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Depth of fiber cones of ideals with almost minimal mixed multiplicity ☆

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ABSTRACT

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$, I an $\mathfrak{m}$-primary ideal with almost minimal mixed multiplicity such that depth $G(I) \geqslant d-1$. We show that $F_{\mathfrak{m}}(I)$ has almost maximal depth (i.e. depth $F_{\mathfrak{m}}(I) \geqslant d-1$).

1. Introduction

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$ having infinite residue field and I an $\mathfrak{m}$-primary ideal of R. The fiber cone of I is the standard graded algebra $F_{\mathfrak{m}}(I) = \bigoplus_{n \geqslant 0} I^n/\mathfrak{m}I^n$ and $G(I) = \bigoplus_{n \geqslant 0} I^n/I^{n+1}$ is the associated graded ring of I. Let $\mu(I) := \lambda(I/\mathfrak{m}I)$ (where λ denotes the length function) denote the minimum number of generators of an ideal I. The Hilbert polynomial of $H_{\mathfrak{m}}(I, n) := \sum_{j=0}^{n} \mu(I^j)$ is denoted by $P_{\mathfrak{m}}(I, n)$ and write

$$P_{\mathfrak{m}}(I, n) = f_0(I)\binom{n+d-1}{d} - f_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d f_d(I).$$

We call $f_i(I)$ the ith fiber coefficient of $F_{\mathfrak{m}}(I)$.

In this paper, we are interested in the depth of $F_{\mathfrak{m}}(I)$.

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In order to state the main theorem of this paper, we recall some necessary definitions first. Bhattacharya in [2] proved that for large values of r and s, the function $\lambda(R/\mathfrak{m}^r I^s)$ is given by a polynomial $P(r,s)$ of total degree d in r and s, we can write this polynomial $P(r,s)$ in the form:

$$P(r,s) = \sum_{i+j\leqslant d} e_{ij}(\mathfrak{m}|I)\binom{r+i}{i}\binom{s+j}{j},$$

where $e_{ij}(\mathfrak{m}|I)$ are certain integers. When $i+j=d$, we set $e_{ij}(\mathfrak{m}|I)=e_j(\mathfrak{m}|I)$ for $j=0,\dots,d$. In this case, these integers are called the mixed multiplicities of $\mathfrak{m}$ and I.

An ideal $J\subseteq I$ is called a reduction of I if there exists a positive integer n such that $I^{n+1}=JI^n$. A multiset of ideals consisting of j copies of I and $d-j$ copies of $\mathfrak{m}$ is denoted by $(I^{[j]}|\mathfrak{m}^{[d-j]})$. Rees in [12] introduced joint reductions to calculate mixed multiplicities. A sequence of elements $a_1,\dots,a_{d-1}\in I$, $a_d\in\mathfrak{m}$ is called a joint reduction of the multiset of ideals $(I^{[d-1]}|\mathfrak{m})$ if the ideal $(a_1,\dots,a_{d-1})\mathfrak{m}+a_d I$ is a reduction of $I\mathfrak{m}$.

We now describe the contents of the paper. For a Cohen–Macaulay local ring $(R,\mathfrak{m})$, the 'Abhyankar–Sally' equality gives that $e(\mathfrak{m})=\mu(\mathfrak{m})-d+1+\lambda(\frac{\mathfrak{m}^2}{J\mathfrak{m}})$, where J is a minimal reduction of $\mathfrak{m}$ and $e(.)$ is the Hilbert–Samuel multiplicity. Rossi and Valla in [14], and H.-J. Wang independently in [15] proved that if J is a minimal reduction of $\mathfrak{m}$ in a Cohen–Macaulay local ring $(R,\mathfrak{m})$ such that $\lambda(\mathfrak{m}^2/J\mathfrak{m})=1$, then depth $G(\mathfrak{m})\geqslant d-1$. Later Rossi extended this result to $\mathfrak{m}$-primary ideals in [13]. She showed that if I is an $\mathfrak{m}$-primary ideal with a minimal reduction J such that $\lambda(I^2/JI)=1$, then depth $G(I)\geqslant d-1$. Jayanthan and Verma in [10] proved that if I is an $\mathfrak{m}$-primary ideal with almost minimal multiplicity (i.e. $\lambda(\mathfrak{m}I/\mathfrak{m}J)=1$ for any minimal reduction J of I) and depth $G(I)\geqslant d-2$, then depth $F_{\mathfrak{m}}(I)\geqslant d-1$. In Section 4, we prove that if I is an ideal with almost minimal mixed multiplicity and depth $G(I)\geqslant d-1$, then depth $F_{\mathfrak{m}}(I)\geqslant d-1$. Our general references for the paper are [1,7–9,11].

