Acta Appl Math (2010) 109: 1165–1178
DOI 10.1007/s10440-008-9375-3

Moderate Deviations for a Class of L-Statistics

Hui Jiang

Received: 28 October 2008 / Accepted: 7 November 2008 / Published online: 15 November 2008

Abstract We study a class of L-statistics based on linear combinations of order statistics divided by the sample mean. The moderate deviation and functional moderate deviation are obtained by the method of Rényi representation. Moreover, we also apply our result to Jackson, Gini and Fortiana-Grané tests and obtain their asymptotic properties.

Keywords Asympotic properties · L-statistics · Large deviations · Moderate deviations · Order statistics · Rényi representation

Mathematics Subject Classification (2000) 62F12 · 62N02 · 60F15

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables and have the density

$$f(s) = e^{-x}, \quad x \geq 0.$$

The order statistics of $X_1, X_2, \ldots, X_n$ are denoted as $X_1^* \leq X_2^* \leq \cdots \leq X_n^*$. Shorack and Wellner [9] proposed a class of L-statistics of the form

$$T_n = \frac{\sum_{i=1}^n w_{i,n} X_i^*}{\sum_{i=1}^n X_i}, \tag{1.1}$$

where $\{w_{i,n}\}$, $i = 1, 2, \ldots, n$, is an array of coefficients. Motivated by their simple, well-studied asymptotic behavior and nice properties of order statistics under exponentiality, the tests T_n is widely used. Moreover, many statistics belong to this class (cf. [3, 4, 9]).

Research supported by the National Natural Science Foundation of China (10571139).

H. Jiang (✉)
School of Science, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
e-mail: huijiang@nuaa.edu.cn

The asymptotic behavior and applications of T_n have been studied widely. Tchirina [11] obtained the consistency and large deviations of T_n. Mason and Shorack [6] proved the necessary and sufficient conditions for the central limit theorem. For further references, on can see [5, 10].

Set

$$T_n(t) = \frac{\sum_{i=1}^{[nt]} w_{i,n} X_i^*}{\sum_{i=1}^{n} X_i}, \quad 0 \le t \le 1.$$

Since the law of large numbers, central limit theorem and large deviation have been well investigated, as a complement, the purpose of this paper is to study further estimations about this estimator, i.e. moderate deviation. We study the moderate deviation for $T_n(t)$ from two points of view: the parametrical statistical one when $t \in [0, 1]$ is fixed and the nonparametrical statistical one when t varies in $[0, 1]$. Moreover, we also apply our result to the statistics in Jackson, Gini and Fortiana-Grané tests (cf. [3, 4, 9]). And their asymptotic properties can be obtained.

More precisely, we are interested in the estimations of

$$P\left(\frac{n}{b_n}\left(T_n(t) - \int_0^t W(u)du\right) \in A\right),$$

where A is a given domain of deviation, $(b(n), n > 0)$ is some sequence denoting the scale of deviation. When $b(n) = \sqrt{n}$, this is exactly the estimation of central limit theorem. When $b(n) = n$, it becomes the large deviations. Furthermore, when $b(n)/\sqrt{n} \to \infty$ and $b(n) = o(n)$, this is the so called moderate deviations. In other words, the moderate deviations investigate the convergence speed between the large deviations and central limit theorem.

Let $\{b_n, n \ge 1\}$ be a sequence of positive numbers satisfying

$$\frac{b_n}{n} \to 0 \quad \text{and} \quad \frac{b_n^2}{n} \to \infty. \tag{1.2}$$

Throughout this paper, we assume that

(a) For $w_n(u) := w_{i,n}, \quad \frac{i-1}{n} \le u \le \frac{i}{n}$, there exists a function $w(u)$ defined on $[0, 1]$ such that

$$w_n(u) - w(u) = o(1), \quad u \in (0, 1);$$
$$\frac{1}{n}\sum_{i=1}^{[nt]} W_{i,n} - \int_0^t W(u)du = o(b_n/n), \quad t \in [0, 1].$$

(b) For any $t \in (0, 1]$

$$0 < \int_0^t W^2(u)du < +\infty,$$

where $W(u) = \frac{1}{1-u}\int_u^1 w(v)dv, 0 \le u < 1$ and $W_{i,n} = \frac{1}{n-i+1}\sum_{j=i}^{n} w_{i,n}, i = 1, \ldots, n$.

Remark 1.1 In Sect. 4, we will find that statistics in the Jackson, Gini and Fortiana-Grané tests satisfy our conditions (a) and (b).

