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# A class of AS-regular algebras of dimension five

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ABSTRACT

We classify 5-dimensional Artin–Schelter regular algebras generated by two generators of degree 1 with three generating relations of degree 4 under a generic condition. All the algebras obtained are proved to be strongly noetherian, Auslander regular and Cohen–Macaulay with respect to the Gelfand–Kirillov dimension.



## 1. Introduction

One of the most important questions in noncommutative algebraic projective geometry is to classify the quantum projective space $\mathbb{P}^n$s—noncommutative analogues of projective $n$-spaces. In fact, this is a challenging and hard project, even for $n=4$. An algebraic approach to construct quantum $\mathbb{P}^n$ is to form the noncommutative projective scheme $\operatorname{Proj} A$ [AZ], where $A$ is a noetherian connected graded Artin–Schelter regular algebras of global dimension $n+1$. Then the question turns out to be the classification of Artin–Schelter regular algebras.

The quantum $\mathbb{P}^2$s were classified by Artin and Schelter [AS] and by Artin, Tate and Van den Bergh [ATV] using geometric method. As to the quantum $\mathbb{P}^3$s, many researchers have studied them in terms of Artin–Schelter regular algebras. The most famous 4-dimensional Artin–Schelter regular algebra is

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the Sklyanin algebra of dimension 4, introduced by Sklyanin [Sk1,Sk2]. Homological properties and the representations of the Sklyanin algebra were studied by Smith and Stafford [SS], Levasseur and Smith [LS] respectively. Normal extensions of 3-dimensional Artin–Schelter regular algebras, which are 4-dimensional Artin–Schelter regular algebras, were studied by Le Bruyn, Smith and Van den Bergh [LSV]. The quantum $2 \times 2$-matrix algebra was studied by Vancliff [Va1,Va2]. Some classes of Artin–Schelter regular algebras containing a commutative quadric were studied by Shelton, Van Rompay, Vancliff, Willaert, etc. [SV1,SV2,VV1,VV2,VVW].

Several years ago, Lu, Palmieri, Zhang and the second author [LPWZ1,LPWZ2,LPWZ3] started the project to classify quantum $\mathbb{P}^3$s, or 4-dimensional Artin–Schelter regular algebras by using $A_\infty$-algebraic methods. In general, 4-dimensional Artin–Schelter regular algebras have three resolution types if they are domains, i.e., the so-called type (12221), (13431) and (14641). Under some generic conditions, Lu, Palmieri, Zhang and the second author classified the type (12221) [LPWZ2], i.e., the type of 4-dimensional Artin–Schelter regular algebras generated by two generators of degree 1 with two relations—one of degree 3, the other of degree 4. Type (13431) is the type of 4-dimensional Artin–Schelter regular algebras generated by three generators of degree 1 with four relations—two of degree 2, the other two of degree 3. This type has been studied by Rogalski and Zhang recently [RZ], where they gave all the families of Artin–Schelter regular algebras which are not normal extensions of 3-dimensional Artin–Schelter regular algebras. Type (14641) is the type of 4-dimensional Artin–Schelter regular algebras generated by four generators of degree 1 with six quadratic relations. Zhang and Zhang introduced a new construction, which is called double Ore extension, and they found some new families of this type (see [ZZ1,ZZ2]).

Recently, Floystad and Vatne studied 5-dimensional Artin–Schelter regular algebras [FV]. All the possible resolution types were given for the trivial modules of all 5-dimensional Artin–Schelter regular algebras generated by two elements of degree 1 which are domains.

**Theorem 1.1.** *(See [FV, Lemma 5.4 and Theorem 5.6].) Let $A$ be an AS-regular algebra of global dimension 5 which is a domain of $\operatorname{GK-dim} A \geqslant 4$. If $A$ has two generators of degree 1, then the minimal resolution of the trivial module $k_A$ has the form*

$$0 \to A(-l) \to A(-l+1)^{\oplus 2} \to \bigoplus_{i=1}^{n} A(a_i - l) \to \bigoplus_{i=1}^{n} A(-a_i) \to A(-1)^{\oplus 2} \to A \to k_A \to 0$$

*for some integers $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $l$, such that one of the following holds:*

(1) $n = 3$ *and* $(a_1, a_2, a_3)$ *is* $(3, 5, 5)$, $(4, 4, 4)$ *or* $(3, 4, 7)$ *with* $l = 11, 10, 12$ *respectively*;
(2) $n = 4$ *and* $(a_1, a_2, a_3, a_4)$ *is* $(4, 4, 4, 5)$ *with* $l = 10$;
(3) $n = 5$ *and* $(a_1, a_2, a_3, a_4, a_5)$ *is* $(4, 4, 4, 5, 5)$ *with* $l = 10$.

There are 5-dimensional AS-regular algebras with the resolution types for $n = 3$, where the first two cases can be realized by the enveloping algebras of 5-dimensional graded Lie algebras, while the third one cannot be realized in such a way [FV, Proposition 3.4]. It is open that whether there is a 5-dimensional AS-regular algebra with the resolution type for $n = 4$ or $n = 5$.

In this paper, we focus on the classification of quantum $\mathbb{P}^4$s and consider the Artin–Schelter regular algebras of global dimension 5. The general ideas used here are similar as in [LPWZ2]. Under a generic condition, we classify 5-dimensional Artin–Schelter regular algebras generated by two generators of degree 1 with three generating relations of degree 4.

**Theorem A.** *There are* 9 *types of Artin–Schelter regular algebras of dimension* 5 *which are generated by two elements of degree* 1 *with three generating relations of degree* 4*, as listed in Section* 4 *as algebras* **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H** *and* **I**. *Under the generic condition* (GM2) (*see Section* 4)*, this is a complete list of* 5*-dimensional Artin–Schelter regular algebras which are domains generated by two generators of degree* 1 *with three generating relations of degree* 4.

The generic condition (GM2) mainly means that the structure matrices $\mathcal{R} = (r_{ij})_{2\times 2}$ and $\mathcal{T} = (t_{ks})_{3\times 3}$ of the corresponding Ext-algebra in (3.1) have distinct eigenvalues.

All these algebras enjoy many nice homological properties.

**Theorem B.** *All the algebras* **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H** *and* **I** *are strongly noetherian, Auslander regular and Cohen–Macaulay* (*see Theorems* 5.4, 5.5, 5.8, 5.9).

**Corollary C.** *Let $\mathcal{A}$ be a* 5*-dimensional AS-regular algebra generated by two elements of degree* 1 *with three relations of degree* 4*. Suppose it is a domain and satisfies the generic condition* (GM2)*. If it is not a normal extension of some* 4*-dimensional AS-regular algebra, then it is either an iterated Ore extension of the polynomial ring in one variable or falls into one of the families* **A**, **B** *and* **F**, *up to isomorphism.*

The paper is organized as follows. In Section 2, we recall the definition of Artin–Schelter regular algebras and their properties. The canonical $A_\infty$-structures on the Yoneda Ext-algebras and the general ideas used for the classification of AS-regular algebras by using $A_\infty$-methods are explained also in this section. In Section 3, we analyze the Frobenius structure and $A_\infty$-structure of the Yoneda Ext-algebras $E(A)$ for 5-dimensional AS-regular algebras $A$ generated by two elements of degree 1 with three relations of degree 4. Several systems of equations satisfied by the structural coefficients are obtained following the Stasheff's identities. In Section 4, we introduce a generic condition (GM2) on the algebra structure on $E$, and give all the possible AS-regular algebras of global dimension 5 of the type considered. In Section 5, we prove all the possible algebras listed in Section 4 are strongly noetherian, Auslander regular and Cohen–Macaulay with respect to the Gelfand–Kirillov dimension, thus proving the main results.

## 2. Preliminaries

Throughout the paper, $k$ is an algebraically closed field of characteristic zero and all algebras are connected graded $k$-algebras generated in degree 1. Now we recall the definition of Artin–Schelter regular algebras.

