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Asymptotic patterns of a reaction–diffusion equation with nonlinear–nonlocal functional response

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A reaction–diffusion system with nonlinear–nonlocal functional response is considered in this article. We develop the finite delays approximation method combining with the spreading speed and travelling wavefronts theory for semi-flow to discuss the existence of spreading speed c^* and travelling wavefronts without the differentiability assumption on the reaction function f. The asymptotic patterns and threshold property for the solutions of the considered system are described clearly according to the threshold parameter c^* which is exactly the minimal wave speed as well. Finally, we give an estimate of the spreading speed and minimal wave speed c^*.

Keywords: reaction–diffusion system; nonlinear–nonlocal functional response; spreading speed; travelling wavefront; minimal wave speed.

1. Introduction

In this article, we consider a reaction–diffusion equation with the following form:

$$\frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) + f\left(u(t,x), \int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\varphi(u(t-s,x-y))\,\mathrm{d}y\,\mathrm{d}s\right), \tag{1.1}$$

where f, g, φ are functions which will be described later. Equation (1.1) is a reaction–diffusion system with double non-linearity. One is the non-linear function $f(r,s)$ and another is $\varphi(u)$. The convolutional term $\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\varphi(u(t-s,x-y))\mathrm{d}y\,\mathrm{d}s$ is in fact a non-local weighted spatial-temporal averaging with spacial diffusion and time delay. $g(s,y)$ is a kernel and $\varphi(u)$ can be regarded as a non-linear functional response, for example, a non-linear birth function or a non-linear response function. In the recent years, there are lot of works focusing on the evolution systems with non-local effects that describe the joint effect of spatial diffusion and time delay (see, e.g., AL-Omari & Gourley, 2005; Wang *et al.*, 2006; Weng *et al.*, 2003; Weng & Wu, 2008; Weng & Xu, 2008; Zhao & Xiao, 2006). But to the best knowledge of the authors, there are no works on the system with double non-linearity on the form of convolution.

Some of the recent works on the asymptotic patterns include the travelling wave solutions and spreading speed for the parabolic partial differential equations. The wave phenomena and propagation have been observed in biology, chemical reaction, epidemiology and physics (Marek & Svobodová, 1975; Murray, 2002; Volpert *et al.*, 1994; Smith, 1995). Since the experimental work of Marek & Svobodová (1975) for Belousov–Zhabotinskii reactions, the investigations on travelling wave solutions and asymptotic speeds of spread (spreading speed) for various evolution systems modelling physical and biological phenomena are developed fast during these 30 years. Travelling waves were studied on non-linear reaction–diffusion equations (Brown & Carr, 1977; Murray, 2002), integral and integro-differential equations (AL-Omari & Gourley, 2005; Diekmann, 1978, 1979; Thieme, 1979; Thieme & Zhao, 2003; Wang *et al.*, 2006), delayed reaction–diffusion equations (Schaaf, 1987; Smith & Zhao, 2000; Wu & Zou, 2001), lattice differential systems (Weng *et al.*, 2003; Wu & Zou, 1997), operator iteration equations (Weinberger, 1982, 2002; Lui, 1989). The concept of asymptotic speed of spread was firstly introduced by Aronson & Weinberger (1975) and further developed by Weinberger (1982, 2002). More works can be found in Diekmann (1979), Lui (1989), Liang & Zhao (2007), Thieme (1979), Tian & Weng (2009), Weng *et al.* (2003), Zhao & Wang (2004) and Zhao & Xiao (2006).

We impose some assumptions on (1.1).

(G1) $g(t,x) \geqslant 0$, $g(t,-x) = g(t,x)$ and $\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\mathrm{d}y\,\mathrm{d}s = 1$;

(G2) for any $c \geqslant 0$, there is $0 < \tilde{\delta}(c) \leqslant \infty$ such that

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\mathrm{e}^{-\lambda(cs+y)}\mathrm{d}y\,\mathrm{d}s < \infty \quad \text{for } \lambda \in [0,\tilde{\delta}(c)) \text{ and}$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\mathrm{e}^{-\tilde{\delta}(c)(cs+y)}\mathrm{d}y\,\mathrm{d}s = \infty;$$

(F1) $\varphi \in C(\mathbb{R},\mathbb{R})$, there is $L_{\varphi R} > 0$ such that $|\varphi(r_1) - \varphi(r_2)| \leqslant L_{\varphi R}|r_1 - r_2|$ for any $r_1, r_2 \in [0, R]$;

(F2) for any $R_1, R_2 > 0$, there are L_{fR_1} and L_{fR_2} such that

$$|f(r_1,s_1) - f(r_2,s_2)| \leqslant L_{fR_1}|r_1 - r_2| + L_{fR_2}|s_1 - s_2| \quad \text{for } (r_i,s_i) \in [0,R_1]\times[0,R_2],\ i = 1,2;$$

(F3) $f \in C(\mathbb{R}^2,\mathbb{R})$, there is $u^+ > 0$ such that $f(0,\varphi(0)) = f(u^+,\varphi(u^+)) = 0$, $f(u,\varphi(u)) > 0$ for $u \in (0,u^+)$ and $f(u,\varphi(u)) < 0$ for $u > u^+$;

(F4) $f(r,s)$ is non-decreasing on s and $f(0,s) > 0$ for $0 < s < \varphi(u^+)$, $\varphi(u) > 0$ and $\varphi(u)$ is non-decreasing on $u \in [0,u^+]$;

(F5) there is a $\eta > 0$ such that for any $\tilde{d} \in (1-\eta,1)$, there exists a unique $\alpha \in (0,u^+)$ with $f(\alpha,\varphi(\alpha)\tilde{d}) = 0$, moreover, $f(u,\varphi(u)\tilde{d}) > 0$ for $0 < u < \alpha$ and $f(u,\varphi(u)\tilde{d}) < 0$ for $u > \alpha$;

From the assumption (F3), the model (1.1) has two steady states: $u \equiv 0$ and $u \equiv u^+$. For convenience, in the following sections, let $\tilde{L} := L_{\varphi u^+}$, $L' := L_{fu^+}$ and $\hat{L} := L_{f\varphi(u^+)}$.

We mention here that model (1.1) includes several important reaction–diffusion systems from the literature (see, e.g. AL-Omari & Gourley, 2005; Faria *et al.*, 2006; Schaaf, 1987; Wang *et al.*, 2006, Weng & Wu, 2008; Wu & Zou, 2001; Zhao & Xiao, 2006). Wang *et al.* (2006) investigated the following system of reaction–diffusion equations with non-local delayed non-linearities:

$$\frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x) + f\Big(u(t,x), \int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,x-y)u(t-s,y)\Big)\mathrm{d}y\mathrm{d}s). \tag{1.2}$$

They obtain the existence of travelling wavefronts in view of the upper-lower solutions method associated with the iteration technique. AL-Omari & Gourley (2005) considered travelling wavefronts for the equation describing the mature population for a single species with stage structure and distributed maturation delay:

$$\frac{\partial u_m(t,x)}{\partial t} = d_m \Delta u_m(t,x) + \int_0^{+\infty}\int_{-\infty}^{+\infty} G(x,y,s) f(s) \mathrm{e}^{-\gamma s} b(u_m(t-s,y)) \mathrm{d}y \mathrm{d}s - d(u_m(t,x)), \tag{1.3}$$

where $b(x)$ is a birth function and $d(x)$ is a death function. In particular, $b(x)$, $d(x)$ and $G(y,s)$ are selected as

$$b(x) = \alpha x, \quad d(x) = \beta x^2, \quad G(y,s) = \frac{1}{\sqrt{4\pi d_i s}} \mathrm{e}^{-\frac{y^2}{4d_i s}}. \tag{1.4}$$

It is easy to verify that (1.2) in Wang *et al.* (2006) as $n = 1$ is our special case, and our assumptions (F1)–(F5) and (G1)–(G2) are satisfied for (1.3) when (1.4) occurs. It is a fact that the investigation on (1.1) seems to be necessary and natural when the non-linearity of function f in (1.2) and the non-linearity of function b in (1.3) occur in a same equation. On the other hand, the differentiability of functions f, b and d is necessary for the proofs in the previous mentioned papers. We wonder whether we can weaken such a differentiability, which is also significant in some situations, while f, b and d are not differentiable at some points. These are the motivation of our work in this article.

Our work in this article includes the existence of travelling wave solutions and spreading speed, the relation between minimal wave speed and spreading speed, the estimation of spreading speed. We develop the finite delays approximation method used in Fang *et al.* (2008) and Zhao & Xiao (2006). The article is organized as follows. In Section 2.1, we consider a transection system (2.1) with a finite delay τ by using the theory of monotone semiflow developed by Liang & Zhao (2007) to obtain the spreading speed and minimal wave speed c_τ^*. We are interested in the relation between the finite delay transection system (2.1) and the infinite delay system (1.1), thus by analysing the properties of c_τ^* delicately, we obtain the conclusion that there is a c^* such that $c^* = \lim_{\tau\to+\infty} c_\tau^*$ being the spread speed and the minimal speed for (1.1) in Section 3. In additional, the existence of the travelling wave fronts for (1.1) is an important result in Section 3. At last, we give the estimation of the spreading speed and minimal wave speed. We want to mention here that only hypotheses (F1)–(F5) and (G1)–(G2) are demanded on the system (1.1) in Sections 2.1 and 3 except for Section 4. This shows that functions f and φ do not posses the differentiability property, therefore the general linearization of (1.1) at the zero solution is impossible, which cause another challenging technique for linearization to be needed. Please see Section 3.2 for the details.

2. The equation with finite delay

We consider the following initial problem:

$$\begin{cases} \dfrac{\partial \mathcal{U}(t,x)}{\partial t} = D\Delta \mathcal{U}(t,x) + f\left(\mathcal{U}(t,x), \displaystyle\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\varphi(\mathcal{U}(t-s,x-y)) \mathrm{d}y\,\mathrm{d}s\right), \\ \mathcal{U}(t,x) = \phi(t,x), \quad t\in[-\tau,0], \quad x\in\mathbb{R}, \end{cases} \tag{2.1}$$

where $\phi(t,x)$ is a given initial function.

2.1 *The monotonic solution semiflow of* (2.1)

Let η be defined in (F5) and $\tau > \tau_0$ satisfy

$$\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\mathrm{d}y\,\mathrm{d}s > \int_0^{\tau_0}\int_{-\infty}^{+\infty} g(s,y)\mathrm{d}y\,\mathrm{d}s > 1-\eta. \tag{2.2}$$

Define $G := \int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\mathrm{d}y\,\mathrm{d}s < 1$, where G depends on τ, but it is fixed if τ is fixed. Then by (F3) and (F5), (2.1) has two steady states $\mathcal{U} \equiv 0$ and $\mathcal{U} \equiv \mathcal{U}^+$ if $\tau > \tau_0$, where $\mathcal{U}^+$ satisfies

$$f\big(\mathcal{U}^+, \varphi(\mathcal{U}^+)G\big) = 0. \tag{2.3}$$

We will establish the existence, uniqueness of solutions and a comparison result for the initial problem (2.1). First, we give some notations.

$\mathbb{X}$: the set of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the supremum norm $\|\cdot\|_{\mathbb{X}}$.
$\mathbb{X}^+ := \{\psi \in \mathbb{X}: \psi(x) \geqslant 0, x \in \mathbb{R}\}$, $[0,r]_{\mathbb{X}} = \{u \in \mathbb{X}: r \geqslant \psi(x) \geqslant 0 \text{ for } x \in \mathbb{R}\}$
$\mathscr{C}$: the set of all bounded and continuous functions from $[-\tau,0]\times\mathbb{R}$ to $\mathbb{R}$, $[0,r]_{\mathscr{C}} = \{u \in \mathscr{C}: r \geqslant u(\theta,x) \geqslant 0 \quad \text{for } (\theta,x) \in [-\tau,0]\times\mathbb{R}\}$.
$\mathcal{C} := C([-\tau,0],\mathbb{X})$, $[0,r]_{\mathcal{C}} := \{\phi \in \mathcal{C}: \phi(\theta) \in [0,r]_{\mathbb{X}} \text{ for } \theta \in [-\tau,0]\}$.
$\mathcal{C}_+ := \{\phi \in \mathcal{C}: \phi(\theta) \geqslant_{\mathbb{X}} 0,\ \forall\theta \in [-\tau,0]\}$, so that $\mathcal{C}$ is a partially ordered Banach space.
$\bar{\mathscr{C}} := C([-\tau,0],\mathbb{R})$, $[0,r]_{\bar{\mathscr{C}}} := \{u \in \bar{\mathscr{C}}: r \geqslant u(\theta) \geqslant 0 \text{ for } \theta \in [-\tau,0]\}$.
$\phi(\theta)(x) = \phi(\theta,x)$, $\mathcal{U}_t(\theta)(x) = \mathcal{U}(t+\theta,x)$, $\theta \in [-\tau,0]$, $x \in \mathbb{R}$.

In view of (F1) and (F2), for convenience, we define

$$\tilde{L}_\tau := L_{\varphi\mathcal{U}^+},\quad L'_\tau := L_{f\mathcal{U}^+},\quad \hat{L}_\tau := L_{f\varphi(\mathcal{U}^+)},\quad [0,\mathcal{U}^+]_{\mathscr{C}} := \mathscr{C}_{(\tau)},\quad [0,\mathcal{U}^+]_{\mathcal{C}} := \mathcal{C}_{(\tau)}.$$

Note that the solution of the initial problem for the parabolic partial differential equation

$$\frac{\partial w(t,x)}{\partial t} = D\frac{\partial^2 w(t,x)}{\partial x^2},\quad (t,x) \in (0,+\infty)\times\mathbb{R},$$
$$w(0,x) = \psi(x),\quad x \in \mathbb{R},$$

is

$$w(t,x) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{-(x-y)^2}{4Dt}}\psi(y)\mathrm{d}y,$$

we define an operator $\mathcal{T}(t)$ as follows:

$$[\mathcal{T}(t)\psi](x) = \begin{cases} w(t,x), & t > 0,\\ \psi(x), & t = 0.\end{cases} \tag{2.4}$$

Obviously, $\mathcal{T}(t)\mathbb{X}^+ \to \mathbb{X}^+$ for all $t > 0$. Define an operator $\mathcal{F}: \mathcal{C} \to \mathbb{X}$ by

$$\mathcal{F}(\phi)(x) = f\left(\phi(0,x), \int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\varphi(\phi(-s,x-y))\mathrm{d}y\,\mathrm{d}s\right). \tag{2.5}$$

If we assume that $\phi \in \mathcal{C}$ is a given continuous initial function, the equivalent abstract form of (2.1) is

$$\mathcal{U}(t) = \mathcal{T}(t)[\phi(0)] + \int_0^t \mathcal{T}(t-\xi)\mathcal{F}(\mathcal{U}_\xi)\mathrm{d}\xi, \quad t > 0,$$
$$\mathcal{U}(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{2.6}$$

In order to apply the results in Martin & Smith (1990), we denote $\mathcal{T}(t,s) = \mathcal{T}(t-s)$ and prove the following lemma.

LEMMA 2.1 $\mathcal{T} = \{\mathcal{T}(t-s)\colon t \geqslant s \geqslant 0\}$ is in accordance with (T1)–(T3) in page 15 of Martin & Smith (1990), therefore, it is a $\mathcal{C}_0$ semigroup.