Now, we state our main results as follows:

Theorem 1.1 *Let $t \in (0, 1]$. For any closed subset $F \subset \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n}\left(T_n(t) - \int_0^t W(u)du\right) \in F\right) \le - \inf_{x\in F} I(x)$$

and for any open subset $G \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n}\left(T_n(t) - \int_0^t W(u)du\right) \in G\right) \ge - \inf_{x\in G} I(x)$$

where $I(x) = \frac{x^2}{2\int_0^t W^2(u)du}$. In particular, for $r > 0$, we have

$$P\left(\frac{n}{b_n}\left|T_n(t) - \int_0^t W(u)du\right| \ge r\right) \sim e^{-\frac{b_n^2 r^2}{2n\int_0^t W^2(u)du}}.$$

Theorem 1.2 *Let $D[0, 1]$ denote the space of right continuous functions with left limits in $[0, 1]$, endowed with the uniformly convergence topology. Suppose that $\frac{0}{0} := 0$. Then for any closed subset $F \subset D[0, 1]$,*

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n}\left(T_n(t) - \int_0^t W(u)du\right)_{t\in[0,1]} \in F\right) \le - \inf_{x\in F} J(x)$$

and for any open subset $G \subset D[0, 1]$,

$$\liminf_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n}\left(T_n(t) - \int_0^t W(u)du\right)_{t\in[0,1]} \in G\right) \ge - \inf_{x\in G} J(x)$$

where

$$J(x) = \begin{cases} \frac{1}{2}\int_0^1 \frac{(x'(t))^2}{W^2(t)}dt, & \text{if } x \in H; \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$H = \Big\{x \in D[0, 1] : x \text{ is absolutely continuous with } x(0) = 0$$
$$\text{and } \int_0^1 \frac{(x'(t))^2}{W^2(t)}dt < +\infty\Big\}.$$

In particular, for $r > 0$, we have

$$P\left(\frac{n}{b_n}\sup_{0\le t\le 1}\left|T_n(t) - \int_0^t W(u)du\right| \ge r\right) \sim e^{-\frac{b_n^2}{n}\inf_{x\in\mathcal{A}} J(x)},$$

where $\mathcal{A} = \{x \in D[0, 1] : \sup_{0\le t\le 1} |x(t)| \ge r\}$.

For convenience, let us introduce some notations in large deviations (cf. [1, 12]).

Definition 1.1 Let $(E, \mathcal{E})$ be a metric space with metric ρ, and let $\{Y_n, n \ge 1\}$ be a family of E-valued random variables. Denoting the law of Y_n by μ_n. Let λ_n be a sequence of positive real numbers satisfying $\lambda(n) \to 0$ as $n \to \infty$.

(1) A function $I(x): E \to [0, +\infty]$ is said to a rate function if it is lower semicontinuous and it is said to a good rate function if its level set $\{x \in E : I(x) \le a\}$ is compact for all $a \ge 0$.
(2) The sequence of probability measures $\{\mu_n, n \ge 1\}$ or the sequence $\{Y_n, n \ge 1\}$ is said to satisfy a large deviation principle (LDP) with speed $\lambda(n)$ and rate function $I(\cdot)$ if for any closed set $F \in \mathcal{E}$

$$\limsup_{n\to\infty} \lambda(n) \log \mu_n(F) \le -\inf_{x\in F} I(x)$$

and for any open set $G \in \mathcal{E}$,

$$\liminf_{n\to\infty} \lambda(n) \log \mu_n(G) \ge -\inf_{x\in G} I(x).$$

(3) The sequence $\{\mu_n, n \ge 1\}$ is exponentially tight if for every $L > 0$, there is a compact set $K_L \in \mathcal{E}$ such that

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \lambda(n) \log P(Y_n \in K_L^c) \le -L.$$

2 Moderate Deviations for $L_n(t)$

In this section, we show Theorem 1.1 by Rényi's representation for exponential order statistics (cf. [2, 8]).

2.1 Rényi's Representation

Let $\{e_n, n \ge 1\}$ be a sequence of i.i.d. exponential random variables with parameter 1, then the density function of $(e_1^*, e_2^*, \dots, e_n^*)$

$$f^*(x_1, x_2, \dots, x_n) = n! e^{-(x_1+x_2+\cdots+x_n)} I_{[0<x_1<\cdots<x_n]}(x_1, \dots, x_n)$$

which implies that the density function of $(e_1^*, e_2^* - e_1^*, \dots, e_n^* - e_{n-1}^*)$ is

$$g(x_1, x_2, \dots, x_n) = \prod_{k=1}^{n} (n-k+1) e^{-(n-k+1)x_k} I_{[0,+\infty)}(x_k).$$

Therefore, $\{(n-k+1)(e_k^* - e_{k-1}^*), k \ge 1\}$ is also a sequence of i.i.d. exponential random variables with parameter 1, where $e_0^* = 0$ which implies that $\{e_k^*, 1 \le k \le n\}$ and $\{\sum_{j=n-k+1}^{n} \frac{e_{n-j+1}}{j}, 1 \le k \le n\}$ are identical distribution, that is

$$\{e_k^*, 1 \le k \le n\}(\mathcal{L}) = \left\{ \sum_{j=n-k+1}^{n} \frac{e_{n-j+1}}{j}, 1 \le k \le n \right\}.$$

Then, we have

$$T_n(t) = \frac{\sum_{i=1}^{[nt]} w_{i,n} X_i^*}{\sum_{i=1}^{n} X_i} (\mathcal{L}) = \frac{\sum_{i=1}^{[nt]} W_{i,n} X_i}{\sum_{i=1}^{n} X_i}. \tag{2.1}$$

2.2 Moderate Deviations for $T_n(t)$

We first state the following lemma which is important to our proof.