### 2.1. Artin–Schelter regular algebras

**Definition 2.1.** A connected graded algebra $A$ is called **Artin–Schelter regular** (AS-regular, for short) if the following three conditions hold:

(AS1) $A$ has finite global dimension $d$,

(AS2) $A$ is Gorenstein, i.e., there exists an integer $l$ such that

$$\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k(l), & i = d, \\ 0, & i \neq d \end{cases}$$

where $k$ is the trivial left or right $A$-module $A/A_{\geqslant 1}$, and the notation $(l)$ is the degree $l$-shifting on graded modules,

(AS3) $A$ has finite Gelfand–Kirillov dimension (GK dimension).

### 2.2. $A_\infty$-algebras

We recall the definition of $A_\infty$-algebras and the $A_\infty$-structure on the Yoneda Ext-algebras in this subsection.

**Definition 2.2.** An $A_\infty$**-algebra** over $k$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded $k$-linear maps $m_n : A^{\otimes n} \to A$ of degree $2 - n$ $(n \geqslant 1)$, such that the following Stasheff identities SI$(n)$ hold:

$$\sum(-1)^{r+st} m_u\big(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}\big) = 0 \qquad \mathrm{SI}(n)$$

for all $n \geqslant 1$, where the sum runs over all the decompositions $n = r + s + t$ ($r, t \geqslant 0$ and $s \geqslant 1$) and $u = r + 1 + t$.

We assume that every $A_\infty$-algebra in this paper satisfies the **strictly unital condition**: there is an element $1 \in A^0$, which is called the *strict unit* or *identity* of $A$, such that

- 1 is an identity with respect to the multiplication $m_2$, and
- if $n \neq 2$ and $a_i = 1$ for some $i$, then $m_n(a_1, \ldots, a_n) = 0$.

Note that when the formulas are applied to elements, additional signs appear due to the Koszul sign rule as usual in the graded setting.

A differential graded algebra $(A, d)$ can be viewed as an $A_\infty$-algebra by setting $m_1 = d$, $m_2$ be the multiplication and $m_n = 0$ for all $n \geqslant 3$. On the other hand, for any differential graded algebra $A$, there is a canonical $A_\infty$-algebra structure on its cohomology algebra $HA$ which is unique in some sense [Ka,Me].

Let $A$ be a connected graded algebra, and $k_A$ be the trivial $A$-module. The Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$, viewed as the cohomology algebra of some differential graded algebra, is equipped with an $A_\infty$-algebra structure. We use $\mathrm{Ext}^*_A(k_A, k_A)$ to denote both the usual associative Ext-algebra and the Ext-algebra with the canonical $A_\infty$-structure. Occasionally we use $E$ also to denote Ext with its $A_\infty$-algebra structure.

We assume also that the $A_\infty$-algebras in this paper are $\mathbb{Z}^2$-graded. In fact, the Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ of a connected graded algebra $A$ is a typical example of $\mathbb{Z}^2$-graded $A_\infty$-algebras; the first grading, written as a superscript, is the homological one, and the other grading, which is sometimes called the Adams grading, written as subscript, is induced by the grading on the original graded algebra $A$. The degree of the multiplication maps $m_n$ in $\mathbb{Z}^2$-graded $A_\infty$-algebras is $(2 - n, 0)$, i.e., $m_n$ preserves the Adams grading. For the construction of the $A_\infty$-structure of the Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$, see also [LPWZ3, Proposition 1.2].

### *2.3. $A_\infty$-Ext-algebras of AS-regular algebras*

The following theorem is one bridge for the classification of AS-regular algebras by $A_\infty$-methods.

**Theorem 2.3.** *(See [LPWZ1, Theorem 12.9] or [LPWZ4, Corollary D].) Let $A$ be a connected graded algebra and let $E$ be the* Ext*-algebra of $A$. Then $A$ satisfies the conditions* (AS1) *and* (AS2) *in Definition* 2.1 *if and only if $E$ is a Frobenius algebra.*

This was proved by using $A_\infty$-algebra methods. Theorem 2.3 is a generalization of a result of Smith in [Sm], where $A$ is assumed to be noetherian Koszul.

If $A$ is not Koszul, then the associative algebra $\mathrm{Ext}^*_A(k_A, k_A)$ does not contain enough information to recover $A$ (see, say, [LPWZ1, Ex. 13.4]). Generally speaking, the information from the $A_\infty$-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ is sufficient to recover $A$. This is the main point of the following theorem, which serves as another bridge for the classification of AS-regular algebras by $A_\infty$-methods.

**Theorem 2.4.** *(See [LPWZ3, Corollary B].) Let $A$ be a connected graded algebra which is finitely generated in degree* 1, *and let $E$ be the corresponding $A_\infty$-algebra $\mathrm{Ext}^*_A(k_A, k_A)$. Let $R = \bigoplus_{n \geqslant 2} R_n$ be the minimal graded space of relations of $A$ such that $R_n \subset A_1 \otimes A_{n-1} \subset A_1^{\otimes n}$. Let $i : R_n \to A_1^{\otimes n}$ be the inclusion map and $i^*$ be its $k$-linear dual. Then the multiplication $m_n$ of $E$ restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i^* : \big(E^1\big)^{\otimes n} = \big(A_1^*\big)^{\otimes n} \to R_n^* \subset E^2.$$

Keller stated the result for quiver algebras $kQ/I$ where $Q$ is a finite quiver and $I$ is an admissible ideal of $kQ$ [Ke, Proposition 2]. Theorem 2.4, giving an explicit correspondence between the minimal graded space of relations of $A$ and the $A_\infty$-multiplications of the Ext-algebra $\mathrm{Ext}_A^*(k_A, k_A)$, works for graded algebras generated in degree 1.

Let us give an example to illustrate this.

**Example 1.** Let $A$ be a 3-dimensional AS-regular algebra of Type A in Artin–Schelter's classification [AS], i.e.,

$$A = k\langle x, y\rangle \Big/ \begin{pmatrix} x^3 + axy^2 + ay^2x + byxy \\ y^3 + ayx^2 + ax^2y + bxyx \end{pmatrix}$$

with $a, b \in k \setminus \{0\}$. Then minimal projective resolution of the trivial module $k_A$ has the following form

$$0 \to A(-4) \to A(-3)^{\oplus 2} \to A(-1)^{\oplus 2} \to A \to k_A \to 0,$$

and the Yoneda Ext-algebra $E = \mathrm{Ext}_A^*(k_A, k_A) = k \oplus E_{-1}^1 \oplus E_{-3}^2 \oplus E_{-4}^3$ as a $\mathbb{Z}^2$-graded vector space, where the lower index is the Adams grading and the upper index is the homological grading. The dimensions of the subspaces are

$$\dim E_{-1}^1 = \dim E_{-3}^2 = 2, \qquad \dim E_{-4}^3 = 1.$$

By choosing the basis suitably, let $E_{-1}^1 = k\alpha_1 \oplus k\alpha_2$, $E_{-3}^2 = k\beta_1 \oplus k\beta_2$, and $E_{-4}^3 = k\delta$. Then the $A_\infty$-multiplication $m_3$ on $(E^1)^{\otimes 3}$ is

$$\begin{aligned}
&m_3(\alpha_1 \otimes \alpha_1 \otimes \alpha_1) = \beta_1, && m_3(\alpha_1 \otimes \alpha_1 \otimes \alpha_2) = a\beta_2,\\
&m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_1) = b\beta_2, && m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_2) = a\beta_1,\\
&m_3(\alpha_2 \otimes \alpha_1 \otimes \alpha_1) = a\beta_2, && m_3(\alpha_2 \otimes \alpha_1 \otimes \alpha_2) = b\beta_1,\\
&m_3(\alpha_2 \otimes \alpha_2 \otimes \alpha_1) = a\beta_1, && m_3(\alpha_2 \otimes \alpha_2 \otimes \alpha_2) = \beta_2.
\end{aligned}$$

The following is also needed later in the classification.

**Theorem 2.5.** *(See [Ke, Proposition 1].) As an $A_\infty$-algebra, $E = E(A)$ can be generated by $E^0$ and $E^1$, i.e., $E$ itself is the smallest k-subspace of $E$ which is closed under the $A_\infty$-multiplications $m_n$'s containing $E^0$ and $E^1$.*

The process of recovering the algebra from its Ext-algebra is the main idea used in [LPWZ2] to classify a type of 4-dimensional AS-regular algebras. This is also the idea in this paper. We analyze the $A_\infty$-structures of the Ext-algebras, then we recover the original algebras and check the homological properties.