Proof. First, we are easy to have the conclusion that $\mathcal{T}(t,t) = \mathcal{T}(0)\psi \equiv \psi$ and $\mathcal{T}(t,s)\mathcal{T}(t)(s,r)\psi = \mathcal{T}(t-s)\mathcal{T}(s-r)\psi = \mathcal{T}(t-r)\psi = \mathcal{T}(t,r)\psi$ for all $t \geqslant s \geqslant r \geqslant 0$. Second, we want to claim that for each $\psi \in \mathbb{X}^+$ the map $(t,s) \to \mathcal{T}(t-s)\psi$ is continuous for $t \geqslant s \geqslant 0$. We only to prove that for each $\psi \in \mathbb{X}^+$

$$\lim_{t\to 0^+} [\mathcal{T}(t)\psi](x) = \lim_{t\to 0^+} w(t,x) = \psi(x). \tag{2.7}$$

Actually, for $\forall\, \epsilon > 0$, we can choose $M > 0$ large enough, such that

$$\left| \int_M^\infty + \int_{-\infty}^{-M} \frac{\mathrm{e}^{-y^2}}{\sqrt{\pi}} \mathrm{d}y \right| < \frac{\epsilon}{4\|\psi\|_{\mathbb{X}}},$$

which leads to

$$\left| \int_M^\infty + \int_{-\infty}^{-M} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-z^2} [\psi(x - \sqrt{4Dt}z) - \psi(x)]\mathrm{d}z \right| < \frac{\epsilon}{2} \quad \text{for } t > 0, \ \ x \in \mathbb{R}.$$

On the other hand, for each $x \in \mathbb{R}$, we have

$$\lim_{t\to 0^+} [\psi(x - \sqrt{4Dt}z) - \psi(x)] = 0 \quad \text{uniformly for } z \in [-M, M],$$

and thus

$$|\psi(x - \sqrt{4Dt}z) - \psi(x)| < \frac{\epsilon}{2} \quad \text{uniformly for } z \in [-M, M]$$

if $t > T(\epsilon)$ large enough. Therefore, we have

$$\begin{aligned}
\left| \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\left(\frac{(x-y)}{\sqrt{4Dt}}\right)^2}}{\sqrt{4\pi Dt}} \psi(y)\mathrm{d}y - \psi(x) \right| &= \left| \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\left(\frac{(x-y)}{\sqrt{4Dt}}\right)^2}}{\sqrt{4\pi Dt}} \psi(y)\mathrm{d}y - \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-y^2}}{\sqrt{\pi}} \psi(x)\mathrm{d}y \right| \\
&= \left| \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-z^2} [\psi(x - \sqrt{4Dt}z) - \psi(x)]\mathrm{d}z \right| \\
&= \left| \int_{-\infty}^{-M} + \int_M^{+\infty} + \int_{-M}^{M} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-z^2} [\psi(x - \sqrt{4Dt}z) - \psi(x)]\mathrm{d}z \right| \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } t > T(\epsilon).
\end{aligned}$$

That is,

$$\lim_{t\to 0^+}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\left(\frac{(x-y)}{\sqrt{4Dt}}\right)^2}}{\sqrt{4\pi Dt}}\psi(y)\mathrm{d}y=\psi(x)=[\mathcal{T}(0)\psi](x),$$

and (2.7) is valid. Finally, we can find that $\sup\{|\mathcal{T}(t,s)\psi|\colon \|\psi\|\leqslant 1\}\leqslant \mathrm{e}^{(t-s)}$ for all $t\geqslant s\geqslant 0$. Thus, (T1)–(T3) in Martin & Smith (1990) are valid for $t\geqslant s\geqslant 0$ and $\mathcal{T}$ is a $\mathcal{C}_0$ semigroup (see the remark on page 15 of Martin & Smith, 1990 after (T1)–(T3)). □

DEFINITION 2.1 A continuous function $\mathcal{V}\colon[-\tau,b)\to\mathbb{X}$ is called a supersolution (subsolution) of (2.1) on $[0,b)$ if

$$\mathcal{V}(t)\geqslant(\leqslant)\mathcal{T}(t)\phi(0,\cdot)+\int_0^t\mathcal{T}(t-r)\mathcal{F}(\mathcal{V}_r)\mathrm{d}r\quad\text{for } 0\leqslant t<b. \tag{2.8}$$

If $\mathcal{V}$ is both a supersolution and subsolution on $[0,b)$, then it is said to be a (mild) solution of (2.1).

REMARK 2.1 Assume that there is a bounded and continuous function $\mathcal{V}\colon[-\tau,b)\times\mathbb{R}\to\mathbb{R}$, with $b>0$ and such that $\mathcal{V}$ is C^1 in $t\in[0,b)$, C^2 in $x\in\mathbb{R}$, and

$$\frac{\partial\mathcal{V}(t,x)}{\partial t}\geqslant(\leqslant)D\frac{\partial^2\mathcal{V}(t,x)}{\partial x^2}+f\left(\mathcal{V}(t,x),\int_0^\tau\int_{-\infty}^{+\infty}g(s,y)\varphi(\mathcal{V}(t-s,x-y))\mathrm{d}y\,\mathrm{d}s\right)$$

for $(t,x)\in(0,b)\times\mathbb{R}$. Then by the fact that $\mathcal{T}(t)\mathbb{X}_+\subset\mathbb{X}_+$, it follows that (2.8) holds and hence $\mathcal{V}(t,x)$ is a supersolution (subsolution) of (2.1) on $[0,b)$.

THEOREM 2.1 For any $\tau_0<\tau<+\infty$ and $\phi\in\mathcal{C}_{(\tau)}$, suppose (F1)–(F5) and (G1) hold, then (2.1) has a unique solution $\mathcal{U}(t,x;\phi)$ on $[0,+\infty)$. For any pair of subsolution $\underline{\mathcal{U}}(t,x)$ and supersolution $\bar{\mathcal{U}}(t,x)$ with $0\leqslant\underline{\mathcal{U}}(t,x)\leqslant\bar{\mathcal{U}}(t,x)\leqslant\mathcal{U}^+$, $t\in[-\tau,0]$, $x\in\mathbb{R}$, $0\leqslant\underline{\mathcal{U}}(t,x)\leqslant\bar{\mathcal{U}}(t,x)\leqslant\mathcal{U}^+$ holds for all $t\geqslant 0$ and $x\in\mathbb{R}$.

Proof. For any $\phi\in\mathcal{C}_{(\tau)}$, we have

$$\phi(0,x)+h\mathcal{F}(\phi)(x)=\phi(0,x)+hf\left(\phi(0,x),\int_0^\tau\int_{-\infty}^{+\infty}g(s,y)\varphi(\phi(-s,x-y))\mathrm{d}y\,\mathrm{d}s\right)\geqslant 0$$

for any $h>0$. On the other hand, we have from (F4) that

$$\begin{aligned}\phi(0,x)+h\mathcal{F}(\phi)(x)&=\phi(0,x)+hf\left(\phi(0,x),\int_0^\tau\int_{-\infty}^{+\infty}g(s,y)\varphi(\phi(-s,x-y))\mathrm{d}y\,\mathrm{d}s\right)\\&\leqslant\phi(0,x)+hf\left(\phi(0,x),\varphi(\mathcal{U}^+)\int_0^\tau\int_{-\infty}^{+\infty}g(s,y)\mathrm{d}y\,\mathrm{d}s\right).\end{aligned}\tag{2.9}$$

If $\phi(0,x)=\mathcal{U}^+$, then we have from (2.9) and (2.3) that

$$\begin{aligned}\phi(0,x)+h\mathcal{F}(\phi)(x)&\leqslant\phi(0,x)+hf\left(\mathcal{U}^+,\varphi(\mathcal{U}^+)G\right)\\&=\phi(0,x)=\mathcal{U}^+\quad\text{for any } h>0.\end{aligned}$$

If $\phi(0,x)<\mathcal{U}^+$, then we have from (2.9) that

$$\phi(0,x)+h\mathcal{F}(\phi)(x)\leqslant\phi(0,x)+hf\left(\phi(0,x),\varphi(\mathcal{U}^+)G\right)\leqslant\mathcal{U}^+,$$

if $h > 0$ is small enough. Therefore, we always have $\phi(0) + h\mathcal{F}(\phi) \in [0, \mathcal{U}^+]_{\mathbb{X}}$. Consequently, we obtain

$$\lim_{h\to 0+} \frac{1}{h}\operatorname{dist}\left(\phi(0) + h\mathcal{F}(\phi); [0, \mathcal{U}^+]_{\mathbb{X}}\right) = 0, \quad \forall\, \phi \in \mathcal{C}_{(\tau)}.$$

By Corollary 4 in Martin & Smith (1990) with $K = \mathcal{C}_{(\tau)}$, $S(t, s) = \mathcal{T}(t - s)$, $B(t, \phi) = \mathcal{F}(\phi)$, we conclude that (2.1) admits a unique mild solution $\mathcal{U}(t, x; \phi)$ with $\mathcal{U}_t(\cdot; \phi) \in \mathcal{C}_{(\tau)}$ for $t \in [0, \infty)$. Moreover, we have from Corollary 2.2.5 in Wu (1996) that $\mathcal{U}(t, x; \phi)$ is a classical solution of (2.1) for $t > \tau$, and $\mathcal{C}_{(\tau)}$ is an invariant subset in $\mathcal{C}^+$ for (2.6).

For any $\phi_1, \phi_2 \in \mathcal{C}$, by (F1) and (F2),

$$\|\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2)\|_{\mathbb{X}} = \sup_{x\in\mathbb{R}} |\mathcal{F}(\phi_1)(x) - \mathcal{F}(\phi_2)(x)| \leqslant l \sup_{\theta\in[-\tau,0]} \|\phi_1(\theta) - \phi_2(\theta)\|_{\mathbb{X}},$$

where $l := L'_\tau + \hat{L}_\tau \tilde{L}_\tau$. Therefore, $\mathcal{F}$ is globally Lipschitz continuous in $\mathcal{C}$ and $\mathcal{F}$ is quasimonotone on $\mathcal{C}_{(\tau)}$ in the sense that

$$\lim_{h\to 0+} \frac{1}{h}\operatorname{dist}([\phi_1(0) - \phi_2(0)] + h[\mathcal{F}(\phi_1) - \mathcal{F}(\phi_2)]; \mathbb{X}^+) = 0 \tag{2.10}$$

for all $\phi_1, \phi_2 \in \mathcal{C}_{(\tau)}$ with $\phi_1 \geqslant \phi_2$. In fact, there are two subcases. If $\phi_1(0, x) = \phi_2(0, x)$, then we have from (F4) that

$$\phi_1(0, x) - \phi_2(0, x) + h[\mathcal{F}(\phi_1)(x) - \mathcal{F}(\phi_2)(x)] \geqslant 0.$$

If $\phi_1(0, x) > \phi_2(0, x)$, then we have (F2) that

$$\begin{aligned}
|\mathcal{F}(\phi_1)(x) - \mathcal{F}(\phi_2)(x)| \leqslant{}& L'_\tau|\phi_1(0, x) - \phi_2(0, x)| \\
&+ \hat{L}_\tau \tilde{L}_\tau \int_0^\tau \int_{-\infty}^{+\infty} g(s, y)|\phi_1(-s, x - y) - \phi_2(-s, x - y)|\,\mathrm{d}y\,\mathrm{d}s \quad (2.11)\\
\leqslant{}& L'_\tau|\phi_1(0, x) - \phi_2(0, x)| + \hat{L}_\tau \tilde{L}_\tau G \sup_{\theta\in[-\tau,0]} \|\phi_1(\theta) - \phi_2(\theta)\|_{\mathbb{X}}, \quad (2.12)
\end{aligned}$$

hence, we obtain

$$\begin{aligned}
\phi_1(0, x) - \phi_2(0, x) + h[\mathcal{F}(\phi_1)(x) - \mathcal{F}(\phi_2)(x)] &\geqslant [1 - hL'_\tau][\phi_1(0, x) - \phi_2(0, x)] \\
&\quad - h\hat{L}_\tau \tilde{L}_\tau G \sup_{\theta\in[-\tau,0]} \|\phi_1(\theta) - \phi_2(\theta)\|_{\mathbb{X}} \\
&\geqslant [1 - hL'_\tau][\phi_1(0, x) - \phi_2(0, x)] - 2h\hat{L}_\tau \tilde{L}_\tau G\mathcal{U}^+ \geqslant 0
\end{aligned}$$

as long as $h > 0$ is small enough. From the above discussion, (2.10) follows.

Assume that $\bar{\mathcal{U}}, \underline{\mathcal{U}}$ is a pair of supersolution and subsolution for (2.1) with $\bar{\mathcal{U}}(t, x), \underline{\mathcal{U}}(t, x) \in [0, \mathcal{U}^+]$ for $(t, x) \in [-r, \infty) \times \mathbb{R}$. We have from Corollary 5 in Martin & Smith (1990) and the fact $\bar{\mathcal{U}}(\theta, x) \geqslant \underline{\mathcal{U}}(\theta, x)$ for $(\theta, x) \in [-r, 0] \times \mathbb{R}$ that the solutions of (2.1) satisfy

$$0 \leqslant \mathcal{U}(t, \cdot; \underline{\mathcal{U}}_0) \leqslant \mathcal{U}(t, \cdot; \bar{\mathcal{U}}_0) \leqslant \mathcal{U}^+, \quad t \geqslant 0.$$

Again applying Corollary 5 in Martin & Smith (1990) with $[v^+(t,\cdot) = \mathcal{U}^+, v^-(t,\cdot) = \underline{\mathcal{U}}(t,\cdot)]$ and $[v^+(t,\cdot) = \bar{\mathcal{U}}(t,\cdot), v^-(t,\cdot) = 0]$, respectively, we obtain

$$\underline{\mathcal{U}}(t,\cdot) \leqslant \mathcal{U}(t,\cdot;\underline{\mathcal{U}}_0) \leqslant \mathcal{U}^+, \quad t \geqslant 0,$$
$$0 \leqslant \mathcal{U}(t,\cdot;\bar{\mathcal{U}}_0) \leqslant \bar{\mathcal{U}}(t,\cdot), \quad t \geqslant 0.$$

Combining the above three inequalities, we have $\underline{\mathcal{U}}(t,x) \leqslant \bar{\mathcal{U}}(t,x)$ for all $(t,x) \in (0,\infty) \times \mathbb{R}$.

Summarizing the above discussion, the proof is complete. □

In the following, we equip $\mathscr{C}$ with the compact open topology. Thus, $v^n \to v$ in $\mathscr{C}$ means that the sequence of functions $v^n(\theta,x)$ converges to $v(\theta,x)$ uniformly for (θ,x) in every compact set. We define the norm $\|v\|$ by

$$\|v\| = \sum_{k=1}^{\infty} \frac{\max\limits_{(\theta)\in[-r,0],|x|\leqslant k} |v(\theta,x)|}{2^k},$$

and let $d(u,v)$ be the metric on $\mathscr{C}$ induced by the norm $\|v\|$. Note $(\mathscr{C}, \|\cdot\|)$ is a normed space, and the topology induced by $\|\cdot\|$ on $\mathscr{C}_{(\tau)}$ is equivalent to the compact open topology on $\mathscr{C}_{(\tau)}$ (see Proposition 5.2 in Liang *et al.*, 2010). Furthermore, $\mathscr{C}$ has a lattice structure. Obviously, every element in $\bar{\mathscr{C}}$ can be regarded as function in $\mathscr{C}$.

Recall that a family of operators $\Sigma_t, t \geqslant 0$, is said to be a semiflow on a metric space $(\mathscr{C}, d)$ with metric d provided Σ_t has the following properties:

(i) $\Sigma_0(v) = v, \quad \forall\, v \in \mathscr{C}$;
(ii) $\Sigma_{t_1}(\Sigma_{t_2}(v)) = \Sigma_{t_1+t_2}(v), \quad \forall\, t_1, t_2 \geqslant 0, \ v \in \mathscr{C}$;
(iii) $\Sigma(t,v) := \Sigma_t(v)$ is continuous in (t,v) on $[0,\infty) \times \mathscr{C}$.

It is easy to see that the property (iii) holds if $\Sigma(\cdot,v)$ is continuous on $[0,+\infty)$ for each $v \in \mathscr{C}$, and $\Sigma(t,\cdot)$ is uniformly continuous for t in bounded intervals in the sense that for any $v_0 \in \mathscr{C}$, bounded interval I and $\epsilon > 0$, there exists $\delta = \delta(v_0, I, \epsilon) > 0$ such that if $d(v,v_0) < \delta$, then $d(\Sigma_t(v), \Sigma_t(v_0)) < \epsilon$ for all $t \in I$.

By Theorem 2.1, we know that (2.1) has a solution map $Q_t \colon \mathscr{C}_{(\tau)} \to \mathscr{C}_{(\tau)} \subseteq \mathscr{C}$ defined by

$$Q_t(\phi)(\theta,x) = \mathcal{U}_t(\theta,x;\phi) = \mathcal{U}(t+\theta,x;\phi), \quad \forall\,(\theta,x) \in [-\tau,0] \times \mathbb{R}, \tag{2.13}$$

where $\mathcal{U}(t,x;\phi)$ is the unique solution of (2.1) with $\mathcal{U}(\theta,x) = \phi(\theta,x), \theta \in [-\tau,0], x \in \mathbb{R}$. Now, we discuss the properties of $\{Q_t\}_{t\geqslant 0}$.

THEOREM 2.2 Suppose (F1)–(F5) and (G1) hold, $\{Q_t\}_{t=0}^{\infty}$ is a monotone semiflow on $\mathscr{C}_{(\tau)}$.

Proof. For any $\phi \in \mathscr{C}_{(\tau)}$, we are easy to have

$$Q_0(\phi) = \phi, \quad Q_{t_1}[Q_{t_2}(\phi)](\theta,x) = Q_{t_1+t_2}(\phi)(\theta,x) \quad \text{for } t_1, t_2 > 0.$$

By Theorem 2.1, we have the conclusions that $Q_t(\mathscr{C}_{(\tau)}) \subset \mathscr{C}_{(\tau)}$ for $t > 0$ and Q_t is monotone. Then the difficulty is to claim that Q_t is continuous with respect to the compact open topology. We divide into two steps to prove it.