2. Preliminaries

An element $a\in I$ is called Rees-superficial for I and $\mathfrak{m}$ if there exists a positive integer r_0 such that for all $r\geqslant r_0$ and all $s\geqslant 0$, $aR\cap I^r\mathfrak{m}^s=aI^{r-1}\mathfrak{m}^s$. A sequence of elements $a_1,\dots,a_{d-1}\in I$, $a_d\in\mathfrak{m}$ is called a Rees-superficial sequence for I and $\mathfrak{m}$ if for all $i=1,\dots,d$, $\overline{a_i}$ is superficial for $\bar{I}$ and $\bar{\mathfrak{m}}$, where "$\bar{\ }$" denotes residue classes in $R/(a_1,\dots,a_{i-1})$. In this case, $(a_1,\dots,a_d)$ is a joint reduction of $(I^{[d-1]}|\mathfrak{m})$ and $e_{d-1}(\mathfrak{m}|I)=\lambda(R/(a_1,\dots,a_d))$ by [12]. In particular, if $a_1,\dots,a_d\in I$ is an R-regular sequence, $e_{d-1}(\mathfrak{m}|I)=e(I)$.

D'Cruz, Raghavan and Verma in [5] showed that for an $\mathfrak{m}$-primary ideal I in a Cohen–Macaulay local ring $(R,\mathfrak{m})$, $e_{d-1}(\mathfrak{m}|I)=\mu(I)-d+1+\lambda(\frac{\mathfrak{m}I}{(a_1,\dots,a_{d-1})\mathfrak{m}+a_dI})$, where $(a_1,\dots,a_d)$ is a joint reduction of $(I^{[d-1]}|\mathfrak{m})$. It follows that $e_{d-1}(\mathfrak{m}|I)\geqslant\mu(I)-d+1$ and the equality occurs if and only if $\mathfrak{m}I=(a_1,\dots,a_{d-1})\mathfrak{m}+a_dI$.

We say that I has minimal mixed multiplicity if $e_{d-1}(\mathfrak{m}|I)=\mu(I)-d+1$ and I has almost minimal mixed multiplicity if $e_{d-1}(\mathfrak{m}|I)=\mu(I)-d+2$ (i.e. $\lambda(\frac{\mathfrak{m}I}{(a_1,\dots,a_{d-1})\mathfrak{m}+a_dI})=1$).

For $a\in I$, let a^* denote its initial form in the associated ring $G(I)$, and a^0 denote its initial form in the fiber cones $F_{\mathfrak{m}}(I)$.

The following lemmas were proved in [4,6,5].

Lemma 2.1. *There exist $a_1,\dots,a_{d-1}\in I$, $a_d\in\mathfrak{m}$ such that $a_1,\dots,a_d$ is a Rees-superficial sequence for I and $\mathfrak{m}$. Suppose that depth $G(I)\geqslant d-1$, we can choose the above $a_1,\dots,a_d$ such that $a_1^*,\dots,a_{d-1}^*$ is a $G(I)$-regular sequence.*

Lemma 2.2. *Let $a_1,\dots,a_{d-1}\in I$, $a_d\in\mathfrak{m}$ be a Rees-superficial sequence for I and $\mathfrak{m}$. Then*

$$f_0(I)=e_{d-1}(\mathfrak{m}|I)-\lim_{n\to\infty}\lambda\left(\frac{\mathfrak{m}I^n}{(a_1,\dots,a_{d-1})\mathfrak{m}I^{n-1}+a_dI^n}\right).$$

Definition 2.3. (See [6, Definition 1.2].) Let $L = (a_1, \ldots, a_d)$ be a joint reduction of $(I^{[d-1]}|\mathfrak{m})$. If there exists an integer n such that $\mathfrak{m}I^n = (a_1, \ldots, a_{d-1})\mathfrak{m}I^{n-1} + a_d I^n$, define $r_L(I|\mathfrak{m})$ to be the smallest such n, otherwise, $r_L(I|\mathfrak{m}) = \infty$. The smallest of all $r_L(I|\mathfrak{m})$ where J is varying is denoted by $r(I|\mathfrak{m})$.

If $f : \mathbb{Z} \to \mathbb{Z}$ is a function, let Δ denote the first difference function defined by $\Delta[f(n)] = f(n) - f(n-1)$, and let Δ^i be defined by $\Delta^i[f(n)] = \Delta^{i-1}[\Delta[f(n)]]$.

Let $a \in I$ be a Rees-superficial element for I and $\mathfrak{m}$, then for all large n, $H_{\overline{\mathfrak{m}}}(\bar{I}, n) = \Delta[H_{\mathfrak{m}}(I, n)]$. In particular, $f_i(\bar{I}) = f_i(I)$ for $i = 0, \ldots, d-1$, where "$\overline{}$" denote the image modulo (a).

Remark 2.4. Let $L = (a_1, \ldots, a_d)$ be a joint reduction of $(I^{[d-1]}|\mathfrak{m})$, and let "$\overline{}$" denote the image modulo (a_1). If $r_L(I|\mathfrak{m}) = \infty$. Then $r_{\bar{L}}(\bar{I}|\overline{\mathfrak{m}}) = \infty$.