## 3. $A_\infty$-structural analysis on $E(A)$

In this paper we focus on the 5-dimensional AS-regular algebras which are generated by two elements with three relations of degree 4. We classify the algebras of this type under a generic condition. Following [FV] (see Theorem 1.1), the proof of the following proposition is an easy exercise.

**Proposition 3.1.** *Let $A$ be a* 5*-dimensional AS-regular algebra which is generated by two elements of degree* 1 *with three generating relations of degree* 4*. Then the minimal resolution of the trivial module $k_A$ is of the following form*:

$$0 \to A(-10) \to A(-9)^{\oplus 2} \to A(-6)^{\oplus 3} \to A(-4)^{\oplus 3} \to A(-1)^{\oplus 2} \to A \to k_A \to 0,$$

*and the Hilbert series of $A$ is $(1-t)^{-2}(1-t^2)^{-1}(1-t^3)^{-2}$. The Yoneda* Ext-*algebra $E$ of $A$ is isomorphic to*

$$k \oplus E^1_{-1} \oplus E^2_{-4} \oplus E^3_{-6} \oplus E^4_{-9} \oplus E^5_{-10}$$

*as a $\mathbb{Z}^2$-graded vector space, where the lower index is the Adams grading inherited from the grading of $A$ and the upper index is the homological grading of the* Ext-*group. The dimensions of the subspaces are*

$$\dim E^1_{-1} = \dim E^4_{-9} = 2, \qquad \dim E^2_{-4} = \dim E^3_{-6} = 3, \qquad \dim E^5_{-10} = 1.$$

*With the canonical $A_\infty$-algebra structure, $E = (E, m_2, m_3, m_4)$, that is, $m_n = 0$ for all $n \geqslant 5$.*

### 3.1. Frobenius algebra structures on $E$

Now we start to classify all possible Frobenius algebra structures on the bigraded space

$$E = k \oplus E^1_{-1} \oplus E^2_{-4} \oplus E^3_{-6} \oplus E^4_{-9} \oplus E^5_{-10}$$

with $\dim E^1_{-1} = \dim E^4_{-9} = 2$, $\dim E^2_{-4} = \dim E^3_{-6} = 3$, $\dim E^5_{-10} = 1$. All possible non-trivial actions of the higher multiplications $m_n$ are listed as follows.

The possible non-trivial actions of $m_2$ on $E^{\otimes 2}$ are

$$E^1_{-1} \otimes E^4_{-9} \to E^5_{-10}, \qquad E^4_{-9} \otimes E^1_{-1} \to E^5_{-10};$$
$$E^2_{-4} \otimes E^3_{-6} \to E^5_{-10}, \qquad E^3_{-6} \otimes E^2_{-4} \to E^5_{-10}.$$

The multiplication $m_2$ gives a Frobenius structure on $E$ if and only if that there exists a basis $\{\alpha_1, \alpha_2\}$ of $E^1_{-1}$, a basis $\{\beta_1, \beta_2, \beta_3\}$ of $E^2_{-4}$, a basis $\{\eta_1, \eta_2, \eta_3\}$ of $E^3_{-6}$, a basis $\{\gamma_1, \gamma_2\}$ of $E^4_{-9}$ and a basis $\{\delta\}$ of $E^5_{-10}$ such that

$$\begin{aligned} &\alpha_i \gamma_j = \delta_{ij}\delta, \qquad \gamma_i \alpha_j = r_{ij}\delta, \quad r_{ij} \in k; \\ &\beta_k \eta_s = \delta_{ks}\delta, \qquad \eta_k \beta_s = t_{ks}\delta, \quad t_{ks} \in k, \end{aligned} \tag{3.1}$$

with the matrices $\mathcal{R} = (r_{ij})_{2\times 2}$ and $\mathcal{T} = (t_{ks})_{3\times 3}$ non-singular.

### 3.2. Non-trivial actions of $m_3$ and $m_4$ on $E$

Possible non-trivial actions of $m_3$ on $E^{\otimes 3}$ are

$$E^1_{-1} \otimes E^1_{-1} \otimes E^2_{-4} \to E^3_{-6}, \qquad E^1_{-1} \otimes E^2_{-4} \otimes E^1_{-1} \to E^3_{-6}, \qquad E^2_{-4} \otimes E^1_{-1} \otimes E^1_{-1} \to E^3_{-6};$$
$$E^1_{-1} \otimes E^2_{-4} \otimes E^2_{-4} \to E^4_{-9}, \qquad E^2_{-4} \otimes E^1_{-1} \otimes E^2_{-4} \to E^4_{-9}, \qquad E^2_{-4} \otimes E^2_{-4} \otimes E^1_{-1} \to E^4_{-9}.$$

Now, for $1 \leqslant i, j \leqslant 2$ and $1 \leqslant k, s \leqslant 3$, we assume that

$$m_3(\alpha_i, \alpha_j, \beta_k) = a_{13ijk}\eta_1 + a_{23ijk}\eta_2 + a_{33ijk}\eta_3,$$
$$m_3(\alpha_i, \beta_k, \alpha_j) = a_{12ijk}\eta_1 + a_{22ijk}\eta_2 + a_{32ijk}\eta_3,$$

$$m_3(\beta_k,\alpha_i,\alpha_j)=a_{11ijk}\eta_1+a_{21ijk}\eta_2+a_{31ijk}\eta_3,$$
$$m_3(\alpha_i,\beta_k,\beta_s)=b_{11iks}\gamma_1+b_{21iks}\gamma_2,$$
$$m_3(\beta_k,\alpha_i,\beta_s)=b_{12iks}\gamma_1+b_{22iks}\gamma_2,$$
$$m_3(\beta_k,\beta_s,\alpha_i)=b_{13iks}\gamma_1+b_{23iks}\gamma_2,$$

where the coefficients are scalars in $k$.

Possible non-trivial actions of $m_4$ on $E^{\otimes 4}$ are

$$\left(E^1_{-1}\right)^{\otimes 4}\to E^2_{-4},$$
$$\left(E^1_{-1}\right)^{\otimes 3}\otimes E^3_{-6}\to E^4_{-9},\qquad \left(E^1_{-1}\right)^{\otimes 2}\otimes E^3_{-6}\otimes E^1_{-1}\to E^4_{-9},$$
$$E^3_{-6}\otimes\left(E^1_{-1}\right)^{\otimes 3}\to E^4_{-9},\qquad E^1_{-1}\otimes E^3_{-6}\otimes\left(E^1_{-1}\right)^{\otimes 2}\to E^4_{-9}.$$

Then, for $1\leqslant i,j,h,m\leqslant 2$ and $1\leqslant s\leqslant 3$, we assume that

$$m_4(\alpha_i,\alpha_j,\alpha_h,\alpha_m)=x_{1ijhm}\beta_1+x_{2ijhm}\beta_2+x_{3ijhm}\beta_3,$$
$$m_4(\alpha_i,\alpha_j,\alpha_h,\eta_s)=y_{14ijhs}\gamma_1+y_{24ijhs}\gamma_2,$$
$$m_4(\alpha_i,\alpha_j,\eta_s,\alpha_h)=y_{13ijhs}\gamma_1+y_{23ijhs}\gamma_2,$$
$$m_4(\eta_s,\alpha_i,\alpha_j,\alpha_h)=y_{11ijhs}\gamma_1+y_{21ijhs}\gamma_2,$$
$$m_4(\alpha_i,\eta_s,\alpha_j,\alpha_h)=y_{12ijhs}\gamma_1+y_{22ijhs}\gamma_2,$$

where all the coefficients are scalars in $k$.

### 3.3. Stasheff identities for the $A_\infty$-algebra $E$

We assume first that the structure matrices $\mathcal{R}$ and $\mathcal{T}$ given in (3.1) are diagonal for simplicity, and let

$$\mathcal{R}=\begin{pmatrix} g_1 & \\ & g_2\end{pmatrix}\quad\text{and}\quad \mathcal{T}=\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3\end{pmatrix}.$$

It is easy to see that SI($n$) holds trivially for $n=1,2,3$ and $n\geqslant 7$. Now we look at SI($n$) for $n=4,5$ and 6.

