustrates that the switching control fails to achieve synchronization, and the impulsive control performance is illustrated in Fig. 7. From the above discussion, we can see that the hybrid control performs better than switching control and impulsive control in achieving synchronization and it can be designed flexibly.

VI. CONCLUSION

This paper has studied the issue on the global exponential stability of discrete switched impulsive systems and its application to hybrid synchronization. By a Lie algebraic condition together with the dwell time design, the explicit stability criteria of discrete switched impulsive system have been derived. Based on the Lie algebraic condition, a new hybrid switching and impulsive control strategy for nonlinear synchronization problem has been proposed. Our results are more general than the existing results and the conditions are easy to check. A typical illustrative example of synchronization for chaotic system has demonstrated the effective control performance.

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[1]J. Sun and H. Lin, Chaos **18**, 033127 (2008).
[2]Z. Ji, Z. Wang, H. Lin, and Z. Wang, Int. J. Control **83**, 371 (2010).
[3]D. Liberzon, J. P. Hespanha, and A. S. Morse, Syst. Control Lett. **37**, 117 (1999).
[4]G. Zhai, X. Xu, H. Lin, and D. Liu, Int. J. Appl. Math Comput. Sci. **17**, 447 (2007).
[5]Z. Sun and S. S. Ge, Automatica **41**, 181 (2005).
[6]B. Liu and D. J. Hill, Int. J. Control **82**, 228 (2009).
[7]Y. Zhang, J. T. Sun, and G. Feng, IEEE Trans. Autom. Control **54**, 871 (2009).
[8]H. Xu, Y. Chen, and K. L. Teo, Appl. Math. Comput. **217**, 537 (2010).
[9]H. Xu and K. L. Teo, Phys. Lett. A **374**, 235 (2009).

[10]W. A. Zhang and L. Yu, Automatica **45**, 2265 (2009).
[11]E. Liz and D. Franco, Chaos **20**, 023124 (2010).
[12]M. S. Peng and X. Z. Yang, Chaos **20**, 013125 (2010).
[13]Z. H. Guan and N. Liu, Chaos **20**, 013135 (2010).
[14]S. L. Li, X. Z. Liu, and L. J. Wang, Dyn. Contin. Discrete Impulsive Syst.: Ser. A - Math. Anal. **13A**, 2101 (2006).
[15]Z. Zhang, Nonlinear Anal. Theory, Methods Appl. **71**, 2790 (2009).
[16]S. W. Zhao and J. T. Sun, Int. J. Bifurcation Chaos **19**, 379 (2009).
[17]H. Lin and P. J. Antsaklis, Int. J. Control **81**, 1114 (2008).
[18]Z. H. Guan, D. J. Hill, and J. Yao, Int. J. Bifurcation Chaos **16**, 229 (2006).
[19]X. M. Sun, G. P. Liu, D. Rees, and W. Wang, Automatica **44**, 2902 (2008).
[20]Y. Zhao, C. Wang, X. Han, and Z. Wang, "T-S fuzzy modeling and asymptotic synchronization of two nonidentical discrete-time hyperchaotic maps," in *International Conference on Mechatronics and Automation, ICMA* (IEEE, 2009), Jilin, China, pp. 1244–1248.
[21]H. K. Lam, Phys. Lett. A **374**, 552 (2010).
[22]J. Chen, J. Lu, X. Wu, and W. X. Zheng, Chaos **19**, 043119 (2009).
[23]G. Zhang, Z. Liu, and Z. Ma, Chaos **17**, 043126 (2007).
[24]C. Li, X. Liao, and X. Zhang, Chaos **15**, 023104 (2005).
[25]L. Zhang, H. Jiang, and Q. Bi, Nonlinear Dyn. **59**, 529 (2010).