Proof. Put $J = (a_1, \ldots, a_{d-1})$. If $r_{\bar{L}}(\bar{I}|\overline{\mathfrak{m}}) < \infty$, then there exists an integer n_0 such that $\overline{\mathfrak{m}}\bar{I}^{n_0} = \bar{J}\overline{\mathfrak{m}}\bar{I}^{n_0-1} + \overline{a_d}\bar{I}^{n_0}$. It follows that $\mathfrak{m}I^n \subseteq J\mathfrak{m}I^{n-1} + a_d I^n + (a_1)$ for all $n \geqslant n_0$. Again, as a_1 is a Rees-superficial element for I and $\mathfrak{m}$, there exists a positive integer n_1 such that $(a_1) \cap \mathfrak{m}I^n = a_1\mathfrak{m}I^{n-1}$ for all $n \geqslant n_1$. Thus for all $n \geqslant \max\{n_0, n_1\}$, we have $\mathfrak{m}I^n = \mathfrak{m}I^n \cap (J\mathfrak{m}I^{n-1} + a_d I^n + (a_1)) = J\mathfrak{m}I^{n-1} + a_d I^n + (a_1) \cap \mathfrak{m}I^n = J\mathfrak{m}I^{n-1} + a_d I^n$, contradicting the assumption that $r_L(I|\mathfrak{m}) = \infty$. $\square$

3. Bounds on reduction numbers

In this section, we will give a bound on the reduction number of an $\mathfrak{m}$-primary ideal. Furthermore, we use this bound to prove the almost maximal depth condition for fiber cone of an ideal with almost minimal mixed multiplicity.

Let $L = (a_1, \ldots, a_d)$ be a joint reduction of $(I^{[d-1]}|\mathfrak{m})$ and $J = (a_1, \ldots, a_{d-1})$.

We firstly consider the sequence of ideals $\{A_n\}_{n \geqslant 0}$ with $A_n = \bigcup_{k \geqslant 1}(\mathfrak{m}I^{n+k} : J^k)$, this filtration of ideals behaves quite similar to the Ratliff–Rush closure of an ideal. We summarize some of its properties.

Proposition 3.1.

(1) $A_n : J = A_{n-1}$ *for all* $n \geqslant 1$;
(2) $A_n = \bigcup_{k \geqslant 1}(\mathfrak{m}I^{n+k} : (a_1^k, \ldots, a_{d-1}^k))$ *for all* $n \geqslant 0$;
(3) *If* $\operatorname{grade}(I) > 0$, *then* $A_n = \mathfrak{m}I^n$ *for* $n \gg 0$.

Proof.

(1) Note that $\mathfrak{m}I^{n+1} : J \subseteq \mathfrak{m}I^{n+2} : J^2 \subseteq \ldots$ is an increasing chain of ideals of R, we get $A_n = \mathfrak{m}I^{n+k} : J^k$ for $k \gg 0$. It follows that for $k \gg 0$,

$$A_n : J = \left(\mathfrak{m}I^{n+k} : J^k\right) : J = \mathfrak{m}I^{n+k} : J^{k+1} = A_{n-1}.$$

(2) Let $(\underline{a}) = (a_1, \ldots, a_{d-1})$ and $(\underline{a})^{[k]} = (a_1^k, \ldots, a_{d-1}^k)$. Obviously $\mathfrak{m}I^{n+k} : J^k \subseteq \mathfrak{m}I^{n+k} : (\underline{a})^{[k]}$. Since R is a Noetherian ring, we have $\bigcup_{k \geqslant 1}(\mathfrak{m}I^{n+k} : (\underline{a})^{[k]}) = \mathfrak{m}I^{n+k} : (\underline{a})^{[k]}$ for $k \gg 0$. Let $z \in \mathfrak{m}I^{n+k} : (\underline{a})^{[k]}$ for $k \gg 0$ and $l \geqslant k(d-1)$. Then

$$\begin{aligned} zJ^l &= \sum_{\alpha_1 + \cdots + \alpha_{d-1} = l} z a_1^{\alpha_1} \ldots a_{d-1}^{\alpha_{d-1}} \\ &\subseteq \sum_{\alpha_1 + \cdots + \alpha_{d-1} = l} \mathfrak{m}I^{n+k} a_1^{\alpha_1} \ldots \widehat{a_i^{\alpha_i}} \ldots a_{d-1}^{\alpha_{d-1}} \quad \text{where } \alpha_i \geqslant k \\ &\subseteq \mathfrak{m}I^{n+l}. \end{aligned}$$

Therefore $z \in \mathfrak{m}I^{n+l} : J^l \subseteq A_n$.

(3) If $\operatorname{grade}(I) > 0$, then by Remark 6.6 of [4] there exists $a_1 \in I$ such that it is Rees-superficial for I and $\mathfrak{m}$ and it is also R-regular. Then $\mathfrak{m}I^{n+1} : a_1 = \mathfrak{m}I^n$ for $n \gg 0$. It follows that $\mathfrak{m}I^n \subseteq \mathfrak{m}I^{n+1} : J \subseteq \mathfrak{m}I^{n+1} : a_1 = \mathfrak{m}I^n$ for $n \gg 0$. Thus we can show by using induction on k that $\mathfrak{m}I^{n+k} : J^k = \mathfrak{m}I^n$ for $n \gg 0$. Therefore $A_n = \mathfrak{m}I^n$ for $n \gg 0$. $\square$

Write

$$P_{\mathfrak{m}}(I, n) = f_0'(I)\binom{n+d}{d} - f_1'(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d f_d'(I).$$

Then, comparing with the earlier notation, we get that $f_0'(I) = f_0(I)$ and $f_i'(I) = f_i(I) + f_{i-1}(I)$, $i = 1, \ldots, d$.