Lemma 2.1 *For any* $t \in [0, 1]$, *we have*

$$\frac{1}{n}\sum_{i=1}^{[nt]} W_{i,n}^2 \to \int_0^t W^2(u)du, \quad n \to \infty.$$

Proof Since $w_n(u) - w(u) = o(1)$ and

$$W_{i,n} = \frac{1}{1-(i-1)/n}\int_{\frac{i-1}{n}}^1 w_n(u)du,$$

we can obtain by conditions (a) and (b) as $n \to \infty$

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{[nt]} W_{i,n}^2 &= \frac{1}{n}\sum_{i=1}^{[nt]}\left(\frac{1}{1-(i-1)/n}\int_{\frac{i-1}{n}}^1 w_n(u)du\right)^2 \\ &= \frac{1}{n}\sum_{i=1}^{[nt]}\left(\frac{1}{1-(i-1)/n}\int_{\frac{i-1}{n}}^1 w(u)du\right. \\ &\quad \left.+\frac{1}{1-(i-1)/n}\int_{\frac{i-1}{n}}^1 (w_n(u)-w(u))du\right)^2 \\ &\to \int_0^t W^2(u)du.\end{aligned}$$

□

Lemma 2.2 *Let* $\{e_i, i \geq 1\}$ *be i.i.d. random variables and exponential with parameter* 1, *and* $\tilde{e}_i = e_i - Ee_i$. *Then* $\{P(\frac{1}{b_n}\sum_{j=1}^{[nt]} W_{i,n}\tilde{e}_i \in \cdot), n \geq 1\}$ *satisfies the large deviations with the speed* $\frac{n}{b_n^2}$ *and the rate function*

$$J_t(x) = \frac{t^2}{2\int_0^t W^2(u)du}x^2.$$

In particular, for any sequence of intervals (x_n, y_n) *satisfying* $x_n \to x$, $y_n \to y$ *and* $x < y$,

$$\begin{aligned}&\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{1}{b_n}\sum_{j=1}^{[nt]} W_{i,n}\tilde{e}_i \in (x_n, y_n)\right) \\ &= \lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{1}{b_n}\sum_{j=1}^{[nt]} W_{i,n}\tilde{e}_i \in [x_n, y_n]\right) = -\min\{J_t(x), J_t(y)\}.\end{aligned}$$

Proof For any $i \geq 1$, we have

$$Ee^{s\tilde{e}_i} = \begin{cases}\frac{1}{1-s}, & \text{if } s < 1;\\ +\infty, & \text{otherwise.}\end{cases} \tag{2.2}$$

Let

$$\Lambda_t(\lambda)=\lim_{n\to\infty}\frac{n}{b_n^2}\log Ee^{\frac{\lambda b_n}{n}\sum_{i=1}^{[nt]}W_{i,n}\tilde{e}_i}.$$

For any $\lambda\in\mathbb{R}$, we can choose n large enough such that $\frac{\lambda b_n W_{i,n}}{n}<1$. By (2.2) and $E(\sum_{i=1}^{[nt]}W_{i,n}e_i)=\sum_{i=1}^{[nt]}W_{i,n}$,

$$\begin{aligned}\Lambda_t(\lambda)&=\lim_{n\to\infty}\frac{n}{b_n^2}\log(e^{-\frac{\lambda b_n}{n}\sum_{i=1}^{[nt]}W_{i,n}}Ee^{\frac{\lambda b_n}{n}\sum_{i=1}^{[nt]}W_{i,n}e_i})\\&=\lim_{n\to\infty}\frac{n}{b_n^2}\left(-\frac{\lambda b_n}{n}\sum_{i=1}^{[nt]}W_{i,n}-\sum_{i=1}^{[nt]}\log\left(1-\frac{\lambda b_n}{n}W_{i,n}\right)\right).\end{aligned}$$

Applying Taylor formula to the function $\log(1-x)$, we have

$$\sum_{i=1}^{n}\log\left(1-\frac{\lambda b_n W_{i,n}}{n}\right)=-\sum_{i=1}^{n}\frac{\lambda b_n W_{i,n}}{n}-\frac{1}{2}\sum_{i=1}^{n}\frac{\lambda^2 b_n^2 W_{i,n}^2}{n^2}+o\left(\frac{b_n^2}{n}\right).\tag{2.3}$$

By Lemma 2.1, we have $\Lambda_t(\lambda)=\frac{1}{2}\lambda^2\int_0^t W^2(u)du$. Since the Legendre transform of $\Lambda_t(\lambda)$ is

$$\Lambda_t^*(x)=\sup_{\lambda\in\mathbb{R}}\{\lambda x-\Lambda_t(\lambda)\}=\frac{x^2}{2\int_0^t W^2(u)du},$$