SI(4) is equivalent to

$$m_3\left(m_2\otimes \mathrm{id}^{\otimes 2}-\mathrm{id}\otimes m_2\otimes \mathrm{id}+\mathrm{id}^{\otimes 2}\otimes m_2\right)=m_2(m_3\otimes \mathrm{id}+\mathrm{id}\otimes m_3).$$

By applying to elements, it is easy to see that if one of the components is in $E^0=k$ then the formula holds trivially. If no component is in $E^0=k$, then the action of the left-hand side of the above formula is always zero. The possible non-trivial actions of the right-hand side of the above formula are on

$$E^1_{-1}\otimes E^1_{-1}\otimes E^2_{-4}\otimes E^2_{-4},\quad E^1_{-1}\otimes E^2_{-4}\otimes E^1_{-1}\otimes E^2_{-4},\quad E^1_{-1}\otimes E^2_{-4}\otimes E^2_{-4}\otimes E^1_{-1},$$
$$E^2_{-4}\otimes E^1_{-1}\otimes E^1_{-1}\otimes E^2_{-4},\quad E^2_{-4}\otimes E^1_{-1}\otimes E^2_{-4}\otimes E^1_{-1},\quad E^2_{-4}\otimes E^2_{-4}\otimes E^1_{-1}\otimes E^1_{-1}.$$

By applying SI(4) to $(\alpha_i, \alpha_j, \beta_k, \beta_c)$, $(\alpha_i, \beta_k, \alpha_j, \beta_c)$, $(\alpha_i, \beta_k, \beta_c, \alpha_j)$, $(\beta_k, \alpha_i, \alpha_j, \beta_c)$, $(\beta_k, \alpha_i, \beta_c, \alpha_j)$ and $(\beta_k, \beta_c, \alpha_i, \alpha_j)$, it follows that SI(4) holds if and only if

$$\begin{cases} b_{i1jkc} = a_{c3ijk} t_c, & b_{i2jkc} = a_{c2ijk} t_c, \\ b_{i3jkc} = g_j b_{j1ikc}, & a_{k3ijc} = -a_{c1ijk} t_c, \\ a_{k2ijc} = -g_j b_{j2ikc}, & a_{k1ijc} = -g_j b_{j3ikc}. \end{cases} \tag{3.2}$$

It follows from (3.2) that for any $i, j \in \{1, 2\}$ and $k, c \in \{1, 2, 3\}$ by eliminating the $b$'s,

$$\begin{cases} a_{k1ijc} + g_i g_j t_c a_{c3ijk} = 0, \\ a_{k1ijc} t_k + a_{c3ijk} = 0, \\ g_j t_c a_{c2jik} + a_{k2ijc} = 0. \end{cases} \tag{3.3}$$

SI(5) is equivalent to

$$\begin{aligned} & m_3\big(m_3 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_3 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_3\big) \\ & \quad = m_2(\mathrm{id} \otimes m_4 - m_4 \otimes \mathrm{id}) + m_4\big(\mathrm{id}^{\otimes 3} \otimes m_2 - \mathrm{id}^{\otimes 2} \otimes m_2 \otimes \mathrm{id} + \mathrm{id} \otimes m_2 \otimes \mathrm{id}^{\otimes 2} - m_2 \otimes \mathrm{id}^{\otimes 3}\big). \end{aligned}$$

The left-hand side of the above formula is always zero. If one of the components is in $E^0 = k$, the formula holds trivially by applying it to elements. The possible non-trivial actions of the right-hand side of the above formula are $m_2(\mathrm{id} \otimes m_4 - m_4 \otimes \mathrm{id})$ acting on

$$\begin{gathered} \big(E^1_{-1}\big)^{\otimes 4} \otimes E^3_{-6}, \quad \big(E^1_{-1}\big)^{\otimes 3} \otimes E^3_{-6} \otimes E^1_{-1}, \quad \big(E^1_{-1}\big)^{\otimes 2} \otimes E^3_{-6} \otimes \big(E^1_{-1}\big)^{\otimes 2}, \\ E^1_{-1} \otimes E^3_{-6} \otimes \big(E^1_{-1}\big)^{\otimes 3}, \quad E^3_{-6} \otimes \big(E^1_{-1}\big)^{\otimes 4}. \end{gathered}$$

By applying $m_2(\mathrm{id} \otimes m_4 - m_4 \otimes \mathrm{id})$ to $(\alpha_i, \alpha_j, \alpha_h, \alpha_m, \eta_s)$, $(\alpha_i, \alpha_j, \alpha_h, \eta_s, \alpha_m)$, $(\alpha_i, \alpha_j, \eta_s, \alpha_h, \alpha_m)$, $(\alpha_i, \eta_s, \alpha_j, \alpha_h, \alpha_m)$ and $(\eta_s, \alpha_i, \alpha_j, \alpha_h, \alpha_m)$, it follows that SI(5) holds if and only if, for any $i, j, h, m \in \{1, 2\}$ and $s \in \{1, 2, 3\}$,

$$\begin{gathered} x_{sijhm} = y_{i4jhms}, \qquad g_m y_{m4ijhs} = y_{i3jhms}, \qquad g_m y_{m3ijhs} = y_{i2jhms}, \\ g_m y_{m2ijhs} = y_{i1jhms}, \qquad g_m y_{m1ijhs} = t_s x_{sijhm}. \end{gathered} \tag{3.4}$$

It follows that for any $i, j, h, m \in \{1, 2\}$ and $s \in \{1, 2, 3\}$

$$x_{sijhm}(t_s - g_i g_j g_h g_m) = 0. \tag{3.5}$$

By Theorem 2.4, for any fixed $s \in \{1, 2, 3\}$, there exist some $i$, $j$, $h$ and $m$ such that

$$t_s = g_i g_j g_h g_m. \tag{3.6}$$

SI(6) is equivalent to

$$\begin{aligned} & m_4\big(m_3 \otimes \mathrm{id}^{\otimes 3} + \mathrm{id} \otimes m_3 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id}^{\otimes 2} \otimes m_3 \otimes \mathrm{id} + \mathrm{id}^{\otimes 3} \otimes m_3\big) \\ & \quad = m_3\big(m_4 \otimes \mathrm{id}^{\otimes 2} - \mathrm{id} \otimes m_4 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_4\big). \end{aligned}$$

The possible non-trivial actions of the above formula are on

$$\left(E_{-1}^{1}\right)^{\otimes 6},\quad \left(E_{-1}^{1}\right)^{\otimes 5}\otimes E_{-4}^{2},\quad \left(E_{-1}^{1}\right)^{\otimes 4}\otimes E_{-4}^{2}\otimes E_{-1}^{1},\quad \left(E_{-1}^{1}\right)^{\otimes 3}\otimes E_{-4}^{2}\otimes \left(E_{-1}^{1}\right)^{\otimes 2},$$
$$\left(E_{-1}^{1}\right)^{\otimes 2}\otimes E_{-4}^{2}\otimes \left(E_{-1}^{1}\right)^{\otimes 3},\quad E_{-1}^{1}\otimes E_{-4}^{2}\otimes \left(E_{-1}^{1}\right)^{\otimes 4},\quad E_{-4}^{2}\otimes \left(E_{-1}^{1}\right)^{\otimes 5}.$$