We provide a formula, in dimension 2, for the first fiber coefficient of $F_{\mathfrak{m}}(I)$. This formula is crucial for obtaining the bound on the reduction in Remark 3.5.

Theorem 3.2. *Let $d = 2$, $a_1 \in I$, $a_2 \in \mathfrak{m}$ a Rees-superficial sequence for I and $\mathfrak{m}$ such that a_1^* is a $G(I)$-regular element. Set $L = (a_1, a_2)$. If $r_L(I|\mathfrak{m}) < \infty$. Then*

$$f_1(I) = \sum_{n \geqslant 1} \lambda\big(A_n/(a_1 A_{n-1} + a_2 I^n)\big) - \lambda\left(\frac{R}{A_0}\right).$$

Proof. Consider the exact sequence:

$$0 \to \frac{R}{(I^n : a_1) \cap (A_{n-1} : a_2)} \xrightarrow{\psi} \frac{R}{I^n} \oplus \frac{R}{A_{n-1}} \xrightarrow{\phi} \frac{(a_1, a_2)}{a_2 I^n + a_1 A_{n-1}} \to 0$$

where $\psi(\bar{r}) = (\overline{a_1 r}, \overline{-a_2 r})$ and $\phi(\bar{r}, \bar{s}) = \overline{r a_2 + s a_1}$. It follows that for all $n \geqslant 1$,

$$\begin{aligned} \lambda\left(\frac{R}{I^n}\right) + \lambda\left(\frac{R}{A_{n-1}}\right) &= \lambda\left(\frac{R}{(I^n : a_1) \cap (A_{n-1} : a_2)}\right) + \lambda\left(\frac{(a_1, a_2)}{a_2 I^n + a_1 A_{n-1}}\right) \\ &= \lambda\left(\frac{R}{(I^n : a_1) \cap (A_{n-1} : a_2)}\right) + \lambda\left(\frac{R}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{R}{(a_1, a_2)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} e_1(\mathfrak{m}|I) - \lambda\left(\frac{I^n}{\mathfrak{m}I^n}\right) + \lambda\left(\frac{I^{n-1}}{\mathfrak{m}I^{n-1}}\right) &= \lambda\left(\frac{R}{(I^n : a_1) \cap (A_{n-1} : a_2)}\right) + \lambda\left(\frac{R}{a_2 I^n + a_1 A_{n-1}}\right) \\ &\quad - \lambda\left(\frac{R}{I^n}\right) - \lambda\left(\frac{R}{A_{n-1}}\right) - \lambda\left(\frac{I^n}{\mathfrak{m}I^n}\right) + \lambda\left(\frac{I^{n-1}}{\mathfrak{m}I^{n-1}}\right) \\ &= \lambda\left(\frac{R}{(I^n : a_1) \cap (A_{n-1} : a_2)}\right) + \lambda\left(\frac{R}{A_n}\right) + \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) \\ &\quad - \lambda\left(\frac{R}{A_{n-1}}\right) - \lambda\left(\frac{R}{\mathfrak{m}I^n}\right) + \lambda\left(\frac{R}{\mathfrak{m}I^{n-1}}\right) - \lambda\left(\frac{R}{I^{n-1}}\right) \\ &= \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{A_n}{\mathfrak{m}I^n}\right) + \lambda\left(\frac{A_{n-1}}{\mathfrak{m}I^{n-1}}\right) \\ &\quad - \lambda\left(\frac{(I^n : a_1) \cap (A_{n-1} : a_2)}{I^{n-1}}\right). \end{aligned}$$

Since $r_L(I|\mathfrak{m}) < \infty$, we have that $f_0(I) = e_1(\mathfrak{m}|I)$ by Lemma 2.2 and notice that $\Delta^2[P_\mathfrak{m}(I,n)] = f_0(I)$. It follows that

$$\Delta^2\big[P_\mathfrak{m}(I,n) - H_\mathfrak{m}(I,n)\big] = \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{A_n}{\mathfrak{m} I^n}\right) + \lambda\left(\frac{A_{n-1}}{\mathfrak{m} I^{n-1}}\right) - \lambda\left(\frac{(I^n : a_1) \cap (A_{n-1} : a_2)}{I^{n-1}}\right).$$

As a_1^* is a $G(I)$-regular element, we have $I^n : a_1 = I^{n-1}$ for all $n \geqslant 1$. Hence for all $n \geqslant 1$,

$$\Delta^2\big[P_\mathfrak{m}(I,n) - H_\mathfrak{m}(I,n)\big] = \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{A_n}{\mathfrak{m} I^n}\right) + \lambda\left(\frac{A_{n-1}}{\mathfrak{m} I^{n-1}}\right).$$