Step 1. For each $\phi \in \mathscr{C}_{(\tau)}$, we claim that Q_t is continuous in t. Let $Q(t,\phi) = Q_t(\phi)$. Suppose $t_1 > t_2 > 0$. There are three cases.

(1) $t_1+\theta > t_2+\theta \geqslant 0$. By (2.6)

$$
\begin{aligned}
|Q(t_1,\phi)(\theta,x) - Q(t_2,\phi)(\theta,x)| &= |\mathcal{U}(t_1+\theta,x;\phi) - \mathcal{U}(t_2+\theta,x;\phi)| \\
&\leqslant |\mathcal{T}(t_1+\theta) - \mathcal{T}(t_2+\theta)|\phi(0,x) \\
&\quad + \left|\int_{t_2+\theta}^{t_1+\theta} [\mathcal{T}(t_1+\theta-r)\mathcal{F}(\mathcal{U}_r)](x)\mathrm{d}r\right| \\
&\quad + \int_0^{t_2+\theta} |[\{\mathcal{T}(t_1+\theta-r) - \mathcal{T}(t_2+\theta-r)\} \\
&\qquad \mathcal{F}(\mathcal{U}_r)](x)|\mathrm{d}r,
\end{aligned}
$$

where $\mathcal{T}$ is defined by (2.4), $\mathcal{F}$ is defined by (2.5). Since $\mathcal{T}(t)$ is continuous for $t>0$ and $\mathcal{F}$ is bounded, then for any $\varepsilon > 0$, we can choose $\delta > 0$ such that

$$|Q_{t_1}(\phi) - Q_{t_2}(\phi)| < \varepsilon$$

holds for $(\theta,x) \in [-\tau,0]\times\mathbb{R}$ provided $|t_1-t_2| < \delta$, $t_1,t_2 \geqslant 0$.

(2) $t_2+\theta < t_1+\theta \leqslant 0$. Since $\phi(\theta,x)$ is continuous in θ, then we can choose $\delta > 0$ such that $|\mathcal{U}(t_1+\theta,x) - \mathcal{U}(t_2+\theta,x)| = |\phi(t_1+\theta,x) - \phi(t_2+\theta,x)| < \varepsilon$ holds for $(\theta,x)\in[-\tau,0]\times\mathbb{R}$ if $|t_1-t_2|<\delta$.

(3) $t_1+\theta > 0 > t_2+\theta$. For any $\varepsilon > 0$, by (1), there is $\delta_1 > 0$ such that $|\mathcal{U}(t_1+\theta,x) - \mathcal{U}(0,x)| < \frac{\varepsilon}{2}$ if $t_1+\theta < \delta_1$; by (2), there is $\delta_2 > 0$ such that $|\mathcal{U}(t_2+\theta,x) - \mathcal{U}(0,x)| < \frac{\varepsilon}{2}$ if $t_2+\theta > -\delta_2$. Then let $\delta = \min\{\delta_1,\delta_2\}$, if $t_1 - t_2 < \delta$, we have $|\mathcal{U}(t_1+\theta,x) - \mathcal{U}(t_2+\theta,x)| < \varepsilon$ holds for $\theta\in[-\tau,0]$.

Summarizing (1)–(3), we claim that Q_t is continuous in t, i.e., for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|Q_{t_1}(\phi)(\theta,x) - Q_{t_2}(\phi)(\theta,x)| < \varepsilon \tag{2.14}$$

holds uniformly for $(\theta,x)\in[-\tau,0]\times\mathbb{R}$ if $|t_1-t_2|<\delta$, $t_1,t_2\geqslant 0$.

Step 2. For any $t_0 > 0$, we claim that $Q(t,\phi)$ is continuous in ϕ uniformly for $t\in[0,t_0]$. Suppose $K \subset [-\tau,0]\times\mathbb{R}$ is a compact set, $\|\phi\|_K = \sup_{(\theta,x)\in K}|\phi(\theta,x)|$. For $t\in[0,t_0]$, $(\theta,x)\in K$, $t+\theta>0$, there is

$$
\begin{aligned}
|Q_t(\phi_1)(\theta,x) - Q_t(\phi_2)(\theta,x)| &= |\mathcal{U}(t+\theta,x;\phi_1) - \mathcal{U}(t+\theta,x;\phi_2)| \\
&\leqslant \|\phi_1-\phi_2\|_K + \int_0^{t+\theta} \mathcal{T}(t+\theta-r)[L'_\tau + \hat{L}_\tau\tilde{L}_\tau]\|\mathcal{U}_r(\phi_1) \\
&\qquad - \mathcal{U}_r(\phi_2)\|_K\,\mathrm{d}r,
\end{aligned}
$$

that is,

$$\|\mathcal{U}_t(\phi_1) - \mathcal{U}_t(\phi_2)\|_K \leqslant \|\phi_1-\phi_2\|_K + \int_0^t [L'_\tau + \hat{L}_\tau\tilde{L}_\tau]\|\mathcal{U}_r(\phi_1) - \mathcal{U}_r(\phi_2)\|_K\,\mathrm{d}r.$$

Let $a := L'_\tau + \hat{L}_\tau \tilde{L}_\tau$. By using the Gronwall inequality, we have

$$\|\mathcal{U}_t(\phi_1) - \mathcal{U}_t(\phi_2)\|_K \leqslant \|\phi_1 - \phi_2\|_K \exp\left(\int_0^t a \,\mathrm{d}r\right) = \|\phi_1 - \phi_2\|_K \exp\{at\}.$$

For $t + \theta \leqslant 0$, we have

$$|Q_t(\phi_1)(\theta, x) - Q_t(\phi_2)(\theta, x)| \leqslant \|\phi_1 - \phi_2\|_K.$$

Thus, for any $\varepsilon > 0$, there is $\eta = \varepsilon \exp(-(L'_\tau + \hat{L}_\tau \tilde{L}_\tau)t_0) > 0$ such that if $\|\phi_1 - \phi_2\|_K \leqslant \eta$, then

$$\|Q_t(\phi_1) - Q_t(\phi_2)\|_K < \varepsilon \tag{2.15}$$

uniformly holds for $t \in [0, t_0]$. This means that $Q(t, \phi)$ is continuous in ϕ with respect to the compact open topology uniformly for $t \in [0, t_0]$.

Summarizing the above discussion and (2.14), (2.15), we have the conclusion that $Q_t(\phi) = Q(t, \phi)$ is continuous in (t, ϕ) with respect to the compact open topology, thus the proof is complete. □

2.2 *Asymptotic speed of spread and minimal speed for* (2.1)

In this subsection, we discuss the asymptotic speed of spread and the minimal speed for (2.1). Roughly speaking, if c^*_τ is the asymptotic speed of spread for (2.1), then $\lim_{t\to\infty,|x|\geqslant ct} \mathcal{U}(t, x) = 0 (c > c^*_\tau)$ and $\lim_{t\to\infty,|x|\leqslant ct} \mathcal{U}(t, x) = \mathcal{U}^+ (0 < c < c^*_\tau)$. We say that c^*_τ is the minimal speed in the sense that (2.1) has a travelling wave solution connecting 0 and u^+_τ for $c \geqslant c^*_\tau$, while no such travelling wave solution exists for $0 < c < c^*_\tau$.

For any $\phi \in \mathscr{C}$, define the reflection operator $\mathcal{R}$ by $\mathcal{R}(\phi)(\theta, x) = \phi(\theta, -x)$. Given $z \in \mathbb{R}$, define the translation operator T_z by $T_z(\phi)(\theta, x) = \phi(\theta, x - z)$. $W \subset \mathscr{C}$ is said T-invariant if $T_z W = W$ for all $z \in \mathbb{R}$.

To study the asymptotic speed of spread and travelling wave solutions, we will apply the theorems in Liang & Zhao (2007), which require some hypotheses on a map Q (see Liang & Zhao, 2007). Let $\beta \in \bar{\mathscr{C}}$ with $\beta(\theta) > 0$ for $\theta \in [-\tau, 0]$ and Q: $[0, \beta]_{\mathscr{C}} \to [0, \beta]_{\mathscr{C}}$. The following hypotheses on Q are needed:

- (A1) $Q[\mathcal{R}[\phi]] = \mathcal{R}[Q[\phi]]$, $T_z[Q[\phi]] = Q[T_z[\phi]]$ for $z \in \mathbb{R}$;
- (A2) Q: $[0, \beta]_{\mathscr{C}} \to [0, \beta]_{\mathscr{C}}$ is continuous with respect to the compact open topology;
- (A3) One of the following two properties holds:
  - (a) $\{Q[\phi](\cdot, x) : \phi \in [0, \beta]_{\mathscr{C}}, x \in \mathbb{R}\}$ is a precompact subset of $\bar{\mathscr{C}}$;
  - (b′) The set $Q[[0, \beta]_{\mathscr{C}}](0, \cdot)$ is precompact in $\mathbb{X}$, and there is a positive number $\zeta \leqslant \tau$ such that $Q[\phi](\theta, x) = \phi(\theta + \zeta, x)$ for $-\tau \leqslant \theta \leqslant -\zeta$, and the operator

$$S[\phi](\theta, x) = \begin{cases} \phi(0, x), & -\tau \leqslant \theta < -\zeta, \\ Q[\phi](\theta, x), & -\zeta \leqslant \theta \leqslant 0, \end{cases} \tag{2.16}$$

    has the property that $\{S[\phi](\cdot, 0) : \phi \in D\}$ is a precompact subset of $\bar{\mathscr{C}}$ for any T-invariant set $D \subset [0, \beta]_{\mathscr{C}}$ with $D(0, \cdot)$ precompact in $\mathbb{X}$.
- (A4) Q: $[0, \beta]_{\mathscr{C}} \to [0, \beta]_{\mathscr{C}}$ is monotone (order-preserving) in the sense that $Q[\phi] \geqslant Q[\psi]$ whenever $\phi \geqslant \psi$ in $[0, \beta]_{\mathscr{C}}$;

(A5) Q: $[0, \beta]_{\mathscr{C}} \to [0, \beta]_{\mathscr{C}}$ admits exactly two fixed points 0 and β, and for any positive number ϵ, there is $\alpha \in [0, \beta]_{\mathscr{C}}$ with $\|\alpha\| < \epsilon$ such that $Q[\alpha](\theta, x) > \alpha(\theta, x)$ for $(\theta, x) \in [-\tau, 0] \times \mathbb{R}$;

(A6) One of the following two properties holds:

(a) $Q[[0, \beta]_{\mathscr{C}}]$ is precompact in $[0, \beta]_{\mathscr{C}}$;

(b′) The set $Q[[0, \beta]_{\mathscr{C}}](0, \cdot)$ is precompact in $\mathbb{X}$, and there is a positive number $\zeta \leqslant \tau$ such that $Q[\phi](\theta, x) = \phi(\theta + \zeta, x)$ for $-\tau \leqslant \theta \leqslant -\zeta$, and the operator defined by (2.16) has the property that $S[D]$ is a precompact subset of $[0, \beta]_{\mathscr{C}}$ for any T-invariant set $D \subset [0, \beta]_{\mathscr{C}}$ with $D(0, \cdot)$ precompact in $\mathbb{X}$.

Let $\beta(\theta) \equiv \mathcal{U}^+$. In Section 2.1, we define a solution map Q_t of (2.1) by (2.13) and prove that $\{Q_t\}_{t\geqslant 0}^{\infty}$ is a monotone semiflow on $\mathscr{C}_{(\tau)}$. In the followings, we try to show that for each $t > 0$, Q_t has the properties (A1)–(A6). First, it is easy to obtain the following Lemma 2.2 by Theorem 2.2.

Lemma 2.2 Suppose (F1)–(F5) and (G1) hold, then (A1), (A2) and (A4) are satisfied.

Lemma 2.3 Suppose (F1)–(F5) and (G1) hold, then (A3) and (A6) are satisfied.

Proof. Let $t > 0$ be fixed. We divide into two steps.

Step 1. If $t > \tau$, we claim that Q_t is precompact. Thus, (A6)(a) is satisfied. It is obviously that $Q[\mathscr{C}_{(\tau)}] \subset \mathscr{C}_{(\tau)}$. Let $\mathcal{T}(t)$ be defined by (2.4), then $Q_t[\phi](\cdot, x) = \mathcal{U}(t + \cdot, x)$ and

$$\mathcal{U}(t+\cdot, x) = \mathcal{T}(t+\cdot)\phi(0, x) + \int_0^{t+\cdot} [\mathcal{T}(t+\cdot-r)\mathcal{F}(\mathcal{U}_r)](x)\mathrm{d}r, \quad (t, x) \in (0, \infty) \times \mathbb{R},$$
$$\mathcal{U}(t, x) = \phi(t, x), \quad (t, x) \in [-\tau, 0] \times \mathbb{R}.$$

Note that $\{\mathcal{T}(t)\}_{t>0}$ is a $\mathcal{C}_0$ semigroup by Lemma 2.1. Furthermore, for any given $t > 0$ and any uniformly bounded subset $\mathbb{M} \subset \mathbb{X}$, $\mathcal{T}(t)(\mathbb{M})$ is precompact with respect to the compact open topology. Together with the boundedness of $\mathcal{F}$, we know that Q_t is precompact for each $t > \tau$.

Step 2. If $0 < t \leqslant \tau$, we shall show that Q_t satisfies (A6)(b′). First, we want to show that for $0 < t_0 \leqslant \tau$ and any given compact interval $I \subset \mathbb{R}, \mathcal{U}(t, x; \phi)$ is equi-continuous in $(t, x) \in [0, t_0] \times I$ for all $\phi \in D$, where $D \subset [0, \beta]_{\mathscr{C}}$ is any T-invariant set with $D(0, \cdot)$ precompact in $\mathbb{X}$.

Note $\mathcal{T}(t)$ is a bounded linear operator. By using the boundedness of $\mathcal{F}(\mathcal{U}_r)(x)$, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $|\int_0^t [\mathcal{T}(t-r)\mathcal{F}_r(\mathcal{U}_r)](x)\mathrm{d}r| < \frac{\varepsilon}{12}$ for any $(t, x) \in [0, \delta_0] \times I$. Since $\{\mathcal{T}(t)\}_{t>0}$ is a a $\mathcal{C}_0$ semigroup on $\mathbb{X}$, choose $0 < \delta_1 < \delta_0$, for $(t, x) \in [0, \delta_1] \times I$ and for $\phi \in D$, we have

$$\begin{aligned} |\mathcal{U}(t, x; \phi) - \phi(0, x)| &\leqslant |(\mathcal{T}(t) - \mathcal{T}(0))\phi(0, x)| + \left|\int_0^t [\mathcal{T}(t-r)\mathcal{F}(\mathcal{U}_r)](x)\mathrm{d}r\right| \\ &< \frac{\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{\varepsilon}{6}. \end{aligned} \tag{2.17}$$

Since $D(0, \cdot)$ is precompact in $\mathbb{X}$, then there is $\delta_2 > 0$ such that

$$|\phi(0, x_1) - \phi(0, x_2)| < \frac{\varepsilon}{6} \quad \text{for all } \phi \in D, \tag{2.18}$$

where $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta_2$.

Thus, by (2.17), (2.18), for any $t_1, t_2 \in [0, \delta_1]$, $x_1, x_2 \in I$ with $|x_1 - x_2| < \delta_2$, there is

$$|\mathcal{U}(t_1, x_1; \phi) - \mathcal{U}(t_2, x_2; \phi)| \leqslant |\mathcal{U}(t_1, x_1; \phi) - \phi(0, x_1)| + |\mathcal{U}(t_2, x_2; \phi) - \phi(0, x_2)| + |\phi(0, x_1) - \phi(0, x_2)| \leqslant \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}. \tag{2.19}$$

Note that $Q_t[D]$ is precompact for $t > \tau$, it follows that $\mathcal{U}(t, x; \phi)$ is equi-continuous in $(t, x) \in [\delta_1, t_0] \times I$ for all $\phi \in D$, that means there is $\delta_3 > 0$ such that

$$|\mathcal{U}(t_1, x_1; \phi) - \mathcal{U}(t_2, x_2; \phi)| < \frac{\varepsilon}{2} \tag{2.20}$$

for all $\phi \in D$, when $t_1, t_2 \in [\delta_1, t_0]$ and $x_1, x_2 \in I$ with $|t_1 - t_2| + |x_1 - x_2| < \delta_3$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, for any $\varepsilon > 0$ and $I \subset \mathbb{R}$, any $\phi \in D$, if $t_1, t_2 \in [0, t_0]$, $x_1, x_2 \in I$ with $|t_1 - t_2| + |x_1 - x_2| < \delta$, by (2.19), (2.20), we have

$$|\mathcal{U}(t_1, x_1; \phi) - \mathcal{U}(t_2, x_2; \phi)| < \varepsilon,$$

Second, we claim that Q_t satisfies (A6)(b′) if $t \in (0, \tau]$. By the conclusion in the first part of Step 2, $Q_t[\mathscr{C}_{(\tau)}](0, \cdot)$ is precompact in $\mathbb{X}$. Let $\zeta = t$ in (A6)(b′). Then $Q_t[\phi](\theta, x) = \mathcal{U}(t + \theta, x)$ for $-\tau \leqslant \theta \leqslant -t$. Define

$$S[\phi](\theta, x) = \begin{cases} \mathcal{U}(0, x), & -\tau \leqslant \theta < -t, \\ Q_t[\phi](\theta, x), & -t \leqslant \theta \leqslant 0. \end{cases}$$

We obtain from the above expression that $S(\phi)$ is continuous on $\mathscr{C}_{(\tau)}$, and we can show that $S(D)$ is precompact in $\mathscr{C}_{(\tau)}$ for any T-invariant set $D \subset \mathscr{C}_{(\tau)}$ with $D(0, \cdot)$ precompact in $\mathbb{X}$ by a method similar to Theorem 6.1 in Hale & Lunel (1993).