Gärtner-Ellis theorem implies the conclusion of the lemma. □

We can obtain the following result easily,

Corollary 2.1 *Let $\{e_i,i\geq 1\}$ be i.i.d random variables and exponential with parameter* 1, *and $\tilde{e}_i=e_i-Ee_i$. Then $\{P(\frac{1}{b_n}\sum_{j=1}^{[nt]}\tilde{e}_i\in\cdot),n\geq 1\}$ satisfies the large deviations with the speed $\frac{n}{b_n^2}$ and the rate function $\hat{J}_t(x)=\frac{t}{2}x^2$.*

Now, we prove the exponential equivalence of

$$\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{\sum_{i=1}^{n}X_i}-\int_0^t W(u)du,\qquad \frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{n}-\frac{\sum_{i=1}^{[nt]}W_{i,n}}{n}.$$

The we can study the moderate deviation of $\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{n}-\frac{\sum_{i=1}^{[nt]}W_{i,n}}{n}$ instead of $\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{\sum_{i=1}^{n}X_i}-\int_0^t W(u)du$.

Lemma 2.3 *For any $r>0$, we have*

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{n}{b_n}\left|\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{\sum_{i=1}^{n}X_i}-\int_0^t W(u)du-\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{n}+\frac{\sum_{i=1}^{[nt]}W_{i,n}}{n}\right|\geq r\right)=-\infty.$$

Proof By simple calculation, we have

$$\left\{\frac{n}{b_n}\left|\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{\sum_{i=1}^{n}X_i}-\int_0^t W(u)du-\frac{\sum_{i=1}^{[nt]}W_{i,n}X_i}{n}+\frac{\sum_{i=1}^{[nt]}W_{i,n}}{n}\right|\geq r\right\}$$

$$\subset \left\{ \frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i}{\sum_{i=1}^{n} X_i} - \frac{\sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i}{n} \right| \geq \frac{r}{2} \right\} \cup \left\{ \frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n}}{\sum_{i=1}^{n} X_i} - \int_0^t W(u) du \right| \geq \frac{r}{2} \right\},$$

where $\tilde{X}_i = X_i - EX_i$. Then for any $L > 0$,

$$\begin{aligned} &\left\{ \frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i}{\sum_{i=1}^{n} X_i} - \frac{\sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i}{n} \right| \geq \frac{r}{2} \right\} \\ &\quad \subset \left\{ \frac{1}{b_n} \left| \sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i \right| \geq L \right\} \cup \left\{ \left| \frac{1}{\frac{1}{n} \sum_{i=1}^{n} X_i} - 1 \right| \geq \frac{r}{2L} \right\}. \end{aligned}$$

By Lemma 2.2 and Corollary 2.1, we have

$$\lim_{L\to\infty} \lim_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{1}{b_n} \left| \sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i \right| \geq L \right) = -\infty,$$

and

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log P\left(\left| \frac{1}{\frac{1}{n} \sum_{i=1}^{[nt]} X_i} - 1 \right| \geq \frac{r}{2L} \right) = -\infty.$$

Consequently

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i}{\sum_{i=1}^{n} X_i} - \frac{\sum_{i=1}^{[nt]} W_{i,n} \tilde{X}_i}{n} \right| \geq \frac{r}{2} \right) = -\infty. \tag{2.4}$$

Furthermore, for any $1 > \delta > 0$, we have that

$$\begin{aligned} &\left\{ \frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n}}{\sum_{i=1}^{n} X_i} - \int_0^t W(u) du \right| \geq \frac{r}{2} \right\} \\ &\quad \subset \left\{ \frac{1}{n} \left| \sum_{i=1}^{n} \tilde{X}_i \right| \geq \delta \right\} \cup \left\{ \frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n}}{n(1-\delta)} - \int_0^t W(u) du \right| \geq \frac{r}{2} \right\} \\ &\quad \cup \left\{ \frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n}}{n(1+\delta)} - \int_0^t W(u) du \right| \geq \frac{r}{2} \right\}. \end{aligned}$$

By hypothesis (a), Lemma 2.1 and Corollary 2.1, we have

$$\lim_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n} \left| \frac{\sum_{i=1}^{[nt]} W_{i,n}}{\sum_{i=1}^{n} X_i} - \int_0^t W(u) du \right| \geq \frac{r}{2} \right) = -\infty. \tag{2.5}$$

We can immediately complete the proof of this lemma from (2.4) and (2.5). □

Proof of Theorem 1.1 By Lemma 2.2 and Lemma 2.3, Theorem 1.1 can be obtained immediately. □

3 Functional Moderate Deviations

In this section, we prove the functional moderate deviation theorem (Theorem 1.2). It is well known that the moderate deviation of finite dimensional distributions

$$P\left(\frac{n}{b_n}\left(T_n(t_1)-\int_0^{t_1} W(u)du,\ldots,T_n(t_k)-\int_0^{t_1} W(u)du\right)\right),\quad 0<t_1<\cdots<t_k\le 1,\ k\ge 1$$

and the following exponential tightness: for any $t\in[0,1]$ and $\eta>0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\log P\left(\frac{n}{b_n}\sup_{0\le s\le\delta}\left|T_n(t+s)-\int_0^{t+s} W(u)du-T_n(t)+\int_0^t W(u)du\right|>\eta\right)=-\infty$$

are sufficient for the moderate deviation of $\{T_n(t),t\in[0,1]\}$ (cf. [1, 12]).