Write $P_\mathfrak{m}(I,n) = f_0'(I)\binom{n+2}{2} - f_1'(I)(n+1) + f_2'(I)$, we have $\sum_{n\geqslant 0} \Delta^2[P_\mathfrak{m}(I,n)]t^n = \frac{f_0(I)}{(1-t)}$. Let $\sum_{n\geqslant 0} H_\mathfrak{m}(I,n)t^n = \frac{f(t)}{(1-t)^3}$. Then $f_1'(I) = f'(1)$ by Proposition 4.1.9 of [3].

Note that $H_\mathfrak{m}(I,n) = 1$ for all $n \leqslant 0$. We have that

$$\begin{aligned}
\frac{f_0(I) - f(t)}{(1-t)} &= \sum_{n\geqslant 0} \Delta^2\big[P_\mathfrak{m}(I,n)\big]t^n - \big(1 - 2t + t^2\big)\sum_{n\geqslant 0} H_\mathfrak{m}(I,n)t^n \\
&= \sum_{n\geqslant 0} \Delta^2\big[P_\mathfrak{m}(I,n)\big]t^n - \sum_{n\geqslant 0} \Delta^2\big[H_\mathfrak{m}(I,n)\big]t^n - 2H_\mathfrak{m}(I,-1) \\
&\quad + H_\mathfrak{m}(I,-2) + tH_\mathfrak{m}(I,-1) \\
&= \sum_{n\geqslant 0} \Delta^2\big[P_\mathfrak{m}(I,n) - H_\mathfrak{m}(I,n)\big]t^n - (1-t).
\end{aligned}$$

Set $v_n = \Delta^2[P_\mathfrak{m}(I,n) - H_\mathfrak{m}(I,n)]$, we have that

$$\begin{aligned}
v_0 &= \Delta^2\big[P_\mathfrak{m}(I,0) - H_\mathfrak{m}(I,0)\big] = \Delta^2\big[P_\mathfrak{m}(I,0)\big] - \Delta^2\big[H_\mathfrak{m}(I,0)\big] = f_0(I) - 1, \\
v_n &= \Delta^2\big[P_\mathfrak{m}(I,n) - H_\mathfrak{m}(I,n)\big] = \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{A_n}{\mathfrak{m} I^n}\right) + \lambda\left(\frac{A_{n-1}}{\mathfrak{m} I^{n-1}}\right).
\end{aligned}$$

Therefore $f_0(I) - f(t) = (1-t)\sum_{n\geqslant 0} v_n t^n - (t^2 - 2t + 1)$ and hence $f(t) = f_0(I) - (1-t)\sum_{n\geqslant 0} v_n t^n + t^2 - 2t + 1$. It follows that

$$f'(t) = \sum_{n\geqslant 0} v_n t^n - (1-t)\sum_{n\geqslant 0} n v_n t^{n-1} + 2t - 2.$$

Hence

$$\begin{aligned}
f_1'(I) = f'(1) &= \sum_{n\geqslant 0} v_n = v_0 + \sum_{n\geqslant 1} v_n \\
&= f_0(I) - 1 + \lambda\left(\frac{A_1}{a_2 I + a_1 A_0}\right) - \lambda\left(\frac{A_1}{\mathfrak{m} I}\right) + \lambda\left(\frac{A_0}{\mathfrak{m}}\right) \\
&\quad + \cdots + \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{A_n}{\mathfrak{m} I^n}\right) + \lambda\left(\frac{A_{n-1}}{\mathfrak{m} I^{n-1}}\right) + \cdots
\end{aligned}$$

$$= f_0(I) + \sum_{n\geqslant 1} \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{R}{A_0}\right).$$

It follow that

$$f_1(I) = f_1'(I) - f_0(I) = \sum_{n\geqslant 1} \lambda\left(\frac{A_n}{a_2 I^n + a_1 A_{n-1}}\right) - \lambda\left(\frac{R}{A_0}\right). \qquad \square$$

Let $R(I) = \bigoplus_{n\geqslant 0} I^n t^n$ denote the Rees algebra of I. For an $R(I)$-module M, put $Ann_{I^\nu}(M) = \{x \in I^\nu \mid xt^\nu M = 0\}$.

Lemma 3.3. *(See [13].) Let I and J be ideals of a Noetherian local ring R with $J \subseteq I$, M an $R(I)$-module of finite length as R-module. Let ν be the minimum number of generators of $M/R(J)_+M$ as an R-module. Then*

$$I^\nu = JI^{\nu-1} + Ann_{I^\nu}(M).$$

We now give a bound for the reduction number of an $\mathfrak{m}$-primary ideal.