Summarizing the conclusions in the above two steps, we know that (A6) is satisfied for Q_t with $t > 0$. Note that (A6) implies (A3), thus the proof is complete. □

It is difficult to discuss whether (A5) is valid when we omit the differentiability of function f. We have three lemmas to solve this problem. To find the α in (A5), we discuss the solution $\mathcal{U}(t, x)$ of (2.1) with the form $\mathcal{U}(t, x) = \mathcal{U}(t)$ without x. Let $k(s) = \int_{-\infty}^{+\infty} g(s, y)\mathrm{d}y$, then $\mathcal{U}(t)$ satisfies:

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = f\left(\mathcal{U}(t), \int_0^\tau k(s)\varphi(\mathcal{U}(t - s))\mathrm{d}s\right). \tag{2.21}$$

It is easy to see that $0 \leqslant \mathcal{U}(t) \leqslant \mathcal{U}^+$ if $0 \leqslant \phi(s) \leqslant \mathcal{U}^+$ for $s \in [-\tau, 0]$. What we care is the limit of the solution when $t \to +\infty$, the following two lemmas give us the answer.

LEMMA 2.4 Let (F1)–(F5) and (G1) hold, $\mathcal{U}(t)$ is the solution of (2.21), if $\mathcal{U}(t) \geqslant 0$ holds for $-\tau \leqslant t \leqslant 0$ and $\mathcal{U}(0) > 0$, then $\mathcal{U}(t) > 0$ for $t > 0$.

Proof. Suppose $\bar{t}$ satisfies $\mathcal{U}(\bar{t}) = 0$ and $\mathcal{U}(t) > 0$ for $0 \leqslant t < \bar{t}$, thus

$$\int_0^\tau k(s)\varphi(\mathcal{U}(\bar{t} - s))\mathrm{d}s > 0 \quad \text{and} \quad (\mathcal{U}(t))'|_{t=\bar{t}} \leqslant 0,$$

but by (F4) and (F5), we have

$$\begin{aligned}\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}\Big|_{t=\bar{t}} &= f\left(\mathcal{U}(\bar{t}), \int_0^\tau k(s)\varphi(\mathcal{U}(\bar{t}-s))\mathrm{d}s\right)\\ &= f\left(0, \int_0^\tau k(s)\varphi(\mathcal{U}(\bar{t}-s))\mathrm{d}s\right) > 0,\end{aligned}$$

which is a contradiction. Therefore, $\mathcal{U}(t) > 0$ holds for all $t > 0$. The proof is complete. □

LEMMA 2.5 Let (F1)–(F5) and (G1) hold, $\mathcal{U}(t)$ is the solution of (2.21), if $\mathcal{U}^+ > \mathcal{U}(t) \geqslant 0$ for $-\tau \leqslant t \leqslant 0$ with $\mathcal{U}(0) > 0$, then $\mathcal{U}(t) \to \mathcal{U}^+$ as $t \to +\infty$.

Proof. We divide it into three cases to consider.

(I) $\mathcal{U}(t) = \mathcal{U}^+$ for t sufficiently large. The theorem is valid obviously.

(II) $\mathcal{U}(t) < \mathcal{U}^+$ is eventually monotonic. We claim that $\mathcal{U}(t)$ would not be eventually decreasing. Otherwise, we can choose suitable $\tilde{t} > 0$ with

$$(\mathcal{U}(t))'|_{t=\tilde{t}} \leqslant 0, \mathcal{U}(t) \geqslant \mathcal{U}(\tilde{t}) > 0 \quad \text{for } \tilde{t}-\tau \leqslant t < \tilde{t}.$$

By (F4) and (F5),

$$\begin{aligned}(\mathcal{U}(\tilde{t}))' &= f\left(\mathcal{U}(\tilde{t}), \int_0^\tau k(s)\varphi(\mathcal{U}(\tilde{t}-s))\mathrm{d}s\right)\\ &\geqslant f\left(\mathcal{U}(\tilde{t}), \varphi(\mathcal{U}(\tilde{t}))G\right) > 0,\end{aligned}$$

which is a contradiction. Hence, $\mathcal{U}(t)$ is eventually non-decreasing. Let $b = \lim_{t\to\infty} \mathcal{U}(t)$, then $b \leqslant \mathcal{U}^+$. We claim that $b = \mathcal{U}^+$. In fact, if t is large enough, we have from the monotonic property of $\mathcal{U}(t)$ and (F4), (F5) that

$$(\mathcal{U}(t))' = f\left(\mathcal{U}(t), \int_0^\tau k(s)\varphi(\mathcal{U}(t-s))\mathrm{d}s\right) \leqslant f\left(\mathcal{U}(t), \varphi(\mathcal{U}(t))G\right),$$

which leads to

$$0 = \lim_{t\to\infty}(\mathcal{U}(t))' \leqslant \lim_{t\to\infty} f\left(\mathcal{U}(t), \varphi(\mathcal{U}(t))G\right) = f\left(b, \varphi(b)G\right) > 0$$

provided $b < \mathcal{U}^+$. This is a contradiction. Therefore, $b = \mathcal{U}^+$.

(III) $\mathcal{U}(t)$ is not eventually monotonic. This is a difficult task. We divide it into two steps.

Step 1. Let $\tilde{\alpha} = \inf_{0\leqslant t\leqslant\tau}\{\mathcal{U}(t)\}$, by Lemma 2.4, $\tilde{\alpha} > 0$. We claim that

$$\mathcal{U}(t) > \tilde{\alpha} \quad \text{for } t > \tau. \tag{2.22}$$

If it is not true, suppose $t_1 \geqslant \tau$ is the first point such that $\mathcal{U}(t_1) = \tilde{\alpha}$, then

$$\mathcal{U}(t) \geqslant \mathcal{U}(t_1) = \tilde{\alpha} > 0 \quad \text{for } 0 \leqslant t \leqslant t_1 \quad \text{and} \quad (\mathcal{U}(t_1))' \leqslant 0.$$

But by (F4) and (F5), together with $0 < \tilde{\alpha} < \mathcal{U}^+$, there is

$$\begin{aligned}\frac{\mathrm{d}\mathcal{U}(t_1)}{\mathrm{d}t} &= f\left(\mathcal{U}(t_1), \int_0^\tau k(s)\varphi(\mathcal{U}(t_1 - s))\mathrm{d}s\right)\\ &\geqslant f\,(\mathcal{U}(t_1), \varphi(\mathcal{U}(t_1))G) = f\,(\tilde{\alpha}, \varphi(\tilde{\alpha})G) > 0,\end{aligned}$$

which is a contradiction, thus (2.22) is valid.

Step 2. Let $\{t_j\}$ be the local minimal point of $\mathcal{U}(t)$, then $\big(\mathcal{U}(t_j)\big)' = 0$. There are two subcases of $\{t_j\}$.

(i) There is a subsequence $\{s_j\} = \{t_{k_j}\} \subset \{t_j\}$ such that

$$\mathcal{U}(s_1) = \mathcal{U}(t_{k_1}) = \inf_{t \geqslant t_1}\{\mathcal{U}(t)\}, \mathcal{U}(s_2) = \mathcal{U}(t_{k_2}) = \inf_{t \geqslant t_{k_1+1}}\{\mathcal{U}(t)\},$$

$$\ldots, \quad \mathcal{U}(s_j) = \mathcal{U}(t_{k_j}) = \inf_{t \geqslant t_{k_{j-1}+1}}\{\mathcal{U}(t)\}, \quad \lim_{j\to\infty} s_j = \infty.$$

Then $\mathcal{U}(s_j)$ is non-decreasing as j increasing. Let $\{s_{j_k}\} \subset \{s_j\}$ such that $s_{j_{k-1}} < s_{j_k} - \tau$, and $\alpha_k := \mathcal{U}(s_{j_k})$, then α_k is increasing as k is increasing and there is α such that

$$\lim_{k\to\infty} \alpha_k = \alpha = \lim_{j\to+\infty} \mathcal{U}(s_j).$$

Note that

$$\begin{aligned}0 = \frac{\mathrm{d}(\mathcal{U}(s_{j_k}))}{\mathrm{d}t} &= f\left(\mathcal{U}(s_{j_k}), \int_0^\tau k(s)\varphi(\mathcal{U}(s_{j_k} - s))\mathrm{d}s\right)\\ &\geqslant f\left(\mathcal{U}(s_{j_k}), \varphi(\mathcal{U}(s_{j_{k-1}}))G\right),\end{aligned}$$

then $f\,(\alpha, \varphi(\alpha)G) \leqslant 0$. Since $0 < \tilde{\alpha} \leqslant \alpha \leqslant \mathcal{U}^+$, by (F5), we only have $\alpha = \mathcal{U}^+$, thus $\lim_{t\to+\infty} \mathcal{U}(t) = \mathcal{U}^+$.

(ii) If (i) is not valid, then there is a subsequence $\{\tilde{t}_j\} \subset \{t_j\}$ such that

$$\mathcal{U}(\tilde{t}_j) \leqslant \mathcal{U}(\tilde{t}_{j-1}) \quad \text{and} \quad \mathcal{U}(\tilde{t}_j) \leqslant \mathcal{U}(t) \quad \text{for } t \leqslant \tilde{t}_j. \tag{2.23}$$

But this case will be ruled out. Let $\hat{\alpha} = \lim_{j\to\infty} \mathcal{U}(\tilde{t}_j)$ since

$$\begin{aligned}0 &= f\left(\mathcal{U}(\tilde{t}_j), \int_0^\tau k(s)\varphi(\mathcal{U}(\tilde{t}_j - s))\mathrm{d}s\right)\\ &\geqslant f\left(\mathcal{U}(\tilde{t}_j), \varphi(\mathcal{U}(\tilde{t}_j))G\right),\end{aligned}$$

then $f\big(\hat{\alpha}, \varphi(\hat{\alpha})G\big) \leqslant 0$. This together with $\hat{\alpha} \geqslant \tilde{\alpha} > 0$, we have $\hat{\alpha} = \mathcal{U}^+$. Equation (2.23) implies that $\mathcal{U}(t) = \mathcal{U}^+$, which is a contradiction. Therefore, only the subcase (i) will occur, and the conclusion of this lemma is true.

Summarizing the arguments of (I), (II) and (III), the proof is complete. □

By the above two lemmas, we have the following lemma.

LEMMA 2.6 Suppose (F1)–(F5) and (G1) hold, (A5) is satisfied.

Up to now, we have proved that (A1)–(A6) are satisfied. From Theorems 2.11 and 2.15 in Liang & Zhao (2007), it follows that the map Q_t has an asymptotic speed of spread $c_\tau^* > 0$ in the following sense.

THEOREM 2.3 Assume (F1)–(F5) and (G1) hold, then there is c_τ^* such that the following statements are valid:

(1) if $\phi \in \mathscr{C}_{(\tau)}$ with $0 \leqslant \phi < \mathcal{U}^+$ and $\phi(\theta, x) = 0$ for $\theta \in [-\tau, 0]$ and x outside a bounded set, then

$$\lim_{t\to+\infty,|x|\geqslant ct} \mathcal{U}(t, x; \phi) = 0 \quad \text{for any } c > c_\tau^*;$$

(2) for any $\sigma \in [0, \mathcal{U}^+]_{\bar{\mathscr{C}}}$ with $\sigma(\theta) > 0$ for $\theta \in [-\tau, 0]$, there is a positive number r_σ such that if $\phi \in \mathscr{C}_{(\tau)}$ with $\phi(\cdot, x) > \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then

$$\lim_{t\to+\infty,|x|\leqslant ct} \mathcal{U}(t, x; \phi) = \mathcal{U}^+ \quad \text{for any } c < c_\tau^*.$$

REMARK 2.2 If we impose another assumption on f and φ: $f(u, w)$ and $\varphi(w)$ are sublinear, i.e., $f(\rho u, \rho w) \geqslant \rho f(u, w)$ and $\varphi(\rho w) \geqslant \rho\varphi(w)$ for $\rho \in [0, 1]$ and $w \in [0, \mathcal{U}^+]$; then it follows that Q_t is subhomogeneous in the meaning that $Q_t[\rho\phi] \geqslant \rho Q_t(\phi]$ for $\rho \in [0, 1]$ and $\phi \in \mathscr{C}_{(\tau)}$, and thus the conclusion (2) in Theorem 2.3 becomes (see Theorem 2.17 in Liang & Zhao, 2007)

(2) if $\phi \in \mathscr{C}_{(\tau)}$ with $\phi(\theta, \cdot) \not\equiv 0$ for $\theta \in [-\tau, 0]$, then

$$\lim_{t\to+\infty,|x|\leqslant ct} \mathcal{U}(t, x; \phi) = \mathcal{U}^+ \quad \text{for any } c < c_\tau^*.$$

A travelling wave solution of (2.1) is a solution with the form $\mathcal{U}(t, x) = U^{(\tau)}(x + ct)$, where $c > 0$ is the wave speed. Let $z = x + ct$, then the profile function of travelling wave is $U^{(\tau)}(z)$ which satisfies the equation

$$c\frac{\mathrm{d}}{\mathrm{d}z}U^{(\tau)}(z) = D\frac{\mathrm{d}^2}{\mathrm{d}^2 z}U^{(\tau)}(z) + f\left(U^{(\tau)}(z), \int_0^\tau \int_{-\infty}^{+\infty} g(s, y)\varphi(U^{(\tau)}(z - y - cs))\mathrm{d}y\, \mathrm{d}s\right). \tag{2.24}$$

We are concerned with the monotone travelling waves which connects the two equilibria 0 and $\mathcal{U}^+$:

$$U^{(\tau)}(-\infty) = 0, \quad U^{(\tau)}(+\infty) = \mathcal{U}^+. \tag{2.25}$$

According to Theorem 4.3–4.4 in Liang & Zhao (2007), we have the following theorem about the existence of travelling waves for (2.1).

THEOREM 2.4 Let c_τ^* be defined in Theorem 2.3 and suppose (F1)–(F5), (G1) hold, then the following two statements are valid:

(i) for any $c \geqslant c_\tau^*$, (2.1) has a monotone travelling wave solution $U^{(\tau)}(x + ct)$ connecting 0 and $\mathcal{U}^+$;

(ii) for any $c \in (0, c_\tau^*)$, (2.1) admits no monotone travelling wave solution $U^{(\tau)}(x + ct)$ connecting 0 and $\mathcal{U}^+$.

3. The equation with infinite delay

In this section, we shall discuss the properties of the solution, the asymptotic speed, the existence of travelling wavefront and the minimal wave speed of (1.1). Consider the relationship between the system (1.1) and (2.1). It is seen that as $\tau \to \infty$, the limit system of (2.1) is (1.1). This fact reminds us to discuss the limit of c_τ^* as $\tau \to \infty$.

3.1 *Properties for solutions of* (1.1)

In this subsection, we discuss the existence, uniqueness of solutions for the initial problem of (1.1). Furthermore, we establish a comparison result for the solutions of (1.1) with values in between the two steady states $u \equiv 0$ and $u \equiv u^+$.

Let $\bar{\mathbb{X}} = \mathrm{BUC}(\mathbb{R}, \mathbb{R})$ be the space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|$, $\bar{\mathbb{X}}_+ = \{\psi \in \bar{\mathbb{X}};\ \psi(x) \geqslant 0, x \in \mathbb{R}\}$. Then $\bar{\mathbb{X}}_+$ is a closed cone of $\bar{\mathbb{X}}$ and $\bar{\mathbb{X}}$ is a Banach lattice under the partial ordering $\leqslant_{\bar{\mathbb{X}}}$ induced by $\bar{\mathbb{X}}_+$.

Suppose h: $(-\infty, 0] \to [1, \infty)$ is a function satisfying the following conditions:

(H1) h is continuous, non-increasing and $h(0) = 1$;

(H2) $\dfrac{h(s+\theta)}{h(s)} \to 1$ uniformly for $s \in (-\infty, 0]$ as $\theta \to 0^+$;

(H3) $h(s) \to \infty$ as $s \to -\infty$.