3.1 Moderate Deviations of Finite Dimensional distributions

Lemma 3.1 *For* $0=t_0<t_1<\cdots<t_k\le 1,k\ge 1$, *set*

$$Y_n(t_1,\ldots,t_k)=\left(T_n(t_1)-\int_0^{t_1} W(u)du,\ldots,T_n(t_k)-\int_0^{t_1} W(u)du\right)\tag{3.1}$$

and

$$Z_n(t_1,\ldots,t_k)=\left(\frac{\sum_{i=1}^{[nt_1]} W_{i,n}X_i}{n}-\frac{1}{n}\sum_{i=1}^{[nt_1]} W_{i,n},\ldots,\frac{\sum_{i=1}^{[nt_k]} W_{i,n}X_i}{n}-\frac{1}{n}\sum_{i=1}^{[nt_k]} W_{i,n}\right).\tag{3.2}$$

Then for any $r>0$, *we have*

$$\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{n}{b_n}|Y_n(t_1,\ldots,t_k)-Z_n(t_1,\ldots,t_k)|\ge r\right)=-\infty.$$

Proof By Lemma 2.3,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{n}{b_n^2}\log P\left(\frac{n}{b_n}|Y_n(t_1,\ldots,t_k)-Z_n(t_1,\ldots,t_k)|\ge r\right)\\
&=\lim_{n\to\infty}\frac{n}{b_n^2}\max_{1\le j\le k}\log P\left(\frac{n}{b_n}\left|\frac{\sum_{i=1}^{[nt_j]} W_{i,n}X_i}{\sum_{i=1}^{n} X_i}-\int_0^{t_j} W(u)du-\frac{\sum_{i=1}^{[nt_j]} W_{i,n}X_i}{n}\right.\right.\\
&\quad\left.\left.+\frac{\sum_{i=1}^{[nt_j]} W_{i,n}}{n}\right|\ge\frac{r}{\sqrt{k}}\right)\\
&=-\infty.
\end{aligned}$$

□

Lemma 3.2 *Let* $Z_n(t_1,\ldots,t_k)$ *defined by* (3.2). *Then*

$$\left\{P\left(\frac{n}{b_n}(Z_n(t_1,t_2,\ldots,t_k)-EZ_n(t_1,t_2,\ldots,t_k))\in\cdot\right),n\ge 1\right\}$$

satisfies the large deviations in $\mathbb{R}^k$ *with the speed* $\frac{b_n^2}{n}$ *and the rate function*

$$\hat{J}_{t_1,\dots,t_k}(x) = \sum_{j=1}^{k} \frac{(x_j - x_{j-1})^2}{2\int_{t_{j-1}}^{t_j} W^2(u)du}$$

where $x = (x_1, \dots, x_k), x_0 := 0$.

Proof For n large enough we have $1 < [nt_1] < \cdots < [nt_k] < n$, so by applying the homomorphism

$$\Gamma : (x_1, x_2, \dots, x_k) \to (x_1, x_2 - x_1, \dots, x_k - x_{k-1})$$

Z_n can be mapped to the random vector $V_n(t_1, t_2, \dots, t_k) = \Gamma Z_n(t_1, t_2, \dots, t_k)$ with independent components. Then we consider the large deviations of

$$\frac{n}{b_n}(V_n(t_1, t_2, \dots, t_k) - EV_n(t_1, \dots, t_k)).$$

Similar to the proof of Lemma 2.2, one can get that for any $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$,

$$\Lambda_{t_1,t_2,\dots,t_k}(\lambda) = \lim_{n\to\infty} \frac{n}{b_n^2} \log E_0 e^{\lambda b_n \langle \lambda, Z_n - EZ_n\rangle} = \frac{1}{2}\sum_{j=1}^{k} \lambda_j^2 \int_{t_{j-1}}^{t_j} W^2(u)du.$$

By Gärtner-Ellis theorem,

$$\left\{ P\left(\frac{n}{b_n}(V_n(t_1, t_2, \dots, t_k) - EV_n(t_1, t_2, \dots, t_k)) \in \cdot \right), n \ge 1 \right\}$$

satisfies the large deviations in $\mathbb{R}^k$ with the speed $\frac{b_n^2}{n}$ and the rate function

$$\Lambda^*_{t_1,t_2,\dots,t_k}(x) = \frac{1}{2}\sum_{j=1}^{k} \frac{x_j^2}{\int_{t_{j-1}}^{t_j} W^2(u)du}.$$

Then by the inverse contraction principle,

$$\left\{ P\left(\frac{n}{b_n}(Z_n(t_1, t_2, \dots, t_k) - E_0 Z_n(t_1, t_2, \dots, t_k)) \in \cdot \right), n \ge 1 \right\}$$

satisfies large deviations with speed $\frac{b_n^2}{n}$ and the rate function $\hat{J}_{t_1,t_2,\dots,t_k}(x)$. □

By Lemma 3.1 and Lemma 3.2, we can obtain the finite dimensional moderate deviation of $Y_n(t_1, \dots, t_k)$ defined by (3.1).