By applying SI(6) to $(\alpha_i,\alpha_j,\alpha_h,\alpha_m,\alpha_n,\alpha_l)$, $(\alpha_i,\alpha_j,\alpha_h,\alpha_m,\alpha_n,\beta_k)$, $(\alpha_i,\alpha_j,\alpha_h,\alpha_m,\beta_k,\alpha_n)$, $(\alpha_i,\alpha_j,\alpha_h,\beta_k,\alpha_m,\alpha_n)$, $(\alpha_i,\alpha_j,\beta_k,\alpha_h,\alpha_m,\alpha_n)$, $(\alpha_i,\beta_k,\alpha_j,\alpha_h,\alpha_m,\alpha_n)$, $(\beta_k,\alpha_i,\alpha_j,\alpha_h,\alpha_m,\alpha_n)$, it follows that SI(6) holds if and only if

$$\begin{aligned}
&\sum_{s=1}^{3} x_{sijhm}a_{c1nls}-\sum_{s=1}^{3} x_{sjhmn}a_{c2ils}+\sum_{s=1}^{3} x_{shmnl}a_{c3ijs}=0,\\
&\sum_{s=1}^{3} a_{s3mnk}y_{l4ijhs}+\sum_{s=1}^{3} x_{sijhm}b_{l2nsk}-\sum_{s=1}^{3} x_{sjhmn}b_{l1isk}=0,\\
&\sum_{s=1}^{3} a_{s3hmk}y_{l3ijns}-\sum_{s=1}^{3} a_{s2mnk}y_{l4ijhs}-\sum_{s=1}^{3} x_{sijhm}b_{l3nsk}=0,\\
&\sum_{s=1}^{3} a_{s3jhk}y_{l2imns}-\sum_{s=1}^{3} a_{s2hmk}y_{l3ijns}+\sum_{s=1}^{3} a_{s1mnk}y_{l4ijhs}=0,\\
&\sum_{s=1}^{3} a_{s3ijk}y_{l1hmns}-\sum_{s=1}^{3} a_{s2jhk}y_{l2imns}+\sum_{s=1}^{3} a_{s1hmk}y_{l3ijns}=0,\\
&\sum_{s=1}^{3} a_{s2ijk}y_{l1hmns}-\sum_{s=1}^{3} a_{s1jhk}y_{l2imns}-\sum_{s=1}^{3} x_{sjhmn}b_{l1iks}=0,\\
&\sum_{s=1}^{3} a_{s1ijk}y_{l1hmns}+\sum_{s=1}^{3} x_{sijhm}b_{l3nks}-\sum_{s=1}^{3} x_{sjhmn}b_{l2iks}=0,
\end{aligned}\tag{3.7}$$

where $i,j,h,m,n,l\in\{1,2\}$ and $k,c\in\{1,2,3\}$.

Using (3.2), (3.3) and (3.4), plugging $b_{lcnsk}$, $a_{c1nls}$ and $y_{lcijns}$ in (3.7), we obtain a system of equations with respect to $a_{c2nls}$, $a_{c3nls}$ and $x_{sjhmn}$ as in the following:

$$\begin{aligned}
&\sum_{s=1}^{3} a_{s3nlc}g_n g_l t_s x_{sijhm}+\sum_{s=1}^{3} a_{c2ils}x_{sjhmn}-\sum_{s=1}^{3} a_{c3ijs}x_{shmnl}=0,\\
&\sum_{s=1}^{3} a_{s3mnk}x_{slijh}+\sum_{s=1}^{3} a_{k2lns}t_k x_{sijhm}-\sum_{s=1}^{3} a_{k3lis}t_k x_{sjhmn}=0,\\
&\sum_{s=1}^{3} a_{s3hmk}g_n x_{snlij}-\sum_{s=1}^{3} a_{s2mnk}x_{slijh}-\sum_{s=1}^{3} a_{k3nls}g_n t_k x_{sijhm}=0,\\
&\sum_{s=1}^{3} a_{s3jhk}g_m x_{smnli}-\sum_{s=1}^{3} a_{s2hmk}x_{snlij}-\sum_{s=1}^{3} a_{k3mns}g_m t_k x_{slijh}=0,
\end{aligned}$$

$$\sum_{s=1}^{3} a_{s3ijk} g_h x_{shmnl} - \sum_{s=1}^{3} a_{s2jhk} x_{smnli} - \sum_{s=1}^{3} a_{k3hms} g_h t_k x_{snlij} = 0,$$

$$\sum_{s=1}^{3} a_{s2ijk} g_m g_h g_n x_{shmnl} + \sum_{s=1}^{3} a_{k3jhs} g_h g_j g_m g_n t_k x_{smnli} - \sum_{s=1}^{3} a_{s3lik} t_s x_{sjhmn} = 0,$$

$$\sum_{s=1}^{3} a_{k3ijs} g_h g_i g_j g_m g_n t_k x_{shmnl} - \sum_{s=1}^{3} a_{s3nlk} g_n t_s x_{sijhm} + \sum_{s=1}^{3} a_{s2lik} t_s x_{sjhmn} = 0 \tag{3.8}$$

where $i, j, h, m, n, l \in \{1, 2\}$ and $k, c \in \{1, 2, 3\}$.

In fact, the seven families of equations in (3.8) is just equivalent to one family by (3.3) and (3.5), say the first one:

$$\sum_{s=1}^{3} a_{s3nlc} g_n g_l t_s x_{sijhm} + \sum_{s=1}^{3} a_{c2ils} x_{sjhmn} - \sum_{s=1}^{3} a_{c3ijs} x_{shmnl} = 0 \tag{3.9}$$

where $i, j, h, m, n, l \in \{1, 2\}$ and $c \in \{1, 2, 3\}$.

## 4. Classifications

### 4.1. A generic condition on the algebra structure of E

Let

$$\mathcal{R} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

be as given in (3.1) and let $g_1$, $g_2$ and $t_1$, $t_2$, $t_3$ be the eigenvalues of $\mathcal{R}$ and $\mathcal{T}$, respectively.

**Lemma 4.1.** *Let $E$ be the Yoneda* Ext*-algebra of $A$ as considered, $\mathcal{R}$ and $\mathcal{T}$ be the structure matrices as given in* (3.1)*. If $\mathcal{R}$ is diagonal, then so is $\mathcal{T}$.*

**Proof.** Let $\{f_1, f_2, f_3\}$ be a minimal generating relation of $A$. We may assume that, for any $1 \leqslant l \leqslant 3$, there exists a monomial in $x$ and $y$ appearing only in $f_l$ (that is, its coefficient is non-zero). Let

$$\mathcal{R} = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}.$$

The first four identities in (3.4) still hold. By applying SI(5) to $E^3_{-6} \otimes (E^1_{-1})^{\otimes 4}$ (see (3.4)), we get $\sum_{c=1}^{3} t_{sc} x_{cijhm} = g_m y_{m1ijhs}$. It follows that $\sum_{c=1}^{3} t_{sc} x_{cijhm} = g_m g_h g_i g_j x_{sijhm}$, that is,

$$(t_{ss} - g_m g_h g_i g_j) x_{sijhm} = \sum_{c \neq s} t_{sc} x_{cijhm}. \tag{4.1}$$

Now for any $1 \leqslant l \leqslant 3$, there exist some $i$, $j$, $h$, $m$ such that $x_{cijhm} = 0$ if and only if $c \neq l$, by Theorem 2.4 and the discussion in the first paragraph.

Taking some $i$, $j$, $h$, $m$ so that $x_{1ijhm} \neq 0$ and $x_{2ijhm} = x_{3ijhm} = 0$, it follows from (4.1) for $s = 2$ (respectively, $s = 3$) that $t_{21} x_{1ijhm} = 0$ (respectively, $t_{31} x_{1ijhm} = 0$). Hence $t_{21} = 0$ (respectively, $t_{31} = 0$). Similarly, we have $t_{12} = t_{32} = t_{13} = t_{23} = 0$. So $\mathcal{T}$ is diagonal. □

We introduce a generic condition (GM2) for $m_2$, which is suggested by (3.6).

$$\left(g_1 g_2^{-1}\right)^i \neq 1 \quad \text{for } 1 \leqslant i \leqslant 4 \quad \text{and} \quad t_s \neq t_j \quad \text{for } 1 \leqslant s \neq j \leqslant 3. \tag{GM2}$$

From now on, we assume that the algebra structure on $E$ satisfies the condition (GM2). Then, without loss of generality, we may assume

$$\mathcal{R} = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}.$$

If $E$ is the Yoneda Ext-algebra of some domain $A$, then (GM2) implies that $t_s \neq g_i^4$ for any $i$ and $s$. Again, by (GM2), without loss of generality, we may assume that

$$t_1 = g_1^3 g_2, \qquad t_2 = g_1^2 g_2^2, \qquad t_3 = g_1 g_2^3. \tag{4.2}$$

By (3.5), (4.2) and (GM2), all other $x_{sijhm}$'s are zero except

$$\begin{gathered} x_{11112}, \quad x_{11121}, \quad x_{11211}, \quad x_{12111}; \\ x_{21122}, \quad x_{21212}, \quad x_{22121}, \quad x_{22211}, \quad x_{21221}, \quad x_{22112}; \\ x_{31222}, \quad x_{32122}, \quad x_{32212}, \quad x_{32221}. \end{gathered}$$

For convenience, let

$$\begin{gathered} x_{11112} = a, \qquad x_{11121} = p, \qquad x_{11211} = q, \qquad x_{12111} = r; \\ x_{21122} = l_1, \qquad x_{21212} = l_2, \qquad x_{22121} = l_3, \qquad x_{22211} = l_4, \qquad x_{21221} = l_5, \qquad x_{22112} = l_6; \\ x_{31222} = b, \qquad x_{32122} = d, \qquad x_{32212} = u, \qquad x_{32221} = v \end{gathered} \tag{4.3}$$

with $a, b, p, q, r, d, u, v, l_1, l_2, l_3, l_4, l_5, l_6 \in k$.

