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A Generalization on the Difference Between an Integer and Its Inverse Modulo Q

Tian Ping ZHANG
Department of Mathematics, Northwest University, Xi'an 710069*, P. R. China*
and
College of Mathematics and Information Science, Shannxi Normal University, Xi'an 710062*, P. R. China*
E-mail: tianpzhang@eyou.com

Wen Peng ZHANG
Department of Mathematics, Northwest University, Xi'an 710069*, P. R. China*
E-mail: wpzhang@nwu.edu.cn

Abstract The main purpose of this paper is to use the properties of the Gauss sums, primitive characters and the mean value of Dirichlet L-functions to study the hybrid mean value of the error term $E(n,1,c,q)$ and the hyper-Kloosterman sums $K(h,n+1,q)$, the asymptotic property of the mean square value $\sum_{c=1}^{p} E^2(n,1,c,p)$, and give two interesting mean value formulae.

Keywords generalization, mean value, hyper-Kloosterman sums

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1 Introduction

Let $q > 2$ and c be two integers with $(q,c) = 1$. For each integer $0 < a < q$ with $(a,q) = 1$, we know that there exists one, and only one, integer $0 < b < q$ with $(b,q) = 1$ such that $ab \equiv c (\mathrm{mod}\ q)$. Let

$$M(k,c,q) = \sum_{\substack{a=1 \\ ab\equiv c(\mathrm{mod}\ q)}}^{q}{}' \sum_{b=1}^{q}{}' (a-b)^{2k},$$

where $\sum_{a=1}^{\prime q}$ denotes the summation over all a such that $(a,q) = 1$. The second author [1] used the estimates for the Kloosterman sums and trigonometric sums to obtain a sharp asymptotic formula for $M(k,c,q)$, and proved the following:

Proposition *Let $q > 2$ and c be two integers with $(q,c) = 1$. Then, for any positive integer k, we have the asymptotic formula*

$$M(k,c,q) = \frac{1}{(2k+1)(k+1)}\phi(q)q^{2k} + O\left(4^k q^{\frac{4k+1}{2}} d^2(q)\ln^2 q\right),$$

where $\phi(q)$ is the Euler function, and $d(q)$ is the divisor function.

For $k = 1$ and any fixed positive integer c with $(q,c) = 1$, let

$$E(1,c,q) = M(1,c,q) - \frac{1}{6}\phi(q)q^2 - \frac{1}{3}q\prod_{p|q}(1-p).$$

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The second author [2] showed that, for any integer $q > 2$, we have the asymptotic formula

$$\sum_{c=1}^{q}{}' E^2(1,c,q) = \frac{5}{36}q^3\phi^3(q)\prod_{p^\alpha \| q}\frac{\frac{(p+1)^3}{p(p^2+1)} - \frac{1}{p^{3\alpha-1}}}{1+\frac{1}{p}+\frac{1}{p^2}} + O\left(q^5\exp\left(\frac{4\ln q}{\ln\ln q}\right)\right),$$

where $\prod_{p^\alpha \| q}$ denotes the product over all prime divisors of q with $p^\alpha | q$ and $p^{\alpha+1} \nmid q$. This proves the error terms in the Proposition are the best possible.

For similar results about the error terms, see reference [3] and [4].

In reference [5], Mordell introduced the high-dimensional generalization of the Kloosterman sums as follows:

$$K(h,n+1,q) = \sum_{a_1=1}^{q}{}'\sum_{a_2=1}^{q}{}'\cdots\sum_{a_n=1}^{q}{}'\mathrm{e}\left(\frac{a_1+a_2+\cdots+a_n+h\cdot\overline{a}_1\overline{a}_2\cdots\overline{a}_n}{q}\right),$$

which is called the hyper-Kloosterman sums. The various properties of the hyper-Kloosterman sums have been investigated by many authors. Applications of the hyper-kloosterman sums were found in the estimation of Fourier coefficients of Maass forms [6] and the work on Selberg's eigenvalue conjecture [7]. On the other hand, Smith [8] has built some interesting connections between the hyper-Kloosterman sums and the Heibronn sums.