Define

$$\mathrm{UC}_h = \left\{ \phi;\ \begin{array}{l} \phi\colon (-\infty, 0] \to \bar{\mathbb{X}} \text{ is continuous,} \\ \dfrac{\phi}{h} \text{is uniformly continuous on} (-\infty, 0],\ \sup_{s \leqslant 0} \dfrac{|\phi(s)|}{h(s)} < \infty \end{array} \right\},$$

$$\mathrm{UC}_h^+ = \left\{ \phi \in UC_h; \phi(\theta) \geqslant_{\bar{\mathbb{X}}} 0 \text{ for } \theta \leqslant 0 \right\}.$$

Similarly, UC_h^+ induces a partial ordering $\leqslant_{\mathrm{UC}_h}$ on UC_h. Let UC_h be equipped with the norm $|\phi|_h = |\phi|_{\mathrm{UC}_h} := \sup_{s \leqslant 0} \dfrac{|\phi(s)|_{\bar{\mathbb{X}}}}{h(s)}$ for $\phi \in UC_h$. According to Ruan & Wu (1994), $(\mathrm{UC}_h, |\cdot|_h)$ is a Banach space. As usual, we identify an element $\phi \in \mathrm{UC}_h$ as a function from $(-\infty, 0] \times \mathbb{R} \to \mathbb{R}$ by $\phi(s, x) = \phi(s)(x)$. For any continuous function u: $(-\infty, b) \to \bar{\mathbb{X}}$, where $b > 0$, we define u_t by $u_t(s) = u(t+s)$, $s \in (-\infty, 0]$. We define some subsets of $\bar{\mathbb{X}}$ and UC_h by

$$[0, u^+]_{\bar{\mathbb{X}}} := \left\{ \psi \in \bar{\mathbb{X}};\ 0 \leqslant_{\bar{\mathbb{X}}} \psi(x) \leqslant_{\bar{\mathbb{X}}} u^+, x \in \mathbb{R} \right\},$$

$$[0, u^+]_{\mathrm{UC}_h} := \left\{ \phi \in \mathrm{UC}_h;\ 0 \leqslant_{\mathrm{UC}_h} \phi(\theta) \leqslant_{\mathrm{UC}_h} u^+ \text{ for } \theta \leqslant 0 \right\}.$$

Rewrite (1.1) as

$$\frac{\partial u(t,x)}{\partial t} - D\frac{\partial^2 u(t,x)}{\partial x^2} + L'u(t,x) = L'u(t,x) + f(u(t,x), (g * \varphi(u))(t,x)). \tag{3.1}$$

Define a functional F: $\mathrm{UC}_h \to \bar{\mathbb{X}}$ as follows:

$$\begin{aligned} F(\phi)(x) &= f(\phi(0,x), (g * \varphi(\phi))(0,x)) + L'\phi(0,x) \\ &= f\left(\phi(0,x), \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, x-y)\varphi(\phi(-s,y))\mathrm{d}y\,\mathrm{d}s\right) + L'\phi(0,x). \end{aligned} \tag{3.2}$$

Let

$$[T(t)\psi](x) := \frac{\mathrm{e}^{-L't}}{\sqrt{4\pi Dt}} \int_{\mathbb{R}} \mathrm{e}^{\frac{-(x-y)^2}{4Dt}} \psi(y)\mathrm{d}y \quad \text{for } t > 0;$$
$$[T(0)\psi](x) = \psi(x). \tag{3.3}$$

We can prove $T(t)$: $\bar{\mathbb{X}} \to \bar{\mathbb{X}}$ is an $\mathcal{C}_0$ semigroup with $T(t)\bar{\mathbb{X}}_+ \subset \bar{\mathbb{X}}_+$ for all $t > 0$ by the similar way in Lemma 2.1. Thus, the conditions (T1)–(T3) in Ruan & Wu (1994) are satisfied.

Let $\phi \in [0, u^+]_{\mathrm{UC}_h}$ be a given initial function, then by (3.1)–(3.2), the equivalent abstract integral form of (1.1) is

$$u(t) = T(t)\phi(0,\cdot) + \int_0^t T(t-r)F(u_r)\mathrm{d}r, \quad t > 0,$$
$$u(t) = \phi(t,\cdot), \qquad t \in (-\infty, 0]. \tag{3.4}$$

DEFINITION 3.1 A continuous function v: $(-\infty, b) \to \bar{\mathbb{X}}$ is called a supersolution (subsolution) of (1.1) on $[0, b)$ if

$$v(t) \geqslant (\leqslant) T(t)\phi(0,\cdot) + \int_0^t T(t-r)F(v_r)\mathrm{d}r \quad \text{for } 0 \leqslant t < b. \tag{3.5}$$

If v is both a supersolution and subsolution on $[0, b)$, then it is said to be a (mild) solution of (1.1).

REMARK 3.1 Assume that there is a bounded and continuous function v: $\mathbb{R} \times (-\infty, b) \to \mathbb{R}$, with $b > 0$ and such that v is C^2 in $x \in \mathbb{R}$, C^1 in $t \in (0, b)$, and

$$\frac{\partial v(t,x)}{\partial t} \geqslant (\leqslant) D\frac{\partial^2 v(t,x)}{\partial x^2} + f(v(t,x), (g * \varphi(v))(t,x))$$

for $(t,x) \in (0,b) \times \mathbb{R}$. Then by the fact that $T(t)\bar{\mathbb{X}}_+ \subset \bar{\mathbb{X}}_+$, it follows that (3.5) holds and hence $v(t,x)$ is a supersolution (subsolution) of (1.1) on $[0, b)$.

The main result of this section is the following.

THEOREM 3.1 Suppose that the hypotheses (F1)–(F4) and (G1) hold, then the following conclusions are valid.

(i) For any $\phi \in [0, u^+]_{\mathrm{UC}_h}$, (1.1) has a unique solution $u(t,x) = u(t,x;\phi)$ defined on $[0,\infty)$ such that $u(t) \in [0, u^+]_{\bar{\mathbb{X}}}$, $u_t \in [0, u^+]_{\mathrm{UC}_h}$ for $t \geqslant 0$.

(ii) For any pair of supersolution $\bar{v}(t,x)$ and subsolution $\underline{v}(t,x)$ of (1.1) on $\mathbb{R} \times \mathbb{R}$ with $0 \leqslant \bar{v}(t,x), \underline{v}(t,x) \leqslant u^+$ for $(t,x) \in \mathbb{R} \times \mathbb{R}$ and $0 \leqslant \underline{v}(s,x) \leqslant \bar{v}(s,x) \leqslant u^+$ for $(s,x) \in (-\infty, 0] \times \mathbb{R}$, there holds $0 \leqslant \underline{v}(t,x) \leqslant \bar{v}(t,x) \leqslant u^+$ for $(t,x) \in [0,+\infty) \times \mathbb{R}$.

Proof. Let $T(t,s) = T(t-s)$, $S(t,s) = T(t-s)$. Then one can verify the conditions (T1)–(T4), (S1)–(S2) in Ruan & Wu (1994) are satisfied.

Define $L = 2L' + \tilde{L}\hat{L}$ and

$$D := [0,\infty) \times [0, u^+]_{\bar{X}}, \quad D(t) := [0, u^+]_{\bar{X}} \quad \text{for } t \in [0,\infty),$$
$$\mathcal{D} := [0,\infty) \times [0, u^+]_{\mathrm{UC}_h}, \quad \mathcal{D}(t) := [0, u^+]_{\mathrm{UC}_h} \quad \text{for } t \in [0,\infty).$$

By calculation and (F1) and (F2), we obtain that

$$|F(\phi_1) - F(\phi_2)|_{\bar{\mathbb{X}}} = \sup_{x\in\mathbb{R}} |F(\phi_1)(x) - F(\phi_2)(x)| \leqslant L|\phi_1 - \phi_2|_{\mathrm{UC}_h}$$

as long as $|\phi_1|_{\mathrm{UC}_h} \leqslant u^+$, $|\phi_2|_{\mathrm{UC}_h} \leqslant u^+$. Therefore, the condition (F) in Ruan & Wu (1994) is satisfied.

It is obvious that $u \equiv u^+$ and $u \equiv 0$ is a pair of supersolution and subsolution of (1.1). Let $v^+ \equiv u^+, v^- \equiv 0$, then (C3)–(C4) in Ruan & Wu (1994) are satisfied. Furthermore, if $\phi_1, \phi_2 \in [0, u^+]_{\mathrm{UC}_h}$, and $\phi_1 \geqslant_{\mathrm{UC}_h} \phi_2$, then we have from (F2) and (F4) that

$$\begin{aligned}
F(\phi_1)(x) - F(\phi_2)(x) &= f(\phi_1(0,x), (g * \varphi(\phi_1))(0,x)) - f(\phi_2(0,x), (g * \varphi(\phi_2))(0,x)) \\
&\quad + L'(\phi_1(0,x) - \phi_2(0,x)) \\
&= f(\phi_1(0,x), (g * \varphi(\phi_1))(0,x)) - f(\phi_2(0,x), (g * \varphi(\phi_1))(0,x)) \\
&\quad + f(\phi_2(0,x), (g * \varphi(\phi_1))(0,x)) - f(\phi_2(0,x), (g * \varphi(\phi_2))(0,x)) \\
&\quad + L'(\phi_1(0,x) - \phi_2(0,x)) \\
&\geqslant f(\phi_2(0,x), (g * \varphi(\phi_1))(0,x)) - f(\phi_2(0,x), (g * \varphi(\phi_2))(0,x)) \geqslant 0,
\end{aligned}$$

thus $F(\phi)$ is non-decreasing on $\phi \in [0, u^+]_{\mathrm{UC}_h}$. For any $\phi_1, \phi_2 \in [0, u^+]_{\mathrm{UC}_h}$ with $\phi_1 \geqslant_{\mathrm{UC}_h} \phi_2$, the inequality

$$[\phi_1(0) - \phi_2(0)] + \nu[F(\phi_1) - F(\phi_2)] \geqslant_{\bar{\mathbb{X}}} 0 \quad \text{for } \nu \geqslant 0$$

leads to

$$\lim_{\nu\to 0^+} \operatorname{dist}\left\{[\phi_1(0) - \phi_2(0)] + \nu[F(\phi_1) - F(\phi_2)], \bar{\mathbb{X}}_+\right\} = 0.$$

For each $b > 0$, the existence and uniqueness of a solution $u(t, x; \phi)$ on $[0, b)$ follows from Theorem 5.2 in Ruan & Wu (1994) with $S(t, s) = T(t, s) = T(t - s)$ for $t \geqslant s \geqslant 0$ and $v^+ \equiv u^+, v^- \equiv 0$. Note $0 \leqslant u(t, \cdot; \phi) \leqslant u^+$ on $[0, b)$, hence the maximal interval of existence is $[0, \infty)$.

We now prove the conclusion (ii). Some $\bar{v}, \underline{v} \in [0, u^+]_{\mathrm{UC}_h}$ and $\underline{v} \leqslant_{\mathrm{UC}_h} \bar{v}$, it follows from Theorem 5.2 in Ruan & Wu (1994) that

$$0 \leqslant u(t, x; \underline{v}) \leqslant u(t, x; \bar{v}) \leqslant u^+ \quad \text{for } x \in \mathbb{R}, t \geqslant 0.$$

Again by applying Theorem 5.2 in Ruan & Wu (1994) with

$$v_-(t, x) = \underline{v}(t, x), v^+(t, x) \equiv u^+;$$

$$v_-(t, x) \equiv 0, v^+(t, x) = \bar{v}(t, x),$$

respectively, we get

$$\underline{v}(t, x) \leqslant u(t, x; \underline{v}) \leqslant u^+ \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R},$$

$$0 \leqslant u(t, x; \bar{v}) \leqslant \bar{v}(t, x) \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}$$

from which it follows that $\underline{v}(t, x) \leqslant \bar{v}(t, x)$ for all $(t, x) \in [0, \infty) \times \mathbb{R}$. □

3.2 *Asymptotic speed and minimal speed for* (1.1)

In this section, we rewrite $u^{(\tau)}(t,x) = \mathcal{U}(t,x)$ and $u_\tau^+ = \mathcal{U}^+$ to address the dependence on τ of $\mathcal{U}(t,x)$ and $\mathcal{U}^+$. Furthermore, we use $\phi^{(\tau)}(\theta,x), \theta \in [-\tau,0], x \in \mathbb{R}$ to denote the initial function $\phi(\theta,x)$ in (2.1) to address that $\phi(\theta,x)$ is defined on $[-\tau,0]$ for the first variable θ.

From the Section 2.2, we know that there is c_τ^* being the asymptotic speed for (2.1) as well as the minimal speed. It is natural to ask if there is a c^* being such a speed for (1.1)? What is the relation between c^* and c_τ^*? Since (1.1) is the limit system of (2.1), we wonder if $c^* = \lim_{\tau\to+\infty} c_\tau^*$? In this subsection, we try to answer these interesting questions positively by using the finite delays approximation method which is introduced in Fang *et al.* (2008) and Zhao & Xiao (2006). We suppose (F1)–(F5) and (G1)–(G2) are satisfied throughout this subsection. Note that the differentiability of the functions f, ϕ is lost, so we have to do much preparation work, before the main results comes out, which are Lemmas 3.1–3.6 and Theorem 3.2.

LEMMA 3.1 u_τ^+ is increasing on τ, thus $\lim_{\tau\to+\infty} u_\tau^+ = u^+$.

Proof. Assume $\tau_1 < \tau_2$, by (2.1) and (F4), we have

$$\begin{aligned} 0 &= f\left(u_{\tau_2}^+, \varphi(u_{\tau_2}^+)\int_0^{\tau_2}\int_{-\infty}^{+\infty} g(s,y)\mathrm{d}y\,\mathrm{d}s\right) \\ &\geqslant f\left(u_{\tau_2}^+, \varphi(u_{\tau_2}^+)\int_0^{\tau_1}\int_{-\infty}^{+\infty} g(s,y)\mathrm{d}y\,\mathrm{d}s\right), \end{aligned}$$

then (F5) implies that $u_{\tau_2}^+ \geqslant u_{\tau_1}^+$, thus the lemma is valid. □

LEMMA 3.2 c_τ^* is increasing on τ.

Proof. Assume the claim is not true, then there is $0 < \tau_1 < \tau_2$ such that $c_{\tau_1}^* > c_{\tau_2}^*$. For a given $\sigma \in [0, u_{\tau_2}^+]_{\bar{\mathscr{C}}}$ with $u_{\tau_1}^+ > \sigma(\theta) > 0$ for $\theta \in [-\tau_2, 0]$, in view of Theorem 2.3, there is a positive number r_σ, and thus an interval $I = [-r_\sigma, r_\sigma]$. Define $\phi^{(\tau_2)} \in \mathscr{C}_{(\tau_2)}$ satisfying

$$u_{\tau_1}^+ \geqslant \phi^{(\tau_2)}(\theta,x) > \sigma(\theta) \quad \text{for } \theta \in [-\tau_2, 0], \quad x \in I. \tag{3.6}$$

Furthermore, let $\phi^{(\tau_1)}(\theta,x) := \phi^{(\tau_2)}(\theta,x)$ for $\theta \in [-\tau_1, 0], x \in \mathbb{R}$. Choose $c \in (c_{\tau_2}^*, c_{\tau_1}^*)$, by Theorem 2.3 and (3.6), we have

$$\lim_{t\to+\infty,|x|=ct} u^{(\tau_1)}(t,x;\phi^{(\tau_1)}) = u_{\tau_1}^+ \quad \text{and} \quad \lim_{t\to+\infty,|x|=ct} u^{(\tau_2)}(t,x;\phi^{(\tau_2)}) = 0. \tag{3.7}$$

On the other hand, $u^{(\tau_2)}(t,x;\phi^{(\tau_2)})$ is the supersolution for (2.1) with $\tau = \tau_1$, then we have from Theorem 2.1 that $u^{(\tau_2)}(t,x;\phi^{(\tau_2)}) \geqslant u^{(\tau_1)}(t,x;\phi^{(\tau_1)})$ for $t > 0, x \in \mathbb{R}$, which is a contradiction with (3.7). Thus, c_τ^* is increasing on τ. The proof is complete. □

Now, we study the boundedness of $\{c_\tau^*\}_{\tau=\tau_0}^{+\infty}$. Under the assumption (G1) and (G2), one can define a function p with two parameters λ and c as follows:

$$p(\lambda,c) := D\lambda^2 - c\lambda + L' + \hat{L}\tilde{L}\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\mathrm{e}^{-\lambda(cs+y)}\mathrm{d}y\,\mathrm{d}s, \tag{3.8}$$

where $\lambda \in [0, \tilde{\delta}(c))$, p is continuous and differentiable on λ and c. Note that we only assume that $f \in C(\mathbb{R}^2,\mathbb{R})$ and $\varphi \in C(\mathbb{R},\mathbb{R})$, thus $p(\lambda,c)$ defined above is in fact a generalized eigenfunction.