Theorem 3.1 *Suppose* $0 = t_0 < t_1 < t_2 < \cdots < t_k \le 1$. *For any closed subset* $F \subset \mathbb{R}^k$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n}(Y_n(t_1, \dots, t_k) - EY_n(t_1, \dots, t_k)) \in F \right) \le -\inf_{x\in F} J_{t_1,\dots,t_k}(x)$$

and for any open subset $G \subset \mathbb{R}^k$,

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n}(Y_n(t_1,\ldots,t_k) - EY_n(t_1,\ldots,t_k)) \in G\right) \geq -\inf_{x\in G} J_{t_1,\ldots,t_k}(x)$$

where

$$J_{t_1,\ldots,t_k}(x) = \sum_{j=1}^{k} \frac{(x_j - x_{j-1})^2}{2\int_{t_{j-1}}^{t_j} W^2(u)du}$$

and $x = (x_1,\ldots,x_k)$, $x_0 := 0$.

3.2 Exponential Tightness

We first state the following maximum inequality, which will be used soon.

Lemma 3.3 *Suppose* $\{\xi_n, n \geq 1\}$ *be a sequence of i.i.d random variables and* $S_n = \xi_1 + \cdots + \xi_n, n \geq 1$. *Then for all* $r > 0$,

$$P\left(\max_{1\leq j\leq n} |S_j| > 3r\right) \leq 3 \max_{1\leq j\leq n} P(|S_j| > r).$$

Lemma 3.4 *For any* $t \in [0,1]$ *and* $\eta > 0$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{b_n} \sup_{0\leq s\leq\delta} \left|\frac{\sum_{i=1}^{[nt+ns]} W_{i,n}\tilde{X}_i}{\sum_{i=1}^n X_i} - \frac{\sum_{i=1}^{[nt]} W_{i,n}\tilde{X}_i}{\sum_{i=1}^n X_i}\right| > \eta\right) = -\infty,$$

where $\tilde{X}_i = X_i - E_0X_i$.

Proof By simple calculation, for $L > 1$,

$$\begin{aligned}&\left\{\frac{n}{b_n} \sup_{0\leq s\leq\delta} \left|\frac{\sum_{i=1}^{[nt+ns]} W_{i,n}\tilde{X}_i}{\sum_{i=1}^n X_i} - \frac{\sum_{i=1}^{[nt]} W_{i,n}\tilde{X}_i}{\sum_{i=1}^n X_i}\right| > \eta\right\}\\ &\quad\subset \left\{\frac{1}{b_n} \sup_{0\leq s\leq\delta} \left|\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right| > \eta/L\right\} \cup \left\{\frac{n}{\sum_{i=1}^n X_i} > L\right\}.\end{aligned}$$

By Corollary 2.1, we have

$$\limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{n}{\sum_{i=1}^n X_i} > L\right) = \limsup_{n\to\infty} \frac{n}{b_n^2} \log P\left(\frac{1}{n}\sum_{i=1}^{[nt]} \tilde{X}_i \leq \frac{1}{L} - 1\right) = -\infty. \quad (3.3)$$

By the Ottaviani inequality (Lemma 3.3), for any $\eta > 0$,

$$P\left(\frac{1}{b_n} \sup_{0\leq s\leq\delta} \left|\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right| \geq \eta/L\right) \leq 3 \sup_{0\leq s\leq\delta} P\left(\left|\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right| \geq b_n\eta/3L\right),$$

and the Chebyshev inequality,

$$
\begin{aligned}
&P\left(\left|\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right| \geq b_n\eta/3L\right)\\
&\quad\leq \exp\left\{-\frac{\lambda b_n^2\eta}{3Ln}\right\}\left(E_0\exp\left\{\lambda\frac{b_n}{n}\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right\} + E_0\exp\left\{-\lambda\frac{b_n}{n}\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right\}\right).
\end{aligned}
$$

By simple calculation

$$
E_0\exp\left\{\lambda\frac{b_n}{n}\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right\} = \prod_{i=[nt]}^{[nt+ns]}\left(1+\frac{\lambda^2 b_n^2}{n^2}W_{i,n}^2 + o\left(\frac{b_n^2}{n^2}\right)\right).
$$

Consequently,

$$
\lim_{\lambda\to+\infty}\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\frac{n}{b_n^2}\log P\left(\left|\sum_{i=[nt]}^{[nt+ns]} W_{i,n}\tilde{X}_i\right| \geq b_n\eta/3L\right) = -\lim_{\lambda\to+\infty}\frac{\lambda\eta}{3L} = -\infty. \tag{3.4}
$$

The proof of this lemma can be completed by (3.3) and (3.4). □

Applying Lemma 2.1 and Lemma 3.4, the following exponential tightness can be obtained.