Now we consider a generalization on the problem in the Proposition. For any integers $q > 2$ and $n \geq 1$, letting

$$M(n,k,c,q) = \sum_{\substack{a_1=1\\ a_1\cdots a_n b\equiv c(\bmod q)}}^{q}{}'\cdots\sum_{a_n=1}^{q}{}'\sum_{b=1}^{q}{}'(a_1\cdots a_n - b)^{2k},$$

for $n = 1$, we have $M(1,k,c,q) = M(k,c,q)$, which has been stated above; while for $n > 1$ and $k = 1$, we still know nothing about it yet, that is,

$$M(n,1,c,q) = \sum_{\substack{a_1=1\\ a_1\cdots a_n b\equiv c(\bmod q)}}^{q}{}'\cdots\sum_{a_n=1}^{q}{}'\sum_{b=1}^{q}{}'(a_1\cdots a_n - b)^{2}.$$

We now define

$$E(n,1,c,q) = M(n,1,c,q) + \frac{\phi^n(q)q^{n+1}}{2^n} - \left(\frac{\phi(q)q^2}{3} + \frac{q}{6}\prod_{p|q}(1-p)\right)^n$$
$$- \left(\frac{\phi^n(q)q^2}{3} + \frac{\phi^{n-1}(q)}{6}\prod_{p|q}(1-p)\right).$$

In this paper, we shall use the properties of the Gauss sums, primitive characters and the mean value of Dirichlet L-functions to study the hybrid mean value of the error term $E(n,1,c,q)$ and the hyper-Kloosterman sums $K(h,n+1,q)$, the asymptotic property of the mean square value $\sum_{c=1}^{p}E^2(n,1,c,p)$, and give two interesting mean value formulae. That is, we shall prove the following:

Theorem 1 *For any integers q, n with $q > 2$ and $n > 1$, we have the asymptotic formula*

$$\sum_{c=1}^{q}{}' E(n,1,c,q)K(c,n+1,q) = \frac{(-1)^n i^{n+1}q^{2n+1}\phi(q)}{\pi^{n+1}}\prod_{p\|q}\left(1-\frac{p^n-1}{p^n(p-1)^2}\right) + O\left(q^{2n+1+\epsilon}\right),$$

where ε denotes any fixed positive number.

Theorem 2 *For any odd prime p and any integer n with $n > 1$, we have the asymptotic formula*

$$\sum_{c=1}^{p} E^2(n,1,c,p) = \frac{p^{3n+3}}{3}\left(\frac{\pi}{6}\right)^{2n} \prod_{p_1 \neq p}\left(1 - \frac{1 - C_{2n}^n}{p_1^2}\right) + O(p^{3n+2+\varepsilon}),$$

where $\prod_{p_1 \neq p}$ denotes the product over all primes p_1 with $p_1 \neq p$, and $C_m^n = m!/n!(m-n)!$.

Taking $q = p$, an odd prime in our Theorem 1, we immediately deduce the following:

Corollary *For any odd prime number p and positive integer n with $n > 1$, we have*

$$\sum_{c=1}^{p}{}' E(n,1,c,p)K(c,n+1,p) = \frac{(-1)^n i^{n+1} p^{2n+2}}{\pi^{n+1}} + O\left(p^{2n+1+\epsilon}\right).$$

2 Several Lemmas

In this section, we use several lemmas to complete the proof of the theorems. First we have

Lemma 1 *Let χ be a character modulo q, generated by the primitive character χ_m modulo m. Then we have the identity*

$$\tau(\chi) = \chi_m\left(\frac{q}{m}\right)\mu\left(\frac{q}{m}\right)\tau(\chi_m),$$

where $\tau(\chi) = G(1,\chi)$ is the Gauss sum, and $\mu(n)$ is the Möbius function.

Proof See reference [9].

Lemma 2 *For any positive integer q, let χ be a non-primitive character modulo q and $\chi_q \Leftrightarrow \chi_{q^*}^*$. If $(n,q) > 1$, we have*

$$G(n,\chi) = \begin{cases} \overline{\chi}^*\left(\dfrac{n}{(n,q)}\right)\chi^*\left(\dfrac{q}{q^*(n,q)}\right)\mu\left(\dfrac{q}{q^*(n,q)}\right)\phi(q)\phi^{-1}\left(\dfrac{q}{(n,q)}\right)\tau(\chi^*), & q^* = \dfrac{q_1}{(n,q_1)}, \\ 0, & q^* \neq \dfrac{q_1}{(n,q_1)}, \end{cases}$$

where q_1 is the greatest divisor of q that has the same prime factors as q^*.

If $(n,q) = 1$, we have

$$G(n,\chi) = \overline{\chi}^*(n)\chi^*\left(\frac{q}{q^*}\right)\mu\left(\frac{q}{q^*}\right)\tau(\chi^*).$$

Proof See reference [9].