It is easy to obtain

$$\frac{\partial p}{\partial \lambda} = 2D\lambda - c + \hat{L}\tilde{L}\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)[-(cs+y)]\mathrm{e}^{-\lambda(cs+y)}\mathrm{d}y\,\mathrm{d}s,$$

$$\frac{\partial^2 p}{\partial \lambda^2} = 2D + \hat{L}\tilde{L}\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)(cs+y)^2\mathrm{e}^{-\lambda(cs+y)}\mathrm{d}y\,\mathrm{d}s > 0,$$

$$p(0,c) = L' + \hat{L}\tilde{L} > 0,$$

$$p(\lambda,0) = D\lambda^2 + L' + \hat{L}\tilde{L}\int_0^{+\infty}\int_{-\infty}^{+\infty} g(s,y)\mathrm{e}^{-\lambda y}\mathrm{d}y\,\mathrm{d}s > 0,$$

$$p(\lambda,+\infty) = -\infty \quad \text{for any given } \lambda > 0,$$

$$\lim_{\lambda\to\tilde{\delta}(c)} p(\lambda,c) = +\infty \quad \text{for any given } c > 0,$$

$$\frac{\partial p}{\partial c} = -\lambda + \hat{L}\tilde{L}\int_0^{+\infty}\int_{-\infty}^{+\infty} (-\lambda s)g(s,y)\mathrm{e}^{-\lambda(cs+y)}\mathrm{d}y\,\mathrm{d}s < 0 \quad \text{for } \lambda > 0.$$

In view of the above properties of p, we obtain Lemma 3.3.

LEMMA 3.3 There exists a unique pair of $(\lambda_L^*, c_{(L)}^*)$ such that

(i) $p(\lambda_L^*, c_{(L)}^*) = 0, \quad \frac{\partial p}{\partial \lambda}(\lambda_L^*, c_{(L)}^*) = 0;$

(ii) $p(\lambda,c) > 0$ holds for $0 < c < c_{(L)}^*$ and $\lambda \in [0, \tilde{\delta}(c))$;

(iii) for $c > c_{(L)}^*$, there are two zeros $0 < \lambda_{a(c)} < \lambda_{b(c)} < \tilde{\delta}(c)$ for $p(\lambda,c) = 0$. Furthermore, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ with $0 < \lambda_{a(c)} < \lambda_{a(c)} + \varepsilon < \lambda_{b(c)}$, we have

$$p(\lambda_{a(c)} + \varepsilon, c) < 0. \tag{3.9}$$

LEMMA 3.4 Let $\Gamma(t-r,x-y) = \frac{\mathrm{e}^{-\gamma(t-r)}}{\sqrt{4\pi D(t-r)}}\mathrm{e}^{-\frac{(x-y)^2}{4D(t-r)}}$, then

$$\int_{-\infty}^{+\infty} \Gamma(t-r,x-y)\mathrm{e}^{-\lambda y}\,\mathrm{d}y = \mathrm{e}^{-\lambda x}\,\mathrm{e}^{(D\lambda^2-\gamma)(t-r)}.$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \Gamma(t-r,x-y)\mathrm{e}^{-\lambda y}\mathrm{e}y &= \frac{\mathrm{e}^{-\gamma(t-r)}}{\sqrt{4\pi D(t-r)}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x-y)^2}{4D(t-r)}}\mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&= \frac{\mathrm{e}^{-\gamma(t-r)}\,\mathrm{e}^{-\lambda x}}{\sqrt{4\pi D(t-r)}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(y)^2}{4D(t-r)}+\lambda y}\,\mathrm{d}y \\
&= \frac{\mathrm{e}^{-\gamma(t-r)}\,\mathrm{e}^{-\lambda x}}{\sqrt{4\pi D(t-r)}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{[y-2D(t-r)\lambda]^2}{4D(t-r)}}\,\mathrm{e}^{D(t-r)\lambda^2}\,\mathrm{d}y \\
&= \mathrm{e}^{-\lambda x}\,\mathrm{e}^{(D\lambda^2-\gamma)(t-r)}.
\end{aligned}$$

□

LEMMA 3.5 Assume that the initial function $\phi \in [0, u^+]_{\mathrm{UC}_h}$ satisfies $\phi(\theta, x) = 0$ for (θ, x) outside a bounded set of $(-\infty, 0] \times \mathbb{R}$. Then $\lim\limits_{t\to\infty,|x|>ct} u(t, x) = \lim\limits_{t\to\infty,|x|>ct} u(t, x; \phi) = 0$ holds for $c > c^*_{(L)}$.

Proof. (1) Let $c > c^*_{(L)}$ be fixed and $\bar{c} \in (c^*_{(L)}, c)$, $\bar{\lambda} > 0$ be chosen such that

$$p(\bar{c}, \bar{\lambda}) < 0. \tag{3.10}$$

Let $\bar{u}(t, x) := \min\{\beta\, \mathrm{e}^{\bar{\lambda}(\bar{c}t - \mathrm{sgn}\{x\}x)}, u^+\}$, where $\beta > 0$ is to be decided. It is obvious that $\bar{u}(t, x)$ is not C^2 in $x \in \mathbb{R}$ and C^1 in $t \in \mathbb{R}^+$. We shall prove $\bar{u}(t, x)$ is the supersolution of (1.1) when f satisfies (F1)–(F4) by using the abstract form

$$\bar{u}(t)(\cdot) \geqslant T(t)\phi(0, \cdot) + \int_0^t T(t - r)F(\bar{u}_r)(\cdot)\mathrm{d}r,$$

where F and T is defined by (3.2) and (3.3), respectively.

Since ϕ has a compact support, we can choose β large sufficiently such that

$$\phi(t, x) \leqslant \min\left\{\beta\, \mathrm{e}^{\bar{\lambda}(\bar{c}t - \mathrm{sgn}\{x\}x)}, u^+\right\} = \bar{u}(t, x) \quad \text{ for } (t, x) \in (-\infty] \times \mathbb{R}.$$

If $\bar{c}t - |x| \geqslant \dfrac{1}{\bar{\lambda}} \ln\left(\frac{u^+}{\beta}\right)$, then $\bar{u}(t, x) = u^+$, and obviously, we have

$$T(t)\phi(0, \cdot) + \int_0^t T(t - r)F(\bar{u}_r)(\cdot)\mathrm{d}r \leqslant u^+. \tag{3.11}$$

If $\bar{c}t - |x| < \frac{1}{\bar{\lambda}} \ln\left(\frac{u^+}{\beta}\right)$, then $\bar{u}(t, x) = \beta\, \mathrm{e}^{\bar{\lambda}(\bar{c}t - \mathrm{sgn}\{x\}x)}$, we claim that

$$T(t)\phi(0, x) + \int_0^t T(t - r)F(\bar{u}_r)(x)\mathrm{d}r \leqslant \bar{u}(t, x) = \beta\, \mathrm{e}^{\bar{\lambda}(\bar{c}t - \mathrm{sgn}\{x\}x)}. \tag{3.12}$$

In fact, assuming that $x \geqslant 0$, by Lemma 3.4, we have

$$T(t)\phi(0, x) \leqslant \int_{-\infty}^{+\infty} \Gamma(t, x - y)\beta\, \mathrm{e}^{-\bar{\lambda}y} dy = \beta\, \mathrm{e}^{-\bar{\lambda}x} \mathrm{e}^{(d\bar{\lambda}^2 - L')t}. \tag{3.13}$$

Let

$$A := 2L' + \hat{L}\tilde{L} \int_0^{+\infty} \int_{-\infty}^{+\infty} g(s, y)\mathrm{e}^{-\bar{\lambda}(\bar{c}s + y)}\mathrm{d}s\, \mathrm{d}y.$$

Then by (F2) and Lemma 3.4, we have

$$\begin{aligned}
\int_0^t T(t - r)F(\bar{u}_r)\mathrm{d}r &\leqslant A\beta \int_0^t \int_{-\infty}^{+\infty} \Gamma(t - r, x - y)\mathrm{e}^{\bar{\lambda}(\bar{c}r - y)}\mathrm{d}y\, \mathrm{d}r \\
&= A\beta \int_0^t \mathrm{e}^{\bar{\lambda}\bar{c}r}\, \mathrm{e}^{-\bar{\lambda}x}\, \mathrm{e}^{(d\bar{\lambda}^2 - L')(t - r)}\mathrm{d}r \\
&= A\beta\, \mathrm{e}^{-\bar{\lambda}x} \mathrm{e}^{(d\bar{\lambda}^2 - L')t} \int_0^t \mathrm{e}^{(\bar{\lambda}\bar{c} + L' - d\bar{\lambda}^2)r}\mathrm{d}r \\
&= \frac{A\beta\, \mathrm{e}^{-\bar{\lambda}x}}{\bar{\lambda}\bar{c} + L' - d\bar{\lambda}^2} \mathrm{e}^{(d\bar{\lambda}^2 - L')t}\left[\mathrm{e}^{(\bar{\lambda}\bar{c} + L' - d\bar{\lambda}^2)t} - 1\right].
\end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we obtain

$$T(t)\phi(0,\cdot)+\int_0^t T(t-r)F(\bar u_r)\mathrm{d}r \leqslant \beta\,\mathrm{e}^{-\bar\lambda x}\,\mathrm{e}^{(d\bar\lambda^2-L')t}\left[1-\frac{A}{\bar\lambda\bar c+L'-d\bar\lambda^2}\right]$$
$$+\frac{A\beta}{\bar\lambda\bar c+L'-d\bar\lambda^2}\mathrm{e}^{(-\bar\lambda x+\bar\lambda\bar c t)}. \tag{3.15}$$

By (3.10), there are $\bar\lambda\bar c+L'-D\bar\lambda^2-A=-p(\bar\lambda,\bar c)>0$ and thus $D\bar\lambda^2-L'<\bar c\bar\lambda$, which together with (3.15) leads to

$$T(t)\phi(0,\cdot)+\int_0^t T(t-r)F(\bar u_r)\mathrm{d}r \leqslant \beta\,\mathrm{e}^{-\bar\lambda x}\mathrm{e}^{\bar\lambda\bar c t}\left[\frac{-p(\bar\lambda,\bar c)+A}{\bar\lambda\bar c+L'-D\bar\lambda^2}\right]$$
$$=\beta\,\mathrm{e}^{\bar\lambda(\bar c t-|x|)}.$$

If $x<0$, one can discuss by a similar way. Therefore, (3.12) holds, (3.11) and (3.12) lead to a conclusion that $\bar u(t,x)$ is a supersolution of (1.1) when f satisfies (F1)–(F4). Thus, for any $c>c^*_{(L)}$, we obtain

$$\lim_{t\to\infty,|x|\geqslant ct} u(t,x)\leqslant \lim_{t\to\infty,|x|\geqslant ct}\bar u(t,x)=0.$$

The proof is complete. □

Lemmas 3.2 and 3.5 lead to the following main result.

THEOREM 3.2 There is a $c^*>0$ being the limit of $\{c^*_\tau\}_{\tau=\tau_0}^{+\infty}$ as $\tau\to\infty$.

Proof. For any initial function ϕ satisfying the assumption in Lemma 3.5, we define a function $\phi^{(\tau)}(t,x):=\phi(t,x)|_{t\in[-\tau,0]}$. Noting for any $\tau>0$, $u(t,x;\phi)|_{t\in[-\tau,\infty)}$ can be regard as a supsolution of (2.1). Thus, we have from Theorem 2.1 that

$$u(t,x;\phi)|_{t\in[-\tau,\infty)}\geqslant u^{(\tau)}(t,x;\phi^{(\tau)})\quad \text{for } (t,x)\in[-\tau,\infty)\times\mathbb{R}.$$

For any $c>c^*_{(L)}$, we have from Lemma 3.5 that

$$0\leqslant \lim_{t\to+\infty,|x|>ct}u^{(\tau)}(t,x)\leqslant \lim_{t\to+\infty,|x|>ct}u(t,x)=0.$$

Hence, by Theorem 2.3, we obtain $c^*_\tau\leqslant c^*_{(L)}$ for $\tau\geqslant\tau_0$, $\{c^*_\tau\}_{\tau=\tau_0}^{+\infty}$ is bounded. Noting the conclusion in Lemma 3.2, we complete the proof. □

Is c^* the asymptotic and the minimal speed for (1.1)? Before answering this question, we have to study the solution of (1.1) with the form $u(x,t)=U(x+ct)$, $c>0$, i.e, the travelling wave solution of (1.1). Let $z=x+ct$. Then the wave profile equation is

$$c\,(U(z))'=DU''(z)$$
$$+f\left(U(z),\int_0^{+\infty}\int_{-\infty}^{+\infty}g(s,y)\varphi(U(z-y-cs))\mathrm{d}y\,\mathrm{d}s\right). \tag{3.16}$$

What we are interested in is the monotone travelling wavefront $U(z)$ connecting the two equilibria $U \equiv 0$ and $U \equiv u^+$. That is, we shall discuss the non-decreasing solution of (3.16) with a group of boundary value condition

$$U(-\infty) = 0 \quad \text{and} \quad U(+\infty) = u^+. \tag{3.17}$$

Define

$$\begin{aligned}[QU](z) := {} & L'U(z) \\ & + f\left(U(z), \int_0^{+\infty}\int_{-\infty}^{+\infty} g(s, y)\varphi(U(z - y - cs))\mathrm{d}y\,\mathrm{d}s\right).\end{aligned} \tag{3.18}$$

A equivalent form of (3.16) is

$$U(z) = \int_{-\infty}^{+\infty} k(z, s)[QU](s)\mathrm{d}s, \tag{3.19}$$

where

$$k(z, s) = \begin{cases} \dfrac{1}{\zeta}\mathrm{e}^{\lambda_1(z-s)}, & s \leqslant z, \\ \dfrac{1}{\zeta}\mathrm{e}^{\lambda_2(z-s)}, & s \geqslant z, \end{cases}$$

and

$$\zeta = D(\lambda_2 - \lambda_1), \quad \lambda_1 = \frac{c - \sqrt{c^2 + 4L'D}}{2D}, \quad \lambda_2 = \frac{c + \sqrt{c^2 + 4L'D}}{2D}. \tag{3.20}$$

Theorem 2.4 establishes the existence of the monotone travelling wavefronts for the system (2.1) with finite delay. It is natural to guess that $\lim_{\tau\to\infty} U^{(\tau)}(z) = U(z)$ is a solution of (3.16) and (3.17), where $U^{(\tau)}(x + ct)$ is the monotone travelling wavefront of (2.1) satisfying (2.24) and (2.25). Define

$$\begin{aligned}[Q^{(\tau)}U^{(\tau)}](z) := {} & L'_\tau U^{(\tau)}(z) \\ & + f\left(U^{(\tau)}(z), \int_0^{\tau}\int_{-\infty}^{+\infty} g(s, y)\varphi(U^{(\tau)}(z - y - cs))\mathrm{d}y\,\mathrm{d}s\right).\end{aligned} \tag{3.21}$$

Then $U^{(\tau)}(z)$ satisfies the following integral equation:

$$U^{(\tau)}(z) = \int_{-\infty}^{+\infty} k^{(\tau)}(z, s)[Q^{(\tau)}U^{(\tau)}](s)\mathrm{d}s, \tag{3.22}$$

where

$$k^{(\tau)}(z, s) = \begin{cases} \dfrac{1}{\zeta_\tau}\mathrm{e}^{\lambda_1^{(\tau)}(z-s)}, & s \leqslant z, \\ \dfrac{1}{\zeta_\tau}\mathrm{e}^{\lambda_2^{(\tau)}(z-s)}, & s \geqslant z, \end{cases}$$

and

$$\zeta_\tau = D(\lambda_2^{(\tau)} - \lambda_1^{(\tau)}), \quad \lambda_1^{(\tau)} = \frac{c - \sqrt{c^2 + 4L'_\tau D}}{2D}, \quad \lambda_2^{(\tau)} = \frac{c + \sqrt{c^2 + 4L'_\tau D}}{2D}. \tag{3.23}$$

The following Lemma 3.6 will tell us that $\{U^{(\tau)}(z)\}_{\tau=\tau_0}^{+\infty}$ is equi-continuous. Hence, we can use Arzela–Ascoli theorem and the dominated convergence theorem to obtain the existence for the monotone wavefronts for (3.16) when $c > c^*$.

LEMMA 3.6 $\left\{U^{(\tau)}(z)\right\}_{\tau=\tau_0}^{+\infty}$ is equi-continuous, where τ_0 is defined by (2.2).