So, by Theorem 2.4, the possible AS-regular algebras are of the form $A = k\langle x, y\rangle/(f_1, f_2, f_3)$, with the generating relations $f_1$, $f_2$ and $f_3$ as in the following:

$$\begin{aligned} f_1 &= ax^3y + px^2yx + qxyx^2 + ryx^3, \\ f_2 &= l_1x^2y^2 + l_2xyxy + l_3yxyx + l_4y^2x^2 + l_5xy^2x + l_6yx^2y, \\ f_3 &= bxy^3 + dyxy^2 + uy^2xy + vy^3x. \end{aligned} \tag{4.4}$$

If $A$ is a domain, then $ab \neq 0$, $vr \neq 0$ and none of $(l_1, l_2, l_5)$, $(l_1, l_2, l_6)$, $(l_3, l_4, l_5)$ and $(l_3, l_4, l_6)$ equals $(0, 0, 0)$. We may assume that $a = b = 1$.

### *4.2. Classification under the generic condition (GM2)*

**Proposition 4.2.** *Suppose that $E$ is the Yoneda* Ext*-algebra of some AS-regular algebra considered, satisfying the generic condition* (GM2)*. Then, for some $2 \leqslant n \leqslant 8$,*

$$g_1^n g_2^{10-n} = 1.$$

**Proof.** By Theorem 2.5, the Yoneda Ext-algebra $E$ should be $A_\infty$-generated by $E^0$ and $E^1$. It follows that $m_3$ is non-trivial. So, not all the parameters $a_{scijk}$ and $b_{lciks}$ are zero. By (3.2), not all the parameters $a_{s2ijk}$ and $a_{s3ijk}$ are zero for $i, j \in \{1,2\}$ and $s, k \in \{1,2,3\}$.

If all the $a_{s3ijk}$'s are zero for $i, j \in \{1,2\}$ and $s, k \in \{1,2,3\}$, then there exists $a_{c2mnh} \neq 0$ for some $m, n \in \{1,2\}$ and $c, h \in \{1,2,3\}$. It follows from (3.3) that

$$0 = a_{c2mnh} + g_n t_h a_{h2nmc} = a_{c2mnh} + g_n t_h(-g_m t_c a_{c2mnh}) = (1 - g_n g_m t_h t_c) a_{c2mnh}.$$

So $1 - g_n g_m t_h t_c = 0$, which implies that $g_n g_m g_1^{8-h-c} g_2^{h+c} = 1$ by (4.2).

If there exists $a_{s3ijk} \neq 0$ for some $i, j \in \{1,2\}$ and $k, s \in \{1,2,3\}$, then the first two equations in (3.3) have non-zero solutions. So

$$1 - g_i g_j t_s t_k = 0.$$

It follows that $g_i g_j g_1^{8-k-s} g_2^{k+s} = 1$ by (4.2).

Both of the two cases imply that there exists an integer $n$ with $2 \leqslant n \leqslant 8$ such that $g_1^n g_2^{10-n} = 1$. $\square$

By Proposition 4.2, there are only four cases need to be considered, i.e.,

$$\text{(i)}\quad g_1^2 g_2^8 = 1, \qquad \text{(ii)}\quad g_1^3 g_2^7 = 1, \qquad \text{(iii)}\quad g_1^4 g_2^6 = 1, \qquad \text{(iv)}\quad g_1^5 g_2^5 = 1.$$

**Proposition 4.3.** *Except the case* (iv)*, any other case gives no AS-regular algebras.*

**Proof.** Case (i): By (3.3), $(1 - g_n g_m t_h t_c) a_{c2mnh} = (1 - g_n g_m g_1^{8-h-c} g_2^{h+c}) a_{c2mnh} = 0$. It follows from (GM2) that $g_n g_m g_1^{8-h-c} g_2^{h+c} = 1$ if and only if $h = c = 3$ and $n = m = 2$. So, except $a_{32223}$, all other $a_{c2mnh}$'s are zero.

By the first two equations in (3.3), $(1 - g_i g_j t_c t_k) a_{c3ijk} = 0$. So, except $a_{33223}$, all other $a_{c3ijk}$'s are zero. Since $a_{c3ijk} = -t_k a_{k1ijc}$, all other $a_{k1ijc}$'s are zero except $a_{31223}$.

In summary, except $a_{31223}, a_{32223}, a_{33223}$, all other $a_{csijk}$'s are zero.

It follows that $\eta_1, \eta_2 \in E^3_{-6}$ are not contained in $\operatorname{Im} m_3$. So, $E$ can not be $A_\infty$-generated by $E^0$ and $E^1$, and $E$ is not an Ext-algebra of some AS-regular algebra.

Case (ii): In this case, $g_n g_m g_1^{8-h-c} g_2^{h+c} = 1$ if and only if that $h = c = n + m = 3$ or $h + c = 5$, $n = m = 2$ by (GM2). It follows that except

$$a_{32213}, \quad a_{32123}, \quad a_{32222}, \quad a_{22223},$$

all other $a_{c2mnh}$'s are zero. In particular, $a_{12mnh}$'s are zero.

Similarly, by (GM2) and $(1 - g_i g_j t_c t_k) a_{c3ijk} = 0$, except

$$a_{33213}, \quad a_{33123}, \quad a_{33222}, \quad a_{23223},$$

all other $a_{c3ijk}$'s are zero. In particular, all $a_{13ijk}$'s are zero. Since $a_{c3ijk} = -t_k a_{k1ijc}$, all $a_{11ijc}$'s are zero.

So, $\eta_1 \in E^3_{-6}$ is not contained in $\operatorname{Im} m_3$ and $E$ is not $A_\infty$-generated by $E^0$ and $E^1$. In this case, $E$ is not an Ext-algebra of some AS-regular algebra either.

Case (iii): By (GM2) and $(1 - g_n g_m g_1^{8-h-c} g_2^{h+c}) a_{c2mnh} = 0$, except

$$a_{32113}, \quad a_{22123}, \quad a_{22213}, \quad a_{32122}, \quad a_{32212}, \quad a_{12223}, \quad a_{32221}, \quad a_{22222},$$

all other $a_{c2mnh}$'s are zero. In particular, except $a_{12223}$ all $a_{12mnh}$'s are zero.

Similarly, by (GM2) and $(1 - g_i g_j t_c t_k)a_{c3ijk} = 0$, except

$$a_{33113}, \quad a_{23123}, \quad a_{23213}, \quad a_{33122}, \quad a_{33212}, \quad a_{13223}, \quad a_{33221}, \quad a_{23222},$$

all other $a_{c3ijk}$'s are zero. In particular, except $a_{13223}$ all $a_{13ijk}$'s are zero.

Since $a_{c3ijk} = -t_k a_{k1ijc}$, except $a_{11223}$ all other $a_{11ijc}$'s are zero.

Let $l = k = 1$ and $i = j = h = m = n = 2$ in the second, third and fourth equations of (3.8), we get the following equations:

$$\begin{cases} a_{33221}x_{31222} = 0, \\ a_{33221}x_{32122}g_2 = a_{32221}x_{31222}, \\ a_{33221}x_{32212}g_2 = a_{32221}x_{32122} + a_{13223}x_{31222}g_2t_1. \end{cases}$$

It follows from $x_{31222} = 1$ that

$$a_{33221} = a_{32221} = a_{13223} = 0.$$

Since $a_{33221} = -t_1 a_{11223}$ and $a_{12223} = -g_2 t_3 a_{32221}$, $a_{11223} = a_{12223} = a_{13223} = 0$. Hence all $a_{1sijk}$'s are zero.