Lemma 3 *Let q, n be any integers with $q > 2, n > 1$. Then, for any positive integer c with $(q,c) = 1$, we have*

$$E(n,1,c,q) = -\frac{2i^{n+1}q^{n+1}}{\pi^{n+1}\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c)\left(\sum_{t=1}^{\infty}\frac{G(t,\chi)}{t}\right)^{n+1},$$

where χ denotes a Dirichlet character modulo q with $\chi(-1) = -1$.

Proof From the orthogonality relation for a character modulo q we have

$$M(n,1,c,q) = \sum_{\substack{a_1=1 \\ a_1\cdots a_n b \equiv c (\bmod q)}}^{q}{}' \cdots \sum_{a_n=1}^{q}{}' \sum_{b=1}^{q}{}' (a_1\cdots a_n - b)^2$$

$$= \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(c) \sum_{a_1=1}^{q} \cdots \sum_{a_n=1}^{q} \sum_{b=1}^{q} \chi(a_1 \cdots a_n b) \left(a_1 \cdots a_n - b\right)^2$$

$$= \Big(\sideset{}{'}\sum_{a=1}^{q} a^2 \Big)^n + \phi^{n-1}(q) \sideset{}{'}\sum_{b=1}^{q} b^2 - \frac{2}{\phi(q)} \Big(\sideset{}{'}\sum_{a=1}^{q} a \Big)^{n+1} - \frac{2}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(c) \Big(\sum_{a=1}^{q} a\chi(a) \Big)^{n+1}$$

$$= \left(\frac{\phi(q)q^2}{3} + \frac{q}{6} \prod_{p|q} (1-p) \right)^n + \frac{\phi^n(q)q^2}{3} + \frac{\phi^{n-1}(q)}{6} \prod_{p|q} (1-p) - \frac{\phi^n(q)q^{n+1}}{2^n} + E(n,1,c,q),$$

where we have used the identities (see reference [10])

$$\sideset{}{'}\sum_{a=1}^{q} a = \frac{\phi(q)q}{2}, \qquad \sideset{}{'}\sum_{a=1}^{q} a^2 = \frac{\phi(q)q^2}{3} + \frac{q}{6} \prod_{p|q} (1-p).$$

Now if $\chi(-1) = 1$ and $\chi \neq \chi_0$, then we have

$$\sum_{a=1}^{q} a\chi(a) = \sum_{a=1}^{q} (q-a)\chi(q-a) = \sum_{a=1}^{q} (q-a)\chi(a) = -\sum_{a=1}^{q} a\chi(a) = 0.$$

On the other hand,

$$\sum_{b=1}^{q} \chi(b) \left(\frac{b}{q} - \frac{1}{2} \right) = \frac{1}{q} \sum_{b=1}^{q} b\chi(b), \quad \text{if} \quad \chi(-1) = -1.$$

We may immediately get

$$E(n,1,c,q) = -\frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c) \Big(\sum_{a=1}^{q} a\chi(a) \Big)^{n+1}$$

$$= -\frac{2q^{n+1}}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c) \Big(\sum_{a=1}^{q} \chi(a) \Big(\frac{a}{q} - \frac{1}{2} \Big) \Big)^{n+1}$$

$$= -\frac{2q^{n+1}}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c) \Big(\sum_{a=1}^{q} \chi(a) \Big(\Big(\frac{a}{q} \Big) \Big) \Big)^{n+1}$$

$$= -\frac{2q^{n+1}}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c) \Big(\frac{-1}{\pi} \sum_{t=1}^{\infty} \frac{\sum_{a=1}^{q} \chi(a) \sin(2\pi ta/q)}{t} \Big)^{n+1}$$

$$= -\frac{2q^{n+1}}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c) \Big(\frac{-1}{2\pi i} \sum_{t=1}^{\infty} \frac{G(t,\chi) - G(-t,\chi)}{t} \Big)^{n+1}$$

$$= -\frac{2i^{n+1}q^{n+1}}{\pi^{n+1}\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(c) \Big(\sum_{t=1}^{\infty} \frac{G(t,\chi)}{t} \Big)^{n+1},$$

where we have used the identities

$$((x)) = -\frac{1}{\pi} \sum_{t=1}^{\infty} \frac{\sin(2\pi tx)}{n}, \qquad \sin x = \frac{1}{2i} \left(\mathrm{e}^{xi} - \mathrm{e}^{-xi} \right).$$

This proves Lemma 3.