Proof. For any $z, z' \in \mathbb{R}$, we assume $z > z'$. Note that there is a unique $z_1 \in (z', z)$ such that $\mathrm{e}^{\lambda_1^{(\tau)}(z-z_1)} = \mathrm{e}^{\lambda_2^{(\tau)}(z'-z_1)}$, that is

$$\mathrm{e}^{\lambda_1^{(\tau)}(z-s)} < \mathrm{e}^{\lambda_2^{(\tau)}(z'-s)} \quad \text{for } s < z_1, \quad \mathrm{e}^{\lambda_1^{(\tau)}(z-s)} > \mathrm{e}^{\lambda_2^{(\tau)}(z'-s)} \quad \text{for } s > z_1.$$

Noting that there is $\bar{M} > 0$ such that $[Q^{(\tau)}U^{(\tau)}](z) \leqslant \bar{M}$ for any $z \in \mathbb{R}$, $\tau \geqslant \tau_0$, we obtain

$$\begin{aligned}
&\left|\int_{-\infty}^{+\infty} [k^{(\tau)}(z,s) - k^{(\tau)}(z',s)][Q^{(\tau)}U^{(\tau)}](s)\mathrm{d}s\right| \\
&\leqslant \bar{M}\int_{-\infty}^{+\infty} |k^{(\tau)}(z,s) - k^{(\tau)}(z',s)|\mathrm{d}s \\
&\leqslant \frac{\bar{M}}{\zeta_\tau}\left[\int_{-\infty}^{z'}\left(\mathrm{e}^{\lambda_1^{(\tau)}(z'-s)} - \mathrm{e}^{\lambda_1^{(\tau)}(z-s)}\right)\mathrm{d}s + \int_{z'}^{z_1}\left(\mathrm{e}^{\lambda_2^{(\tau)}(z'-s)} - \mathrm{e}^{\lambda_1^{(\tau)}(z-s)}\right)\mathrm{d}s\right] \\
&\quad + \frac{\bar{M}}{\zeta_\tau}\left[\int_{z_1}^{z}\left(\mathrm{e}^{\lambda_1^{(\tau)}(z-s)} - \mathrm{e}^{\lambda_2^{(\tau)}(z'-s)}\right)\mathrm{d}s + \int_{z}^{+\infty}\left(\mathrm{e}^{\lambda_2^{(\tau)}(z-s)} - \mathrm{e}^{\lambda_2^{(\tau)}(z'-s)}\right)\mathrm{d}s\right] \\
&= \frac{\bar{M}}{\zeta_\tau}\left(-\frac{1}{\lambda_1^{(\tau)}} + \frac{1}{\lambda_1^{(\tau)}}\mathrm{e}^{\lambda_1^{(\tau)}(z-z')} - \frac{1}{\lambda_2^{(\tau)}}\mathrm{e}^{\lambda_2^{(\tau)}(z'-z_1)} + \frac{1}{\lambda_2^{(\tau)}} + \frac{1}{\lambda_1^{(\tau)}}\mathrm{e}^{\lambda_1^{(\tau)}(z-z_1)} - \frac{1}{\lambda_1^{(\tau)}}\mathrm{e}^{\lambda_1^{(\tau)}(z-z')}\right) \\
&\quad + \frac{\bar{M}}{\zeta_\tau}\left(-\frac{1}{\lambda_1^{(\tau)}} + \frac{1}{\lambda_1^{(\tau)}}\mathrm{e}^{\lambda_1^{(\tau)}(z-z_1)} + \frac{1}{\lambda_2^{(\tau)}}\mathrm{e}^{\lambda_2^{(\tau)}(z'-z)} - \frac{1}{\lambda_2^{(\tau)}}\mathrm{e}^{\lambda_2^{(\tau)}(z'-z_1)} + \frac{1}{\lambda_2^{(\tau)}} - \frac{1}{\lambda_2^{(\tau)}}\mathrm{e}^{\lambda_2^{(\tau)}(z'-z)}\right) \\
&= \frac{2\bar{M}}{\zeta_\tau\lambda_1^{(\tau)}}\left(\mathrm{e}^{\lambda_1^{(\tau)}(z-z_1)} - 1\right) + \frac{2\bar{M}}{\zeta_\tau\lambda_2^{(\tau)}}\left(1 - \mathrm{e}^{\lambda_2^{(\tau)}(z'-z_1)}\right).
\end{aligned}$$

By (3.23) and (3.20), $\lambda_1 < \lambda_1^{(\tau)} < \lambda_1^{(\tau_0)} < 0 < \lambda_2^{(\tau_0)} < \lambda_2^{(\tau)} < \lambda_2$ and $\zeta > \zeta_\tau > \zeta_{\tau_0} > 0$ hold. It follows that

$$\left|\int_{-\infty}^{+\infty}\left[k^{(\tau)}(z,s)-k^{(\tau)}(z',s)\right][Q^{(\tau)}U^{(\tau)}](s)\mathrm{d}s\right| \leqslant \frac{2\bar{M}}{\zeta_{\tau_0}\lambda_1^{(\tau_0)}}\left(\mathrm{e}^{\lambda_1(z-z_1)}-1\right)+\frac{2\bar{M}}{\zeta_{\tau_0}\lambda_2^{(\tau_0)}}\left(1-\mathrm{e}^{\lambda_2(z'-z_1)}\right)$$
$$\leqslant \frac{2\bar{M}}{\zeta_{\tau_0}\lambda_1^{(\tau_0)}}\left(\mathrm{e}^{\lambda_1(z-z')}-1\right)+\frac{2\bar{M}}{\zeta_{\tau_0}\lambda_2^{(\tau_0)}}\left(1-\mathrm{e}^{\lambda_2(z'-z)}\right).$$

Then, for any $\varepsilon > 0$, we can choose $\delta(\varepsilon)$ such that if $|z'-z| < \delta$, $|U^{(\tau)}(z)-U^{(\tau)}(z')| < \varepsilon$ holds for any $\tau \geqslant \tau_0$, hence $\left\{U^{(\tau)}(z)\right\}_{\tau=\tau_0}^{+\infty}$ is equi-continuous. □

Based on the above discussion, we use the finite delays approximation method which is introduced in Fang *et al.* (2008) and Zhao & Xiao (2006) to consider the asymptotic speed and the minimal speed of model (1.1) consequently.

THEOREM 3.3 Assume (F1)–(F5) and (G1)–(G2) hold, let c^* be defined in Theorem 3.2, the following statements are valid:

(i) for any $c \geqslant c^*$, (3.16) has a monotone travelling wave solution $U(x+ct)$ connecting 0 and u^+;

(ii) for $\phi \in [0,u^+]_{\mathrm{UC}_h}$ with $\phi(\theta,x)=0$ for (θ,x) outside a bounded set, then

$$\lim_{t\to\infty,|x|\geqslant ct} u(t,x;\phi)=0 \qquad \text{for any } c>c^*,$$

where $u(t,x;\phi)$ is the solution of (1.1) with $u(\theta,x)=\phi(\theta,x)$ for $\theta\in(-\infty,0]$, $x\in\mathbb{R}$.

Proof. (i) For $c \geqslant c^*$, by Lemmas 3.1 and 3.2, we can choose $\tau \geqslant \tau_0$ such that $c \geqslant c^* \geqslant c_\tau^*$ and $\frac{1}{2}u^+ < u_\tau^+$. By Theorem 2.4, there is $U^{(\tau)}(x+ct)=U^{(\tau)}(z)$ satisfying (2.24)–(2.25) and $U^{(\tau)}(0)=\frac{1}{2}u^+$. $\left\{U^{(\tau)}(z)\right\}_{\tau=\tau_0}^{+\infty}$ is uniformly bounded, and moreover, it is equi-continuous by Lemma 3.6. Using Arzela–Ascoli theorem and the standard diagonal method, we obtain a subsequence, without loss of generality, it is still be denoted as $\left\{U^{(\tau)}(z)\right\}_{\tau=\tau_0}^{+\infty}$, which converges to $U^*(z)$ as $\tau\to+\infty$, uniformly for z in any bounded subset of $\mathbb{R}$. Obviously, $\lim_{\tau\to+\infty} Q^{(\tau)}U^{(\tau)}(z)=QU(z)$ provided $\lim_{\tau\to+\infty} U^{(\tau)}(z)=U(z)$. Thus, by the dominated convergence theorem, together with (3.18)–(3.19) and (3.21)–(3.22), we have

$$U^*(z)=\lim_{\tau\to+\infty}U^{(\tau)}(z)=\int_{-\infty}^{+\infty}\lim_{\tau\to+\infty}[k^{(\tau)}(z,s)][Q^{(\tau)}U^{(\tau)}](s)\mathrm{d}s$$
$$=\int_{-\infty}^{+\infty}k(z,s)[QU^*](s)\mathrm{d}s.$$

Hence, $U^*(z)$ is the wavefronts of (3.17)–(3.16) with $U^*(0)=\frac{1}{2}u^+$.

Clearly, $U^*(z)$ must be non-decreasing, otherwise there are $z_1,z_2\in[a,b]$ with $z_1>z_2$ such that $U^*(z_1)<U^*(z_2)$. For $\varepsilon=\frac{U^*(z_2)-U^*(z_1)}{2}>0$, we can find $\tau_1>\tau_0$ such that $|U^*(z)-U^{(\tau_1)}(z)|<\frac{\varepsilon}{2}$ for $z\in[a,b]$. Note that

$$U^*(z_2)-U^*(z_1)=U^*(z_2)-U^{(\tau_1)}(z_2)+U^{(\tau_1)}(z_2)-U^{(\tau_1)}(z_1)+U^{(\tau_1)}(z_1)-U^*(z_1),$$

since $U^{(\tau_1)}(z_2)-U^{(\tau_1)}(z_1)\leqslant 0$, then

$$U^*(z_2)-U^*(z_1)\leqslant U^*(z_2)-U^{(\tau_1)}(z_2)+U^{(\tau_1)}(z_1)-U^*(z_1)<\varepsilon=\frac{U^*(z_2)-U^*(z_1)}{2},$$

which is a contradiction, thus $U^*(z)$ must be non-decreasing.

Next, we claim $\lim_{\tau\to-\infty} U^*(z) = 0$ and $\lim_{\tau\to+\infty} U^*(z) = u^+$. Because $U^*(z)$ is monotone and bounded, we have A_1 and A_2 with $0 \leqslant A_1 \leqslant A_2 \leqslant u^+$ such that $\lim_{\tau\to-\infty} U^*(z) = A_1$ and $\lim_{\tau\to+\infty} U^*(z) = A_2$. Since $\lim_{z\to-\infty} Q[U](z) = [QA_1]$, then by (3.19),

$$\begin{aligned}
A_1 = \lim_{z\to-\infty} U^*(z) &= \lim_{z\to-\infty} \int_{-\infty}^{+\infty} k(z,s)[QU^*](s)\mathrm{d}s \\
&= \lim_{z\to-\infty} \frac{1}{\zeta}\left[\int_z^{+\infty} \mathrm{e}^{\lambda_2(z-s)}[QU^*](s)\mathrm{d}s + \int_{-\infty}^{z} \mathrm{e}^{\lambda_1(z-s)}[QU^*](s)\mathrm{d}s\right] \\
&= \lim_{z\to-\infty} \frac{1}{\zeta}\left[\int_{-\infty}^{0} \mathrm{e}^{\lambda_2 s}[QU^*](z-s)\mathrm{d}s + \int_{0}^{+\infty} \mathrm{e}^{\lambda_1 s}[QU^*](z-s)\mathrm{d}s\right] \\
&= [QA_1]\frac{1}{\zeta}\left(\int_{-\infty}^{0} \mathrm{e}^{\lambda_2 s}\mathrm{d}s + \int_{0}^{+\infty} \mathrm{e}^{\lambda_1 s}\mathrm{d}s\right).
\end{aligned}$$

By (3.18) and (3.20), we have

$$\begin{aligned}
A_1 &= [L'A_1 + f(A_1, \varphi(A_1))]\frac{1}{\zeta}\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) \\
&= [L'A_1 + f(A_1, \varphi(A_1))]\frac{-1}{D}\frac{1}{\lambda_1\lambda_2} \\
&= [L'A_1 + f(A_1, \varphi(A_1))]\frac{1}{L'},
\end{aligned}$$

thus A_1 must be the root of $f(u, \varphi(u)) = 0$. Obviously, $U(0) = \frac{1}{2}$ implies that $A_1 = 0$. By the same way, we can prove that $A_2 = u^+$, then $U^*(z)$ is the solution of (3.16) with the boundary condition (3.17).

Now, we shall show that (ii) is valid. Consider the case $x < -ct$ first. Choose $\bar{c} \in (c^*, c)$, since ϕ has a compact support and $\lim_{z\to+\infty} U(z) = u^+$, we can choose x_1 large sufficiently such that

$$U(x_1 + x + \bar{c}\theta) \geqslant \phi(\theta, x) \quad \text{for } (\theta, x) \in (-\infty, 0] \times \mathbb{R}.$$

Let

$$\bar{\phi}(\theta, x) = U(x_1 + x + \bar{c}\theta), \quad u(t, x; \bar{\phi}) = U(x_1 + x + \bar{c}t).$$

Then

$$\bar{\phi}(\theta, x) \geqslant \phi(\theta, x) \quad \text{for } \theta \in (-\infty, 0], x \in \mathbb{R},$$

and we have from comparison principle

$$u(t, x; \bar{\phi}) \geqslant u(t, x; \phi). \tag{3.24}$$

By (i) and (3.24), there is

$$0 \leqslant \lim_{t\to\infty, x<-ct} u(t,x;\phi) \leqslant \lim_{t\to\infty, x<-ct} u(t,x;\bar{\phi}) = \lim_{t\to\infty, x<-ct} U(x_1 + x + \bar{c}t) = U(-\infty) = 0. \tag{3.25}$$

As for the case $x > ct$, we can show (ii) as follows. Let $\tilde{\phi}(\theta,x) = \phi(\theta,-x)$, $\tilde{u}(t,x;\tilde{\phi}) = u(t,-x;\phi)$, similarly as above, we have

$$\lim_{t\to\infty, x<-ct} \tilde{u}(t,x;\tilde{\phi}) = \lim_{t\to\infty, x<-ct} u(t,-x;\phi) = 0 \quad \text{for } c > c^*,$$

it follows that

$$\lim_{t\to\infty, x>ct} u(t,x;\phi) = 0, \tag{3.26}$$

(3.25) and (3.26) imply that (ii) is valid. The proof is complete. □

THEOREM 3.4 Assume (F1)–(F5) and (G1)–(G2) hold, let c^* be defined in Theorem 3.2, the following statements are valid:

(i) for any $\sigma \in C(-\infty,0]$ with $0 < \sigma(\theta) \leqslant u^+$ for $\theta \in (-\infty,0]$, there is a positive number r_σ such that if $\phi \in [0,u^+]_{\mathrm{UC}_h}$ with $\phi(\cdot,x) > \sigma(\cdot)$ for x on an interval of length $2r_\sigma$, then

$$\lim_{t\to\infty, |x|\leqslant ct} u(t,x;\phi) = u^+ \quad \text{for any } 0 < c < c^*;$$

(ii) for any $c \in (0,c^*)$, (3.16) admits no monotone travelling wave solution $U(x+ct)$ connecting 0 and u^+ with $U(\cdot) \in C_{[0,u^+]}(\mathbb{R},\mathbb{R}) := \{\psi : \psi \in C(\mathbb{R},\mathbb{R}), 0 \leqslant \psi(z) \leqslant u^+ \text{ for } z \in \mathbb{R}\}$.

Proof. First we claim (i) is valid. Note $u_\tau^+ \to u^+$ as $\tau \to \infty$. Let

$$\phi^{(\tau)}(\theta,x) = \min\{\phi(\theta,x), u_\tau^+\}, \quad \sigma^{(\tau)}(\theta) = \min\{\sigma(\theta), u_\tau^+\}, \quad \theta \in [-\tau,0], \quad x \in \mathbb{R}.$$

Then $0 < \sigma^{(\tau)}(\theta) \leqslant u_\tau^+$ for $\theta \in [-\tau,0]$, and we have $\phi^{(\tau)} \in \mathscr{C}_{(\tau)}$ with $\phi^{(\tau)}(\cdot,x) > \sigma^{(\tau)}(\cdot)$ for x on an interval of length $2r_{\sigma^{(\tau)}}$. In view of Theorem 2.3, we have

$$\lim_{t\to+\infty, |x|\leqslant ct} u^{(\tau)}(t,x;\phi^{(\tau)}) = u_\tau^+ \quad \text{for any } c < c_\tau^*. \tag{3.27}$$

Since $u(t,x;\phi)$ is the supersolution for (2.1), by the comparison principles, we have

$$u^+ \geqslant u(t,x;\phi) \geqslant u^{(\tau)}(t,x;\phi^{(\tau)}). \tag{3.28}$$

For any $c < c^*$, by Theorem 3.2, we can choose τ sufficiently large that $c^* > c_\tau^* > c$. Let $\tau \to \infty$, then together with Lemma 3.1 and (3.27), we have $\lim_{t\to+\infty, |x|\leqslant ct} u(t,x;\phi) = u^+$. Therefore, the conclusion of (i) is true.