So, in this case, $\eta_1 \in E^3_{-6}$ is also not contained in $\operatorname{Im} m_3$ and $E$ is not $A_\infty$-generated by $E^0$ and $E^1$. Hence $E$ is not an Ext-algebra of some AS-regular algebra. □

The only interesting case left is the case (iv) $g_1^5 g_2^5 = 1$, which will be discussed in the next subsection.

### 4.3. Case $g_1^5 g_2^5 = 1$

Using the third equation in (3.3) for $a_{k2ijc}$, we have $(1 - g_i g_j t_c t_k)a_{k2ijc} = 0$ with $i, j \in \{1, 2\}$ and $k, c \in \{1, 2, 3\}$. Then we get the following equations:

$$a_{22113} = -g_1^2 g_2^3 a_{32112}, \qquad a_{32121} = -g_1^3 g_2^2 a_{12213}, \qquad a_{22122} = -g_1^2 g_2^3 a_{22212},$$
$$a_{12123} = -g_1 g_2^4 a_{32211}, \qquad a_{12222} = -g_1^2 g_2^3 a_{22221},$$

and all other $a_{k2ijc}$'s are zero.

Solving the first and second equations in (3.3) for $a_{k3ijc}$ with $i, j \in \{1, 2\}$ and $k, c \in \{1, 2, 3\}$, we get all $a_{k3ijc}$'s are zero except

$$a_{23113}, \quad a_{33112}, \quad a_{33121}, \quad a_{13213}, \quad a_{23122}, \quad a_{23212}, \quad a_{13123}, \quad a_{33211}, \quad a_{13222}, \quad a_{23221}.$$

Plugging the $x_{sijhm}$'s with the parameters as listed in (4.3) in the family of Eqs. (3.9), we get the following 50 equations:

$$\begin{aligned}
g_1^3 g_2^2 a_{12213} + l_1 a_{33112} &= 0, & p g_1^3 g_2^2 a_{12213} - g_1^4 g_2^2 a_{13123} + l_2 a_{33112} &= 0, \\
-g_1^4 g_2^2 a_{13213} - l_1 a_{32112} + l_5 a_{33112} &= 0, & -g_1^3 g_2^3 a_{13222} + g_1^2 g_2^3 l_1 a_{22212} + a_{23113} &= 0, \\
q g_1^3 g_2^2 a_{12213} - p g_1^4 g_2^2 a_{13123} + l_6 a_{33112} &= 0, & -p g_1^4 g_2^2 a_{13213} - l_2 a_{32112} + l_3 a_{33112} &= 0, \\
-p g_1^3 g_2^3 a_{13222} + g_1^2 g_2^3 l_2 a_{22212} + d a_{23113} &= 0, & -g_1^4 g_2^2 l_1 a_{23113} - l_5 a_{32112} + l_4 a_{33112} &= 0, \\
g_1^2 g_2^3 l_5 a_{22212} + u a_{23113} - g_1^3 g_2^3 l_1 a_{23122} &= 0, & v a_{23113} - g_1^3 g_2^3 l_1 a_{23212} + g_1^2 g_2^3 a_{32112} &= 0, \\
-g_1^2 g_2^4 l_1 a_{23221} + g_1 g_2^4 a_{32211} &= 0, & r g_1^3 g_2^2 a_{12213} - q g_1^4 g_2^2 a_{13123} + a_{33121} &= 0,
\end{aligned}$$

$$
\begin{aligned}
-qg_1^4g_2^2a_{13213}-l_6a_{32112}+pa_{33121}&=0, & -qg_1^3g_2^3a_{13222}+g_1^2g_2^3l_6a_{22212}+l_1a_{23122}&=0,\\
-g_1^4g_2^2l_2a_{23113}-l_3a_{32112}+qa_{33121}&=0, & g_1^2g_2^3l_3a_{22212}+l_2a_{23122}-g_1^3g_2^3l_2a_{23122}&=0,\\
l_5a_{23122}-g_1^3g_2^3l_2a_{23212}+dg_1^2g_2^3a_{32112}&=0, & a_{13123}-g_1^2g_2^4l_2a_{23221}+dg_1g_2^4a_{32211}&=0,\\
-g_1^4g_2^2l_5a_{23113}-l_4a_{32112}+ra_{33121}&=0, & g_1^2g_2^3l_4a_{22212}-g_1^3g_2^3l_5a_{23122}+l_6a_{23122}&=0,\\
l_3a_{23122}-g_1^3g_2^3l_5a_{23212}+ug_1^2g_2^3a_{32112}&=0, & da_{13123}-g_1^2g_2^4l_5a_{23221}+ug_1g_2^4a_{32211}&=0,\\
l_4a_{23122}+vg_1^2g_2^3a_{32112}-g_1^3g_2^3a_{33112}&=0, & ua_{13123}+vg_1g_2^4a_{32211}-g_1^2g_2^4a_{33121}&=0,\\
va_{13123}-g_1^2g_2^4a_{33211}&=0, & -rg_1^4g_2^2a_{13123}+a_{33211}&=0,\\
-rg_1^4g_2^2a_{13213}-a_{32211}+pa_{33211}&=0, & -rg_1^3g_2^3a_{13222}-a_{22221}+l_1a_{23212}&=0,\\
-g_1^4g_2^2l_6a_{23113}-pa_{32211}+qa_{33211}&=0, & -pa_{22221}-g_1^3g_2^3l_6a_{23122}+l_2a_{23212}&=0,\\
-l_1a_{22212}+l_5a_{23212}-g_1^3g_2^3l_6a_{23212}&=0, & a_{13213}+g_1^2g_2^3l_1a_{22221}-g_1^2g_2^4l_6a_{23221}&=0,\\
-g_1^4g_2^2l_3a_{23113}-qa_{32211}+ra_{33211}&=0, & -qa_{22221}-g_1^3g_2^3l_3a_{23122}+l_6a_{23212}&=0,\\
-l_2a_{22212}+l_3a_{23212}-g_1^3g_2^3l_3a_{23212}&=0, & da_{13213}+g_1^2g_2^3l_2a_{22221}-g_1^2g_2^4l_3a_{23221}&=0,\\
-l_5a_{22212}+l_4a_{23212}-dg_1^3g_2^3a_{33112}&=0, & ua_{13213}+g_1^2g_2^3l_5a_{22221}-dg_1^2g_2^4a_{33121}&=0,\\
-a_{12213}+va_{13213}-dg_1^2g_2^4a_{33211}&=0, & -g_1^4g_2^2l_4a_{23113}-ra_{32211}&=0,\\
-ra_{22221}-g_1^3g_2^3l_4a_{23122}+a_{23221}&=0, & -l_6a_{22212}-g_1^3g_2^3l_4a_{23212}+pa_{23221}&=0,\\
l_1a_{13222}+g_1^2g_2^3l_6a_{22221}-g_1^2g_2^4l_4a_{23221}&=0, & -l_3a_{22212}+qa_{23221}-ug_1^3g_2^3a_{33112}&=0,\\
l_2a_{13222}+g_1^2g_2^3l_3a_{22221}-ug_1^2g_2^4a_{33121}&=0, & -da_{12213}+l_5a_{13222}-ug_1^2g_2^4a_{33211}&=0,\\
-l_4a_{22212}+ra_{23221}-vg_1^3g_2^3a_{33112}&=0, & l_6a_{13222}+g_1^2g_2^3l_4a_{22221}-vg_1^2g_2^4a_{33121}&=0,\\
-ua_{12213}+l_3a_{13222}-vg_1^2g_2^4a_{33211}&=0, & -va_{12213}+l_4a_{13222}&=0.
\end{aligned}
\tag{4.5}
$$

To find all the possible generating relations, it suffices to find all the solutions of the system of equations (4.5) for $p, q, r, d, u, v, l_1, l_2, l_3, l_4, l_5, l_6$ as defined in (4.3). It follows from the middle two equations in (4.5) that if $a_{13123}\neq 0$ then $v=g_1g_2r$.