Lemma 3.5 *For any $t \in [0,1]$ and $\eta > 0$,*

$$
\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\log P\left(\frac{n}{b_n}\sup_{0\leq s\leq\delta}\left|T_n(t+s)-\int_0^{t+s}W(u)du - T_n(t)+\int_0^t W(u)du\right| > \eta\right) = -\infty.
$$

3.3 Proof of Theorem 1.2

By Theorem 3.1 and Lemma 3.5,

$$
\left\{P\left(\frac{n}{b_n}\left(T_n(t)-\int_0^t W(u)du\right)_{t\in[0,1]}\in\cdot\right)\right\}
$$

satisfies the large deviations with the speed $\frac{b_n^2}{n}$ and the rate function

$$
\begin{aligned}
\hat{J}(x) = \sup\{&J_{t_1,\ldots,t_k}\big(x(t_1),\ldots,x(t_k)\big);\\
&0=t_0<t_1<\cdots<t_k\leq 1, k\geq 0\},\ x\in D[0,1].
\end{aligned}
$$

It is enough to prove that $\hat{J}(x) = J(x)$. For $x(t)\in D[0,1]$, set $F(t)=\int_0^t W^2(u)du$ for $t\in[0,1]$. If $x\in H$, then for $0=t_0<t_1<\cdots<t_k\leq 1$,

$$
\begin{aligned}
&\sum_{j=1}^k\frac{(x(t_j)-x(t_{j-1}))^2}{2\int_{t_{j-1}}^{t_j}W^2(u)du}\\
&\quad=\frac{1}{2}\sum_{j=1}^k(F(t_j)-F(t_{j-1}))\left(\frac{x(t_j)-x(t_{j-1})}{F(t_j)-F(t_{j-1})}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{k}(F(t_j)-F(t_{j-1}))\left(\frac{1}{F(t_j)-F(t_{j-1})}\int_{t_{j-1}}^{t_j}\frac{x'(t)}{W^2(t)}dF(t)\right)^2 \\
&\le \frac{1}{2}\int_0^1 \frac{(x'(t))^2}{W^2(t)}dt,
\end{aligned}
$$

which implies that $\hat{J}(x) \le J(x)$ for all $x \in H$.

On the other hand, suppose $\hat{J}(x) < +\infty$. Denote by $\alpha_i^n = \frac{i}{2^n}$, $0 \le i \le 2^n$. By the convergence of martingales,

$$
\psi^n(s) = \sum_{i=0}^{2^n}\frac{x(\alpha_{i+1}^n)-x(\alpha_i^n)}{F(\alpha_{i+1}^n)-F(\alpha_i^n)} I_{[\alpha_i^n,\alpha_i^n)}(s) \to \frac{x'(s)}{F'(s)}, \quad \text{a.s.}
$$

By Fatou lemma,

$$
\frac{1}{2}\int_0^1\left(\frac{x'(s)}{F'(s)}\right)^2 dF(s) \le \frac{1}{2}\limsup_{n\to\infty}\int_0^1 |\psi^n(s)|^2 dF(s) \le \hat{J}(x) < +\infty.
$$

Therefore, $J(x) = \hat{J}(x)$. □

4 Application to Some Examples

Now we apply our results to the three statistics of type (1.1) to get their moderate deviations. We only have to check that the conditions (a) and (b) are satisfied.

4.1 Gini Test

$$
G_n = \frac{\sum_{i,j=1}^n |X_i - X_j|}{2(n-1)\sum_{i=1}^n X_i},
$$

where $\{X_i, i \ge 1\}$ be i.i.d. random variables and exponential with parameter 1. Gail and Gastwirth [4] showed that $G_n - \frac{1}{2}$ is a normalized L-statistics with coefficients

$$
w_{i,n} = 2\frac{i-1}{n-1} - \frac{3}{2}.
$$

Moreover, Tchirina [11] shows that $w_n(u)$ defined in conditions (a) converges to $w(x) = 2x - \frac{3}{2}$ uniformly on [0, 1] and $W(x) = x - \frac{1}{2}$. Then we have

$$
W_{i,n} := \frac{1}{n-i+1}\sum_{j=i}^{n} w_{j,n} = \frac{n-i-2}{n-1} - \frac{3}{2},
$$

which implies that for $t \in [0, 1]$,

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{[nt]} W_{i,n} &= \frac{[nt](1+[nt])}{2n(n-1)} + \frac{n[nt]}{n(n-1)} - \frac{3[nt]}{2n} - \frac{2[nt]}{n(n-1)} \\
&= \int_0^t W(x)dx + o(1/n).
\end{aligned}
$$

Therefore, the conditions (a) and (b) are satisfied. Then

Theorem 4.1 $\{P(\frac{n}{b_n}(G_n - \frac{1}{2}) \in \cdot), n \geq 1\}$ *satisfies the large deviation principle with rate* $\frac{b_n^2}{n}$ *and rate function* $I_1(x) = 6x^2$. *In particular, for* $r > 0$, *we have*

$$P\left(\frac{n}{b_n}\left|G_n - \frac{1}{2}\right| \geq r\right) \sim e^{-\frac{6b_n^2 r^2}{n}}.$$