We use reduction to absurdity to prove (ii) of this theorem. Suppose there is $c_1 < c^*$ such that $U(x+c_1t)$ is the non-decreasing travelling wavefronts connecting 0 to u^+. Let $\sigma(\theta) = \frac{1}{2}U(x+c_1\theta)$

for $\theta \in (-\infty, 0]$ and $\phi(\theta, x) = U(x + c_1\theta)$, then $\phi \in [0, u^+]_{\mathrm{UC}_h}$, $\phi(\cdot, x) = U(x + c_1\cdot) > \sigma(\cdot)$ for x on an interval of length $2r_\sigma$ and $u(t, x; \phi) = U(x + c_1 t)$. By using (i) of this theorem, we have

$$\lim_{t\to\infty, |x|\leqslant ct} u(t, x; \phi) = \lim_{t\to\infty, |x|\leqslant ct} U(x + c_1 t) = u^+ \quad \text{for any } c \in (0, c^*).$$

Let $x = -c_2 t$, $0 < c_1 < c_2 < c^*$, then

$$\lim_{t\to\infty, x=-c_2 t} u(t, x; \phi) = \lim_{t\to+\infty, x=-c_2 t} U(x + c_1 t) = \lim_{t\to+\infty} U((c_1 - c_2)t) = U(-\infty) = 0,$$

which is a contradiction. The proof is complete. □

REMARK 3.2 It is evidently that c^* is the spreading speed as well as the minimal speed for (1.1) by Theorems 3.3 and 3.4.

REMARK 3.3 Similar to Remark 2.2, if we impose another assumption on f and φ: $f(u, w)$ and $\varphi(w)$ are sublinear, then the conclusion (i) in Theorem 3.4 becomes

(i) if $\phi \in [0, u^+]_{UC_h}$ with $\phi(\theta, \cdot) \not\equiv 0$ for $\theta \in (-\infty, 0]$, then

$$\lim_{t\to+\infty, |x|\leqslant ct} u(t, x; \phi) = u^+ \quad \text{for any } c < c^*.$$

4. Estimate of spreading speed c^*

In this section, we shall give an estimate of the spreading speed c^*. In order to achieve this goal, we need some stronger assumptions than (F1)–(F2).

(F6) $f(r, s)$ is differentiable at $(0, 0)$ and $\varphi'(0)$ exists, moreover,

$$f_1'(0, 0) + f_2'(0, 0)\varphi'(0) > 0, \quad f_2'(0, 0)\varphi'(0) > 0.$$

(F7) For $(r, s) \in [0, u^+] \times [0, \varphi(u^+)]$, $f(r, s) \leqslant f_1'(0, 0)r + f_2'(0, 0)s$, $\varphi(r) \leqslant \varphi'(0)r$.

(F8) For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$f(r, s) \geqslant [1 - \varepsilon \operatorname{sgn}(f_1'(0, 0))] f_1'(0, 0)r + (1 - \varepsilon) f_2'(0, 0)s \text{ and } \varphi(r) \geqslant (1 - \varepsilon)\varphi'(0)r$$

hold for $0 \leqslant r \leqslant \delta$ and $0 \leqslant s \leqslant \varphi(\delta)$.

Motivated by Liang & Zhao (2007), here we calculate the spreading speed c_τ^* of (2.1) when (F3)–(F8) and (G1)–(G2) are satisfied. Consider the system

$$\begin{aligned}\frac{\partial u^{(\tau)}}{\partial t} &= D\frac{\partial^2 u^{(\tau)}}{\partial x^2} + f_1'(0, 0)u^{(\tau)}(t, x)\\ &\quad + f_2'(0, 0)\varphi'(0)\int_0^\tau \int_{-\infty}^{+\infty} g(s, y)u^{(\tau)}(t - s, x - y)\mathrm{d}y\,\mathrm{d}s. \qquad (4.1)\end{aligned}$$

Let $u^{(\tau)}(t,x) = \mathrm{e}^{-\mu x} V^{(\tau)}(t)$. Then, we have from (4.1) that

$$\begin{aligned}\frac{\mathrm{d}V^{(\tau)}}{\mathrm{d}t} &= D\mu^2 V^{(\tau)}(t) + f_1'(0,0)V^{(\tau)}(t)\\ &\quad + f_2'(0,0)\varphi'(0)\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)V^{(\tau)}(t-s)\mathrm{e}^{\mu y}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}\tag{4.2}$$

The character equation of (4.2) is

$$\chi - D\mu^2 - f_1'(0,0) - f_2'(0,0)\varphi'(0)\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\mathrm{e}^{-\chi s+\mu y}\,\mathrm{d}y\,\mathrm{d}s = 0. \tag{4.3}$$

It is obvious that $\chi = \chi^{\tau}(\mu)$. Let $M_t^{(\tau)}$ and $B_t^{\tau}(\mu)$ be the linear solution maps defined by (4.1) and (4.2), respectively. For any $\alpha \in [0, u_{\tau}^{+}]_{\bar{\mathscr{C}}}$, then we have

$$B_t^{\tau}(\mu)[\alpha](\theta) := M_t^{(\tau)}[\alpha\,\mathrm{e}^{-\mu x}](\theta,0) = \mathrm{e}^{\chi(\mu)t}\alpha(\theta) \quad \text{for } t > 0.$$

By (F6), we have $\chi^{\tau}(0) > 0$ provided τ large enough. Let $t = 1$. Then $\mathrm{e}^{\chi^{\tau}(0)}$ is the principle eigenvalue of $B_1^{\tau}(0)$ which is greater than 1, and thus the assumption (C7) in Liang & Zhao (2007) is satisfied. Define $\Phi^{\tau}(\mu) = \chi^{\tau}(\mu)/\mu$. Then $\Phi^{\tau}(\mu)$ has the properties $\Phi^{\tau}(\infty) = \infty$ and $\Phi^{\tau}(\mu) \to \infty$ as $\mu \downarrow 0$ because $\chi^{\tau}(\mu) \geqslant D\mu^2 + f_1'(0,0)$. By Theorem 3.10 in Liang & Zhao (2007), we have

$$c_{\tau}^{*} \leqslant \inf_{\mu>0} \Phi^{\tau}(\mu), \tag{4.4}$$

where c_{τ}^{*} is the spreading speed of (2.1) when f and g satisfy (F3)–(F8) and (G1)–(G2).

By (F8), we have $\delta > 0$ such that

$$\begin{aligned} &f\left(u^{(\tau)}(t,x), \int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\varphi(u^{(\tau)}(t-s,x-y))\mathrm{d}s\,\mathrm{d}y\right)\\ &\geqslant [1-\varepsilon\mathrm{sgn}(f_1'(0,0))]f_1'(0,0)u^{(\tau)}(t,x)\\ &\quad +(1-\varepsilon)f_2'(0,0)\varphi'(0)\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)u^{(\tau)}(t-s,x-y)\mathrm{d}s\,\mathrm{d}y \end{aligned}$$

holds for $0 \leqslant u^{(\tau)} \leqslant \delta$. Thus, we consider the system

$$\begin{aligned}\frac{\partial u^{(\tau)}}{\partial t} &= D\frac{\partial^2 u^{(\tau)}}{\partial x^2} + [1-\varepsilon\mathrm{sgn}(f_1'(0,0))]f_1'(0,0)u^{(\tau)}(t,x)\\ &\quad +(1-\varepsilon)f_2'(0,0)\varphi'(0)\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)u^{(\tau)}(t-s,x-y)\mathrm{d}y\,\mathrm{d}s. \end{aligned}\tag{4.5}$$

Let $\{\tilde{M}_t^{(\tau)}\}_{t\geqslant 0}$ be the linear solution map defined by (4.5), $\Phi_{\varepsilon}^{\tau}(\mu) = \chi_{\varepsilon}^{\tau}(\mu)/\mu$, here $\chi_{\varepsilon}^{\tau}(\mu)$ is the root of

$$\chi - D\mu^2 - [1-\varepsilon\mathrm{sgn}(f_1'(0,0))]f_1'(0,0) - (1-\varepsilon)f_2'(0,0)\varphi'(0)\int_0^{\tau}\int_{-\infty}^{+\infty} g(s,y)\mathrm{e}^{-\chi s+\mu y}\mathrm{d}y\,\mathrm{d}s = 0. \tag{4.6}$$

A similar analysis and Theorem 3.10 in Liang & Zhao (2007) lead to

$$c_\tau^* \geqslant \inf_{\mu>0} \Phi_\varepsilon^\tau(\mu),$$

which together with (4.4) implies

$$\inf_{\mu>0} \Phi_\varepsilon^\tau(\mu) \leqslant c_\tau^* \leqslant \inf_{\mu>0} \Phi^\tau(\mu).$$

Let $\varepsilon \to 0$, and we have

$$c_\tau^* = \inf_{\mu>0} \Phi^\tau(\mu) = \inf_{\mu>0} \chi^\tau(\mu)/\mu. \tag{4.7}$$

Summarize the above discussion, we have from Theorem 3.2 the following theorem.

THEOREM 4.1 If (F3)–(F8) and (G1)–(G2) are satisfied, then

$$c^* = \lim_{\tau\to\infty} c_\tau^* = \lim_{\tau\to\infty} \inf_{\mu>0} \chi^\tau(\mu)/\mu.$$

REMARK 4.1 In (3.8), let $L' = f_1'(0,0)$, $\hat{L} = f_2'(0,0)$, $\tilde{L} = \varphi'(0)$. Then in view of Lemma 3.3, there is a pair of (c^*, μ^*) such that

$$p(c^*, \mu^*) = 0, \quad \frac{\partial p}{\partial \mu}(c^*, \mu^*) = 0, \tag{4.8}$$

where c^* is the same constant as in Theorem 4.1.

5. Conclusions and discussions

We are dedicated to the existence of minimal wave speed and spreading speed for (1.1) by using the finite delays approximated method. We confirm that the minimal wave speed is associated with the spreading speed c^* for (1.1). In our discussion, we consider the monotonicity of the sequence $\{c_\tau^*\}_{\tau\geqslant\tau_0}$ in Lemma 3.2 and define a generalized eigenfunction $p(\lambda, c)$, whose coefficients are Lipscitz constants in (3.8), to evaluate the value of $c_{(L)}^*$, the boundedness of the limit of the sequence $\{c_\tau^*\}_{\tau\geqslant\tau_0}$, in Lemma 3.3. Thus, the limit c^* of the sequence $\{c_\tau^*\}_{\tau\geqslant\tau_0}$ is admitted in Theorem 3.2. We believe that the analysis herein is delicate and careful, and the conclusions obtained are general because we replace the differentiability of f and φ with the Lipschitz condition.

We only considered a single equation (1.1). It is natural to ask if the results here can be done for a system with more than one equation as in Wang *et al.* (2006). On the other hand, the monotonicity of $f(r, s)$ on $s \in \mathbb{R}$ and $\varphi(u)$ on $u \in [0, u^+]$ are imposed in (F4). We hope these assumptions can be relaxed in the future works.

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References

AL-OMARI, J. F. M. & GOURLEY, S. A. (2005) A nonlocal reaction-diffusion model for a single species with stage structure and distributed maturation delay. *Eur. J. Appl. Math.*, **16**, 37–51.

ARONSON, D. G. & WEINBERGER, H. F. (1975) Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. *Partial Differential Equations and Related Topics* (J. A. Goldstein ed.). Lecture Notes in Mathematics, vol. 446. Springer, pp. 5–49.

BROWN, K. J. & CARR, J. (1977) Deterministic epidemic waves of critical velocity. *Math. Proc. Cambridge Philos. Soc.*, **81**, 431–433.

DIEKMANN, O. (1978) Thresholds and traveling waves for the geographical spread of infection. *J. Math. Biol.*, **6**, 109–130.

DIEKMANN, O. (1979) Run for your life: a note on the asymptotic speed of propagation of an epidemic. *J. Differ. Equ.*, **33**, 58–73.

FANG, J., WEI, J. J. & ZHAO, X. Q. (2008) Spatial dynamics of a nonlocal and time-delayed reaction-diffusion system. *J. Differ. Equ.*, **245**, 2749–2770.

FARIA, T., HUANG, W. ZH. & WU, J. H. (2006) Traveling waves for delayed reaction-diffusion equations with global response. *Proc. R. Soc. A.*, **4**62, 229–261.

HALE, J. K. & LUNEL, S. M. V. (1993) *Introduction to Functional Differential Equations*. New York: Springer, pp. 90–91.

LIANG, X., LIN, X. T. & MATANO, H. (2010) A variational problem associated with the minimal speed of travelling waves for spatially periodic reaction-diffusion equations. *Trans. Amer. Math. Soc.*, **362**, 5605–5633.

LIANG, X. & ZHAO, X. Q. (2007) Asymptotic speeds of spread and traveling waves for monotone semiflows with applications. *Commun. Pure Appl. Math.*, **60**, 1–40. Errata: **61** (2008) 137–138.

LUI, R. (1989) Biological growth and spread modeled by systems of recurisions. I & II. Mathematical theory. *Math. Biosci.*, **93**, 269–312.

MAREK, M. & SVOBODOVÁ, E. (1975) Nonlinear phenomena in oscillatory systems of homogeneous reactions-experimental observations. *Biophys. Chem.*, **3**, 263–273.

MARTIN, R. H. & SMITH, H. L. (1990) Abstract functional differential equations and reaction-diffusion systems. *Trans. Amer. Math. Soc.*, **321**, 1–44.

MURRAY, J. D. (2002) *Mathematical Biology I & II*. New York: Springer.

RUAN, SH. G & WU, J. H. (1994) Reaction-diffusion equations with infinite delay. *Can. Appl. Math. Q.*, **2**, 485–550.

SCHAAF, K. W. (1987) Asymptotic behavior and traveling wave solutions for parabolic functional differential equations. *Trans. Amer. Math. Soc.*, **302**, 587–615.

SMITH, H. L. (1995) *Monotone Dynamic Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. Mathematical Surveys and Monographs, vol. 41. Providence, RI: American Mathematical Society.

SMITH, H. L. & ZHAO, X. Q. (2000) Global asymptotic stability of traveling waves in delayed reaction-diffusion equations. *SIAM J. Math. Anal.*, **31**, 514–534.

THIEME, H. R. (1979) Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread populations. *J. Reine Angew. Math.*, **306**, 94–121.

THIEME, H. R. & ZHAO, X. Q. (2003) Asymptotic speeds of spread and traveling waves for integral equations and delayed reaction-diffusion models. *J. Differ. Equ.*, **195**, 430–470.

TIAN, Y. L. & WENG P. X. (2009) Asymptotic behaviors and traveling wave front for parabolic functional differential equations with spatial-temporal delays. *Nonlinear Anal.*, **71**, 3347–3348.

VOLPERT, A. I., VITALY VOLPERT, A. & VLADIMIR VOLPERT, A. (1994) *Traveling Wave Solutions of Parabolic Systems*. Translation of Mathematical Monographs, vol. 40. Providence, RI: American Mathematical Society.

WEINBERGER, H. F. (1982) Long-time behavior of a class of biological models. *SIAM J. Math. Anal.*, **13**, 353–396.

WEINBERGER, H. F. (2002) On spreading speeds and traveling waves for growth and migration models in a periodic habitat. *J. Math. Biol.*, **45**, 511–548.

WANG, ZH. CH., LI, W. T. & RUAN, SH. G. (2006) Traveling wave fronts in reaction-diffusion systems with spatio-temporal delays. *J. Differ. Equ.*, **222**, 185–232.

WENG, P. X., HUANG, H. X. & WU, J. H. (2003) Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction. *IMA J. Appl. Math.*, **68**, 409–439.

WENG, P. X. & WU, J. H. (2008) Wavefronts for a nonlocal reaction-diffusion population model with general distributive maturity. *IMA J. Appl. Math.* **73**, 477–495.

WENG, P. X. & XU, ZH. T. (2008) Wavefronts for a global reaction-diffusion population model with infinite distributed delay. *J. Math. Anal. Appl.*, **345**, 522–534.

WU, J. H. & ZOU, X. F. (1997) Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice differential equations. *J. Differ. Equ.*, **135**, 315–357.

WU, J. H. & ZOU, X. F. (2001) Traveling wave fronts of reaction-diffusion systems with delay. *J. Dyn. Differ. Equ.*, **13**, 651–687. doi:10.1023/A:1016690424892. Errata: **13** (2001) 651–687.

ZHAO, X. Q. & WANG, W. D. (2004) Fisher waves in an epidemic model. *Discrete Continuous Dyn. Syst. Ser. B.*, **4**, 1117–1128.

ZHAO, X. Q. & XIAO, D. M. (2006) The asymptotic speed of spread and traveling waves for a vector disease model. *J. Dyn. Differ. Equ.*, **18**, 1001–1019. Errata: **20** (2008) 277–279.