Theorem 3.4. *Let $a_1, \dots, a_{d-1} \in I$, $a_d \in \mathfrak{m}$ be a Rees-superficial sequence for I and $\mathfrak{m}$. Put $L = (a_1, \dots, a_d)$ and $J = (a_1, \dots, a_{d-1})$. If $r_L(I|\mathfrak{m}) < \infty$. Then*

$$r_L(I|\mathfrak{m}) \leqslant \sum_{j\geqslant 1} \lambda\left(\frac{A_j}{JA_{j-1} + a_d I^j}\right) - \lambda\left(\frac{R}{A_0}\right) + 2.$$

Proof. Let $M := \bigoplus_{n\geqslant 0} A_n/\mathfrak{m}I^n$. Then M is a finitely generated $R(I)$-module and $\lambda_R(M) < \infty$ by Proposition 3.1(3). For $j \geqslant 1$, $[\frac{M}{R(J)_+M}]_j = M_j/J^jM_0 + J^{j-1}M_1 + \cdots + JM_{j-1}$ and $[\frac{M}{R(J)_+M}]_0 = \frac{A_0}{\mathfrak{m}}$. For $1 \leqslant i \leqslant j$ and $k \gg 0$, we have

$$\begin{aligned} J^iM_{j-i} &= JJ^{i-1}M_{j-i} \\ &= \frac{JJ^{i-1}\bigcup_{k\geqslant 1}(\mathfrak{m}I^{j-i+k} : J^k) + \mathfrak{m}I^j}{\mathfrak{m}I^j} \\ &\subseteq \frac{J\bigcup_{k\geqslant 1}(\mathfrak{m}I^{j-1+k} : J^k) + \mathfrak{m}I^j}{\mathfrak{m}I^j} \\ &\subseteq \frac{JA_{j-1} + \mathfrak{m}I^j}{\mathfrak{m}I^j} = JM_{j-1}. \end{aligned}$$

Therefore $[\frac{M}{R(J)_+M}]_j \cong A_j/JA_{j-1} + \mathfrak{m}I^j$. We have

$$\lambda\big(A_j/JA_{j-1} + \mathfrak{m}I^j\big) \leqslant \lambda\big(A_j/JA_{j-1} + a_dI^j\big)$$

and equality occurs if and only if $\mathfrak{m}I^j \subseteq JA_{j-1} + a_dI^j$. Since $r_L(I|\mathfrak{m}) < \infty$, there exists an integer n such that $\mathfrak{m}I^n = (a_1, \dots, a_{d-1})\mathfrak{m}I^{n-1} + a_dI^n \subseteq JA_{n-1} + a_dI^n$. Let $k = \min\{j \mid \mathfrak{m}I^j \subseteq JA_{j-1} + a_dI^j\}$, μ_j the minimum number of generators of $[\frac{M}{R(J)_+M}]_j$ as an R-module. Then, for $j \geqslant 1$, $\mu_j \leqslant \lambda(A_j/JA_{j-1} + \mathfrak{m}I^j)$ and $\mu_0 \leqslant \lambda(\frac{A_0}{\mathfrak{m}})$. Let $\mu = \sum_{j\geqslant 0} \mu_j$. Then by Lemma 3.3, $I^\mu = JI^{\mu-1} + Ann_{I^\mu}(M)$. Therefore

$$\begin{aligned}\mathfrak{m}I^{\mu+k} &= \mathfrak{m}I^kI^{\mu} = \mathfrak{m}I^k\big(JI^{\mu-1} + Ann_{I^\mu}(M)\big)\\ &= J\mathfrak{m}I^{\mu+k-1} + \mathfrak{m}I^k Ann_{I^\mu}(M)\\ &\subseteq J\mathfrak{m}I^{\mu+k-1} + \big(JA_{k-1} + a_dI^k\big)Ann_{I^\mu}(M)\\ &\subseteq J\mathfrak{m}I^{\mu+k-1} + a_dI^{\mu+k}\end{aligned}$$

where the last relation holds because of $JA_{k-1}Ann_{I^\mu}(M) \subseteq J\mathfrak{m}I^{\mu+k-1}$. Hence

$$r_L(I|\mathfrak{m}) \leqslant \mu + k = \sum_{j\geqslant 0}\mu_j + k \leqslant \mu_0 + \sum_{j\geqslant 1}\lambda\left(\frac{A_j}{JA_{j-1} + \mathfrak{m}I^j}\right) + k.$$

Note that

$$\lambda\left(\frac{A_j}{JA_{j-1} + \mathfrak{m}I^j}\right) \leqslant \begin{cases}\lambda(\frac{A_j}{JA_{j-1}+a_dI^j}) - 1, & j = 1, \dots, k-1,\\ \lambda(\frac{A_j}{JA_{j-1}+a_dI^j}), & j \geqslant k.\end{cases}$$

Therefore we get that

$$\begin{aligned}r_L(I|\mathfrak{m}) &\leqslant \lambda\left(\frac{A_0}{\mathfrak{m}}\right) + \sum_{j=1}^{k-1}\left[\lambda\left(\frac{A_j}{JA_{j-1} + a_dI^j}\right) - 1\right] + \sum_{j\geqslant k}\left[\lambda\left(\frac{A_j}{JA_{j-1} + a_dI^j}\right)\right] + k\\ &= \sum_{j\geqslant 1}\lambda\left(\frac{A_j}{JA_{j-1} + a_dI^j}\right) - \lambda\left(\frac{R}{A_0}\right) + 2. \qquad \square\end{aligned}$$

Remark 3.5. Let $d = 2$, $a_1 \in I$, $a_2 \in \mathfrak{m}$ a Rees-superficial sequence for I and $\mathfrak{m}$ such that a_1^* is a $G(I)$-regular element. Set $L = (a_1, a_2)$. If $r_L(I|\mathfrak{m}) < \infty$. Then $r_L(I|\mathfrak{m}) \leqslant f_1(I) + 2$.