Lemma 4 *Let q and r be integers with $q \geq 2$ and $(r,q) = 1$, χ be a Dirichlet character*

modulo q. Then we have the identities

$$\sum_{\chi \mod q}^{*} \chi(r) = \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right)\phi(d)$$

and

$$J(q) = \sum_{d|q} \mu(d)\phi\left(\frac{q}{d}\right),$$

where $\sum^{*}_{\chi \mod q}$ denotes the summation over all primitive characters modulo q and $J(q)$ denotes the number of primitive characters modulo q.

Proof See Lemma 3 of [11].

Lemma 5 *Let $q = uv$, where $(u, v) = 1$, u be a square-full number or $u = 1$, v be a square-free number. Then we have the asymptotic formula*

$$\begin{aligned} M = &\sum_{d|v} \sum_{d_1|\frac{v}{d}} \cdots \sum_{d_{n+1}|\frac{v}{d}} \frac{u^{n+1}d^{n+1}\phi(d_1)\cdots\phi(d_{n+1})\mu(d_1)\cdots\mu(d_{n+1})}{d_1\cdots d_{n+1}} \\ &\times \sum_{\substack{\chi \mod ud \\ \chi(-1)=-1}}^{*} \overline{\chi}(d_1\cdots d_{n+1})\, L^{n+1}(1,\overline{\chi}) \\ = &\frac{q^n\phi^2(q)}{2}\prod_{p\|q}\left(1-\frac{p^n-1}{p^n(p-1)^2}\right) + O\left(q^{n+1+\epsilon}\right). \end{aligned}$$

Proof Let $\tau_{n+1}(s)$ denote the $(n+1)$-th divisor function (i.e. the number of positive integer solutions of the equation $s = s_1 s_2 \cdots s_{n+1}$). Then for any parameter $N \geq s$ and non-principal character χ modulo s, applying Abel's identity, we have

$$L^{n+1}(1,\overline{\chi}) = \sum_{s=1}^{\infty} \frac{\overline{\chi}(s)\tau_{n+1}(s)}{s} = \sum_{1\leq s\leq N} \frac{\overline{\chi}(s)\tau_{n+1}(s)}{s} + \int_N^{\infty} \frac{A(y,\overline{\chi})}{y^2}\mathrm{d}y,$$

where $A(y,\overline{\chi}) = \sum_{N<s\leq y} \overline{\chi}(s)\tau_{n+1}(s)$.

Using a similar method to that of proving Lemma 5 in [12], we can easily get

$$\begin{aligned} M = &\sum_{d|v} \sum_{d_1|\frac{v}{d}} \cdots \sum_{d_{n+1}|\frac{v}{d}} \frac{u^{n+1}d^{n+1}\phi(d_1)\cdots\phi(d_{n+1})\mu(d_1)\cdots\mu(d_{n+1})}{d_1\cdots d_{n+1}} \\ &\times \sum_{s=1}^{N} \frac{\tau_{n+1}(s)}{s} \sum_{\substack{\chi \mod ud \\ \chi(-1)=-1}}^{*} \overline{\chi}(d_1\cdots d_{n+1})\,\overline{\chi}(s) + O\left(\frac{q^{n+\frac{5}{2}+\epsilon}}{N^{\frac{1}{n+1}}}\right). \end{aligned}$$

Note that for $(a, q) = 1$, by Lemma 4, we have

$$\begin{aligned} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}}^{*} \chi(a) &= \frac{1}{2}\sum_{\chi \mod q}^{*} (1-\chi(-1))\,\chi(a) \\ &= \frac{1}{2}\sum_{r|(q,a-1)} \mu\left(\frac{q}{r}\right)\phi(r) - \frac{1}{2}\sum_{r|(q,a+1)} \mu\left(\frac{q}{r}\right)\phi(r). \end{aligned}$$