4.2 Fortiana-Grané Test

Fortiana and Grané [3] have represented a characterization-based $1 - FG_n$ in the form of normalized L-statistics with coefficients

$$w_{i,n} = 1 - \log n - (n-i)\log(n-i) + (n-i+1)\log(n-i+1).$$

and

$$W_{i,n} = 1 + \log\left(1 - \frac{i-1}{n}\right).$$

$w_n(u)$ defined in condition (a) converges pointwise to $w(x) = 2 + \ln(1-x)$ in (0, 1) and the corresponding function $W(x) = 1 + \log(1-x)$. It is easy to obtain that

$$\frac{1}{n}\sum_{i=1}^{[nt]} W_{i,n} = \int_0^t W(x)dx + O(1/n)$$

which implies the following result:

Theorem 4.2 $\{P(\frac{n}{b_n}(1 - FG_n) \in \cdot), n \geq 1\}$ *satisfies the large deviation principle with rate* $\frac{b_n^2}{n}$ *and rate function* $I_2(x) = x^2/2$. *In particular, for* $r > 0$, *we have*

$$P\left(\frac{n}{b_n}|1 - FG_n| \geq r\right) \sim e^{-\frac{b_n^2 r^2}{2n}}.$$

4.3 Jackson Test

Jackson statistics [7] is a L-statistics with the expectations of order statistics exponential sample for coefficients:

$$J_n = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{n}{n-i+1}\frac{X_k^*}{\sum_{i=1}^n X_i}.$$

Moreover

$$E_0 J_n = 2 - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \to 2, \quad n \to \infty$$

and

$$W_{i,n} = 1 - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k} + \sum_{k=1}^{n-i+1}\frac{1}{k}.$$

It is easily calculate (cf. [11]),

$$w(x) = 2 + \log(1-x), \qquad W(x) = 1 + \log(1-x).$$

Since for any $t \in (0, 1)$

$$\begin{aligned}\left|\sum_{k=[nt]}^{n} \frac{1}{k} + \log t\right| &\le \left|\sum_{k=[nt]}^{n} \frac{1}{k} - \int_{[nt]}^{n+1} \frac{1}{x} dx\right| + \left|\int_{[nt]}^{n+1} \frac{1}{x} dx + \log t\right| \\ &\le \left(\frac{1}{[nt]} - \frac{1}{n}\right) + \log\left(1 + \frac{1-t}{nt-1}\right) = O(1/n),\end{aligned}$$

we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{[nt]} W_{i,n} &= \frac{[nt]}{n} - \frac{[nt]}{n^2}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n}\sum_{k=1}^{[nt]-1}\sum_{i=k+1}^{[nt]}\frac{1}{n-k+1} \\ &= (1-t)\sum_{k=n-[nt]+2}^{n}\frac{1}{k} + O(\log n/n) \\ &= -(1-t)\log(1-t) + O(\log n/n).\end{aligned}$$

Therefore, the conditions (a) and (b) are satisfied. Then

Theorem 4.3 $\{P(\frac{n}{b_n}(J_n - 2) \in \cdot), n \ge 1\}$ *satisfies the large deviation principle with rate* $\frac{b_n^2}{n}$ *and rate function* $I_3(x) = x^2/2$. *In particular, for* $r > 0$, *we have*

$$P\left(\frac{n}{b_n}|J_n - 2| \ge r\right) \sim e^{-\frac{b_n^2 r^2}{2n}}.$$

Acknowledgements The author thanks Professor Gao Fuqing for a number of fruitful discussions and advice.

References

1. Dembo, A., Zeitouni, D.: Large Deviations Techniques and Applications. Springer, Berlin (1998)
2. Feller, W.: An Introduction to Probability Theory and its Applications. Wiley, New York (1971)
3. Fortiana, G., Grané, A.: A scale-free goodness-of-fit statistic for the exponential distribution based on maximum correlations. J. Stat. Plan. Inference **108**, 85–97 (2002)
4. Gail, M.H., Gastwirth, J.L.: A scale-free goodness-of-fit test for the exponential distribution based on the Gini statistic. J. R. Stat. Soc. **40**(3), 350–357 (1978)
5. Helmers, R.: A Berry-Esseen theorem for linear combinations of order statistics. Ann. Probab. **9**(2), 342–347 (1981)
6. Mason, D.M., Shorack, G.R.: Necessary and sufficient conditions for asymptotic normality of L-statistics. Ann. Probab. **20**(4), 1779–1804 (1992)
7. Jackson, O.A.Y.: An analysis of departures from the exponential distribution. J. R. Stat. Soc. **29**(B), 540–549 (1967)
8. Rényi, A.: On the theory of order statistics. Acta Math. Sci. Hung. **4**, 191–232 (1953)
9. Shorack, G.R., Wellner, J.A.: Empirical Process with Applications to Statistics. Wiley, New York (1986)
10. Stigler, S.M.: Linear functions of order statistics with smooth weight functions. Ann. Stat. **2**, 676–693 (1974)
11. Tchirina, A.V.: Asymptotic properties of exponentiality tests based on L-statistics. Acta. Appl. Math. **97**, 297–309 (2007)
12. Wu, L.: An introduction to large deviations. In: Yan, J.A., Peng, S., Fang, S., Wu, L. (eds.) Several Topics in Stochastic Analysis, pp. 225–336. Academic Press of China, Bejing (1997)