4. Ideals with almost minimal mixed multiplicity

In this section, we prove that fiber cones of ideals with almost minimal mixed multiplicity have high depth. We begin with the following lemma.

Lemma 4.1. *Let $d = 2$ and I an ideal with almost minimal mixed multiplicity. Let $a_1 \in I$, $a_2 \in \mathfrak{m}$ be a Rees-superficial sequence for I and $\mathfrak{m}$ such that a_1^* is a $G(I)$-regular element. Set $L = (a_1, a_2)$. If $r_L(I|\mathfrak{m}) < \infty$. Let "$\overline{}$" denote the image modulo (a_1). Then*

$$r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}}) = r_L(I|\mathfrak{m}) = f_1(I) + 1.$$

Proof. Set $s = r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}})$. Clearly $s \leqslant r_L(I|\mathfrak{m})$. Note that $f_1(\bar{I}) = f_1(I)$, $\dim \bar{R} = 1$ and $s < \infty$, $f_0(\bar{I}) = e(\bar{\mathfrak{m}}) = e(\mathfrak{m})$ by Lemma 2.2. By Theorem 3.3 of [6], we have $f_1'(\bar{I}) = e(\bar{\mathfrak{m}}) - 2 + r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}}) = f_0(\bar{I}) - 2 + r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}})$. Hence $f_1(\bar{I}) = f_1'(\bar{I}) - f_0(\bar{I}) = r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}}) - 2$. Therefore $r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}}) = f_1(I) + 2$.

Since I has almost minimal mixed multiplicity, we get $\mu(I) = e_1(\mathfrak{m}|I)$. By Lemma 2.2, we have $f_0(I) = e_1(\mathfrak{m}|I)$. Thus from Theorem 4.3 of [6], we have $f_1'(I) = \mu(I) - 2 + r_L(I|\mathfrak{m}) = f_0(I) - 2 + r_L(I|\mathfrak{m})$. Hence $f_1(I) = f_1'(I) - f_0(I) = r_L(I|\mathfrak{m}) - 2$. $\square$

Now, we can prove the main result of this section.

Theorem 4.2. *Let $d \geqslant 2$ and I an ideal with almost minimal mixed multiplicity. Let $a_1, \ldots, a_{d-1} \in I$, $a_d \in \mathfrak{m}$ be a Rees-superficial sequence for I and $\mathfrak{m}$ satisfying $a_1^*, \ldots, a_{d-1}^*$ is a regular sequence in $G(I)$. Then depth $F_{\mathfrak{m}}(I) \geqslant d-1$.*

Proof. We apply induction on d. Let $d = 2$. Since I has almost minimal mixed multiplicity, $\lambda(\frac{\mathfrak{m}I^n}{a_1\mathfrak{m}I^{n-1}+a_2I^n}) \leqslant 1$ for all $n \geqslant 1$ by Lemma 2.2 of [6]. Let "$\overline{}$" denote the image modulo (a_1). Then $\lambda(\frac{\bar{\mathfrak{m}}\bar{I}}{\bar{a}_1\bar{\mathfrak{m}}+\bar{a}_2\bar{I}}) = \lambda(\frac{\mathfrak{m}I}{a_1\mathfrak{m}+a_2I}) = 1$.

If $r_L(I|\mathfrak{m}) < \infty$, then by Lemma 4.1, we have that $r_{\bar{L}}(\bar{I}|\bar{\mathfrak{m}}) = r_L(I|\mathfrak{m})$ and hence $\lambda(\frac{\bar{\mathfrak{m}}\bar{I}^n}{\bar{a}_1\bar{\mathfrak{m}}\bar{I}^{n-1}+\bar{a}_2\bar{I}^n}) = \lambda(\frac{\mathfrak{m}I^n}{a_1\mathfrak{m}I^{n-1}+a_2I^n})$ for all $n \geqslant 1$.

Now, if $r_L(I|\mathfrak{m}) = \infty$, then by Remark 2.4, $\lambda(\frac{\bar{\mathfrak{m}}\bar{I}^n}{\bar{a}_1\bar{\mathfrak{m}}\bar{I}^{n-1}+\bar{a}_2\bar{I}^n}) = 1 = \lambda(\frac{\mathfrak{m}I^n}{a_1\mathfrak{m}I^{n-1}+a_2I^n})$ for all $n \geqslant 1$.