Therefore,

$$M = \frac{1}{2}\sum_{d|v} \sum_{d_1|\frac{v}{d}} \cdots \sum_{d_{n+1}|\frac{v}{d}} u^{n+1}d^{n+1}\phi(d_1)\cdots\phi(d_{n+1})\mu(d_1)\cdots\mu(d_{n+1}) \sum_{s=1}^{N} \frac{\tau_{n+1}(s)}{d_1\cdots d_{n+1}s}$$

$$\times\Big[\sum_{r|(ud,d_1\cdots d_{n+1}s-1)}\mu\left(\frac{ud}{r}\right)\phi(r)-\sum_{r|(ud,d_1\cdots d_{n+1}s+1)}\mu\left(\frac{ud}{r}\right)\phi(r)\Big]+O\Big(\frac{q^{n+\frac{5}{2}+\epsilon}}{N^{\frac{1}{n+1}}}\Big)$$

$$=\frac{1}{2}\sum_{d|v}u^{n+1}d^{n+1}J(ud)+O\Big(\frac{q^{n+\frac{5}{2}+\epsilon}}{N^{\frac{1}{n+1}}}\Big)$$

$$+O\Big(\sum_{d|v}\sum_{d_1|\frac{v}{d}}\cdots\sum_{d_{n+1}|\frac{v}{d}}u^{n+1}d^{n+1}\phi(d_1)\cdots\phi(d_{n+1})\sum_{r|ud}\phi(r)\sum_{l=1}^{\frac{Nd_1\cdots d_{n+1}}{r}}(lr+1)^{\epsilon-1}\Big)$$

$$+O\Big(\sum_{d|v}\sum_{d_1|\frac{v}{d}}\cdots\sum_{d_{n+1}|\frac{v}{d}}u^{n+1}d^{n+1}\phi(d_1)\cdots\phi(d_{n+1})\sum_{r|ud}\phi(r)\sum_{l=1}^{\frac{Nd_1\cdots d_{n+1}}{r}}(lr-1)^{\epsilon-1}\Big)$$

$$=\frac{u^n\phi^2(u)}{2}\sum_{d|v}d^{n+1}J(d)+O\Big(\frac{q^{n+\frac{5}{2}+\epsilon}}{N^{\frac{1}{n+1}}}\Big)+O\left(q^{n+1+\epsilon}\right)$$

$$=\frac{u^n\phi^2(u)}{2}\prod_{p|v}\Big[p^n(p-1)^2\Big(1-\frac{p^n-1}{p^n(p-1)^2}\Big)\Big]+O\Big(\frac{q^{n+\frac{5}{2}+\epsilon}}{N^{\frac{1}{n+1}}}\Big)+O\left(q^{n+1+\epsilon}\right)$$

$$=\frac{q^n\phi^2(q)}{2}\prod_{p\|q}\Big(1-\frac{p^n-1}{p^n(p-1)^2}\Big)+O\Big(\frac{q^{n+\frac{5}{2}+\epsilon}}{N^{\frac{1}{n+1}}}\Big)+O\left(q^{n+1+\epsilon}\right),$$

where we have used the estimate $\tau_{n+1}(s)\ll s^{\epsilon}$, the fact that u is a square-full number, v is a square-free number, and the identity $J(u)=\phi^2(u)/u$, where u be a square-full number.

Taking $N=q^{\frac{3(n+1)}{2}}$ in the above, we immediately obtain the following asymptotic formula:

$$M=\frac{q^n\phi^2(q)}{2}\prod_{p\|q}\Big(1-\frac{p^n-1}{p^n(p-1)^2}\Big)+O\left(q^{n+1+\epsilon}\right).$$

This completes the proof of Lemma 5.

Lemma 6 *Let p be any odd prime. Then we have the asymptotic formula*

$$\sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=-1}}|L(1,\chi)|^{2n+2}=\frac{(p-1)}{2}\left(\frac{\pi^2}{6}\right)^{2n+1}\prod_{p_1}\Big(1-\frac{1-C_{2n}^n}{p_1^2}\Big)+O(p^{\varepsilon}),$$

where $C_{2n}^n=\frac{(2n)!}{(n!)^2}$.

Proof See Lemma 6 of [13].

3 Proof of Theorems

In this section, we complete the proof of the theorems. Let q,n be any integers with $q>2,n>1$. Then, for any character χ mod q, we have

$$\sum_{c=1}^{q}\overline{\chi}(c)K(c,n+1,q)=\sum_{a_1=1}^{q}{}'\sum_{a_2=1}^{q}{}'\cdots\sum_{a_n=1}^{q}{}'\sum_{c=1}^{q}\overline{\chi}(c)\mathrm{e}\left(\frac{a_1+a_2+\cdots+a_n+c\cdot\overline{a}_1\overline{a}_2\cdots\overline{a}_n}{q}\right)$$

$$=\tau(\overline{\chi})\Big(\sum_{a=1}^{q}\overline{\chi}(a)\mathrm{e}\left(\frac{a}{q}\right)\Big)^n=\tau^{n+1}(\overline{\chi}).$$