For $n \geqslant 1$, consider the following exact sequence:

$$0 \to \frac{\mathfrak{m}I^n : a_1}{\mathfrak{m}I^{n-1}} \xrightarrow{.a_1} \frac{\mathfrak{m}I^n}{a_1\mathfrak{m}I^{n-1}+a_2I^n} \to \frac{\bar{\mathfrak{m}}\bar{I}^n}{\bar{a}_1\bar{\mathfrak{m}}\bar{I}^{n-1}+\bar{a}_2\bar{I}^n} \to 0.$$

We have that $\mathfrak{m}I^n : a_1 = \mathfrak{m}I^{n-1}$. Therefore a_1^0 is a regular element in $F_{\mathfrak{m}}(I)$ and depth $F_{\mathfrak{m}}(I) \geqslant 1$.

Let $d > 2$. Let "$\overline{}$" denote the images modulo $(a_1, \ldots, a_{d-2})$. Then dim $\bar{R} = 2$ and $\lambda(\frac{\bar{\mathfrak{m}}\bar{I}}{(a_1,\ldots,a_{d-1})\bar{\mathfrak{m}}+\bar{a}_d\bar{I}}) \leqslant 1$.

If $\lambda(\frac{\bar{\mathfrak{m}}\bar{I}}{(a_1,\ldots,a_{d-1})\bar{\mathfrak{m}}+\bar{a}_d\bar{I}}) = 0$, then we get depth $F_{\bar{\mathfrak{m}}}(\bar{I}) \geqslant 1$ by Proposition 5.6 of [6].

Now, if $\lambda(\frac{\bar{\mathfrak{m}}\bar{I}}{(a_1,\ldots,a_{d-1})\bar{\mathfrak{m}}+\bar{a}_d\bar{I}}) = 1$, then $\bar{I}$ has almost minimal mixed multiplicity. Therefore, applying induction assumptions, depth $F_{\bar{\mathfrak{m}}}(\bar{I}) \geqslant 1$. Since $a_1^*, \ldots, a_{d-2}^*$ is a regular sequence in $G(I)$, $F_{\bar{\mathfrak{m}}}(\bar{I}) \cong \frac{F_{\mathfrak{m}}(I)}{(a_1^0,\ldots,a_{d-2}^0)F_{\mathfrak{m}}(I)}$ and hence by Sally machine, depth $F_{\mathfrak{m}}(I) \geqslant d-1$. $\square$

The following example shows that the assumption in Theorem 4.2 that depth $G(I) \geqslant d-1$ cannot be dropped.

Example 4.3. Let $R = k[[x, y, z]]$ be a three dimensional regular local ring with k a field and x, y, z indeterminates, $\mathfrak{m} = (x, y, z)$. Let $I = (-x^2 + y^2, -y^2 + z^2, xy, yz, zx)$. It can be seen that $x^2 I \subset I^2$, but $x^2 \notin I$. This shows that the Ratliff–Rush closure $\tilde{I}$ is not equal to I. Hence depth $G(I) = 0$.

Let $L = (-x^2 + y^2, -y^2 + z^2, x)$. Then it is a joint reduction of $(I^{[2]}|\mathfrak{m})$. It can be seen that $\mathfrak{m}I = (-x^2 + y^2, -y^2 + z^2)\mathfrak{m} + xI + (xy^2)$ and $\mathfrak{m}(xy^2) \subset (-x^2 + y^2, -y^2 + z^2)\mathfrak{m} + xI$. Hence I has almost minimal mixed multiplicity. Since I is generated by homogeneous elements of same degree (equal to 2), $F_{\mathfrak{m}}(I) \cong k[-x^2 + y^2, -y^2 + z^2, xy, yz, zx]$. Therefore depth $F_{\mathfrak{m}}(I) \geqslant 1$. Let $\mathfrak{n}$ denote the graded maximal ideal of $F_{\mathfrak{m}}(I)$ and $\bar{\mathfrak{n}}$ the graded maximal ideal of $F_{\mathfrak{m}}(I)/(-x^2 + y^2)F_{\mathfrak{m}}(I)$. Then, it can be easily checked that $\bar{\mathfrak{n}}(\overline{-x^2z^2 + y^2z^2}) = 0$. Note that, since $z^2 \notin F_{\mathfrak{m}}(I)$, $\overline{-x^2z^2 + y^2z^2} \neq 0 \in F_{\mathfrak{m}}(I)/(-x^2 + y^2)F_{\mathfrak{m}}(I)$. Therefore we have produced a nonzero element in $F_{\mathfrak{m}}(I)/(-x^2 + y^2)F_{\mathfrak{m}}(I)$ which is killed by the maximal ideal of $F_{\mathfrak{m}}(I)/(-x^2 + y^2)F_{\mathfrak{m}}(I)$ and hence depth $F_{\mathfrak{m}}(I)/(-x^2 + y^2)F_{\mathfrak{m}}(I) = 0$. This shows that depth $F_{\mathfrak{m}}(I) = 1$.

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