On the other hand, let $k=uv$, where $(u,v)=1$, u is a square-full number or $u=1$, v is a square-free number. Note that $\chi^*\left(\frac{k}{m}\right)\mu\left(\frac{k}{m}\right)\neq 0$ if and only if $m=ud$, where $d\mid v$. So, from

Lemma 1, Lemma 2 and Lemma 3, we have

$$\begin{aligned}
&\sum_{c=1}^{q}{}' E(n,1,c,q)K(c,n+1,q)\\
&\quad= -\frac{2i^{n+1}q^{n+1}}{\pi^{n+1}\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left[\sum_{c=1}^{q}\overline{\chi}(c)K(c,n+1,q)\right]\left[\sum_{t=1}^{\infty}\frac{G(t,\chi)}{t}\right]^{n+1}\\
&\quad= -\frac{2i^{n+1}q^{n+1}}{\pi^{n+1}\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \tau^{n+1}(\overline{\chi})\left[\sum_{t=1}^{\infty}\frac{G(t,\chi)}{t}\right]^{n+1}\\
&\quad= -\frac{2i^{n+1}q^{n+1}}{\pi^{n+1}\phi(q)} \sum_{d|v}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}{}^{*} \overline{\chi}^{n+1}\left(\frac{v}{d}\right)\mu^{n+1}\left(\frac{v}{d}\right)\tau^{n+1}(\overline{\chi})\\
&\qquad\times\left[\sum_{d_1|\frac{v}{d}}\frac{\phi(q)\chi\left(\frac{v}{dd_1}\right)\mu\left(\frac{v}{dd_1}\right)\tau(\chi)L\left(1,\overline{\chi}\right)}{d_1\phi\left(\frac{q}{d_1}\right)}\right]^{n+1}.
\end{aligned}$$

Note the identities

$$\overline{\chi}\left(\frac{v}{d}\right)=\overline{\chi}\left(\frac{v}{dd_1}\right)\overline{\chi}(d_1),\quad \mu\left(\frac{v}{d}\right)=\mu\left(\frac{v}{dd_1}\right)\mu(d_1),\quad \phi(q)=\phi\left(\frac{q}{d_1}\right)\phi(d_1),$$

and

$$\tau(\chi^*)\tau(\overline{\chi}^*)=-m,$$

where χ^* is a primitive character modulo m with $\chi(-1)=-1$.

Then, from Lemma 5, we have

$$\begin{aligned}
&\sum_{c=1}^{q}{}' E(n,1,c,q)K(c,n+1,q)\\
&\quad= \frac{2(-1)^n i^{n+1}q^{n+1}}{\pi^{n+1}\phi(q)}\sum_{d|v}\sum_{d_1|\frac{v}{d}}\cdots\sum_{d_{n+1}|\frac{v}{d}}\sum_{\substack{\chi \bmod ud\\ \chi(-1)=-1}}{}^{*}\frac{\phi(d_1)\cdots\phi(d_{n+1})\mu(d_1)\cdots\mu(d_{n+1})}{d_1\cdots d_{n+1}}\\
&\qquad\times(ud)^{n+1}\overline{\chi}\left(d_1\cdots d_{n+1}\right)L^{n+1}(1,\overline{\chi})\\
&\quad= \frac{(-1)^n i^{n+1}q^{2n+1}\phi(q)}{\pi^{n+1}}\prod_{p\|q}\left(1-\frac{p^n-1}{p^n(p-1)^2}\right)+O\left(q^{2n+1+\epsilon}\right).
\end{aligned}$$

This proves Theorem 1.

From Lemma 3 and Lemma 6, we have

$$\begin{aligned}
\sum_{c=1}^{p}E^2(n,1,c,p)&=\frac{4\times(-1)^{n+1}p^{2n+2}}{\pi^{2n+2}(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\tau^{n+1}(\chi)\tau^{n+1}(\overline{\chi})L^{n+1}(1,\chi)L^{n+1}(1,\overline{\chi})\\
&=\frac{4p^{3n+3}}{\pi^{2n+2}(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^{2n+2}\\
&=\frac{p^{3n+3}}{3}\left(\frac{\pi}{6}\right)^{2n}\prod_{p_1}\left(1-\frac{1-C_{2n}^n}{p_1^2}\right)+O(p^{3n+2+\varepsilon}).
\end{aligned}$$

Thus we complete the proof of Theorem 2.

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