Now we start to solve (4.5) in the following four subcases:

- Subcase $a_{13123}\neq 0$, $l_1=0$.
- Subcase $a_{13123}=0$, $l_1=0$.
- Subcase $a_{13123}=0$, $l_1\neq 0$.
- Subcase $a_{13123}\neq 0$, $l_1\neq 0$.

To save the tedious work, we will just list the relations $f_1$, $f_2$ and $f_3$ in the form as in (4.4).

### 4.4. Subcase $a_{13123}\neq 0, l_1=0$

In this case, the system of equations (4.5) gives only one solution:

$$
\begin{aligned}
&p=0, \quad q=0, \quad r\neq 0, \quad d=0, \quad u=0, \quad v=r;\\
&l_1=0, \quad l_2\neq 0, \quad l_3\neq 0, \quad l_4=0, \quad l_5=0, \quad l_6=0
\end{aligned}
$$

with $v^2 + (l_3/l_2)^3 = 0$, which gives the relations

$$f_1 = x^3y - c^3yx^3,$$
$$f_2 = xyxy - c^2yxyx,$$
$$f_3 = xy^3 - c^3y^3x, \quad c \in k \setminus \{0\}.$$

There are four overlap ambiguities $xyxy^3$, $x^3yxy$, $x^3y^3$ and $xyxyxy$ if one uses the diamond lemma [Be]. The first three are resolvable. Resolving $xyxyxy$ gives a relation $yxyx^2y = xy^2xyx$. It follows that

$$\left(yxyx^2y\right)y = \left(xy^2xyx\right)y = xy^2(xyxy) = c^2xy^3xyx = c^5y^3x^2yx.$$

Then $y(xyx^2y^2 - c^5y^2x^2yx) = 0$ while $xyx^2y^2 - c^5y^2x^2yx \neq 0$. So the given algebra is not a domain.

*4.5. Subcase* $a_{13123} = 0, l_1 = 0$

In this case, except $a_{23122}$ and $a_{23212}$, all other $a_{c3ijk}$'s and all $a_{c2ijk}$'s are zero by solving (4.5). In particular, all $a_{1sijk}$'s and $a_{3sijk}$'s are zero. So, in this case, neither $\eta_1$ nor $\eta_3$ is contained in $\operatorname{Im} m_3$ and $E$ can not be $A_\infty$-generated by $E^0$ and $E^1$. So this case gives no AS-regular algebras.

In fact, if neither $a_{23122}$ nor $a_{23212}$ is zero, then we have $l_4 = 0$, $l_5 = l_6$ and $l_2l_3 = l_5^2$. In this case, $f_2 = l_2xyxy + l_3yxyx + l_5xy^2x + l_5yx^2y = l_2^{-1}(l_2xy + l_5yx)^2$ and this case gives no AS-regular algebras which are domains.

*4.6. Subcase* $a_{13123} = 0, l_1 \neq 0$

Then $a_{33211} = 0$ and we may assume $l_1 = 1$.

If $a_{32211} = 0$, then all the $a_{csijk}$'s are zero by (4.5) and no desired algebra arises in this sub-subcase.

If $a_{32211} \neq 0$ and $l_2 = 0$, there is one solution

$$p = 0, \quad q = 0, \quad r \neq 0, \quad d = 0, \quad u = 0, \quad v = r;$$
$$l_1 = 1, \quad l_2 = 0, \quad l_3 = 0, \quad l_4 \neq 0, \quad l_5 = 0, \quad l_6 = 0$$

with $r^4 + l_4^3 = 0$, which gives the relations

$$f_1 = xy^3 + ry^3x,$$
$$f_2 = x^2y^2 + ly^2x^2,$$
$$f_3 = x^3y + ryx^3$$

where $r, l \in k \setminus \{0\}$ such that $r^4 + l^3 = 0$. The given algebra is not a domain because

$$y^2\left(r^2yx^2 + lx^2y\right) = x^2y^3 + \left(-x^2y^3\right) = 0.$$

If $a_{32211} \neq 0$ and $l_2 \neq 0$, there is one solution

$$p \neq 0, \quad q = p^2, \quad r = p^3, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 = p, \quad l_3 = p^3, \quad l_4 = p^4, \quad l_5 = p^2, \quad l_6 = p^2$$

which gives the relations:

$$f_1 = x^3y + px^2yx + p^2xyx^2 + p^3yx^3,$$
$$f_2 = x^2y^2 + pxyxy + p^3yxyx + p^4y^2x^2 + p^2xy^2x + p^2yx^2y,$$
$$f_3 = xy^3 + pyxy^2 + p^2y^2xy + p^3y^3x, \quad p \in k \setminus \{0\}.$$

By the diamond lemma [Be], a monomial is irreducible if and only if it does not contain $x^3y$, $x^2y^2$ or $xy^3$ as a sub-word. Such monomials are of the form

$$y^i\big(xy^2\big)^{j_1}(xy)^{k_1}\big(x^2y\big)^{l_1}\cdots\big(xy^2\big)^{j_n}(xy)^{k_n}\big(x^2y\big)^{l_n}x^m,$$

where all the power indices are non-negative integers. It follows that the subalgebra generated by $xy^2$, $xy$ and $x^2y$ is a free algebra in three variables. So this solution gives an algebra with infinite GK dimension.

In fact, the Hilbert series of the given algebra is $1 + 2t + 4t^2 + 8t^3 + 13t^4 + 22t^5 + 36t^6 + \cdots$, which is different from the standard Hilbert series $1 + 2t + 4t^2 + 8t^3 + 13t^4 + 20t^5 + 31t^6 + \cdots$ of the 5-dimensional AS-regular algebras considered. So, we can also get that the given algebra is not AS-regular.

### 4.7. *Subcase* $a_{13123} \neq 0$, $l_1 \neq 0$

Without loss of generality we assume that $l_1 = 1$. As we noted before that if $a_{13123} \neq 0$ then $v = g_1g_2r$. By using the first two equations and the last one in (4.5), we know $l_4 \neq 0$. It follows also from the first two equations in (4.5) that $p \neq l_2$. If further $d = p$, then $a_{22212} = 0$ by using the fourth and seventh equations in (4.5). The discussion in this subcase is divided into the following five sub-subcases:

- Sub-subcase $a_{23212} = 0$.
- Sub-subcase $a_{23212} \neq 0$, $d = p$, $q = 0$.
- Sub-subcase $a_{23212} \neq 0$, $d = p$, $q \neq 0$, $l_2 = 0$.
- Sub-subcase $a_{23212} \neq 0$, $d = p$, $ql_2 \neq 0$.
- Sub-subcase $a_{23212} \neq 0$, $d \neq p$.

#### 4.7.1. *Sub-subcase* $a_{23212} = 0$

There is one solution

$$p = 0, \qquad q = 0, \qquad r \neq 0, \qquad d = 0, \qquad u = 0, \qquad v = r;$$
$$l_1 = 1, \qquad l_2 \neq 0, \qquad l_3 \neq 0, \qquad l_4 \neq 0, \qquad l_5 = 0, \qquad l_6 = 0$$

with $l_3 = -r^4g_2^2l_2$, $l_4g_1 = r^3$ and $l_4g_2^2r^2 = -1$ where $g_1g_2 = 1$. Then $r^5 = -g_1^3$. Let $r = t^3$ for some $t \in k \setminus \{0\}$. Then $r = v = t^3$, $l_3 = -t^2l_2$ and $l_4 = -t^4$. This gives an algebra:

Algebra **A**:

$$f_1 = x^3y + t^3yx^3,$$
$$f_2 = x^2y^2 + l_2xyxy - t^2l_2yxyx - t^4y^2x^2,$$
$$f_3 = xy^3 + t^3y^3x, \quad t, l_2 \in k \setminus \{0\}.$$

By the diamond lemma [Be], we have that $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a $k$-linear basis. Algebra **A** is indeed an AS-regular algebra and enjoys many good homological properties as proved in Theorems 5.2 and 5.4.

*4.7.2. Sub-subcase $a_{23212} \neq 0$, $d = p$, $q = 0$*

There is no solution.

*4.7.3. Sub-subcase $a_{23212} \neq 0$, $d = p$, $q \neq 0$, $l_2 = 0$*

If $l_3 = 0$, then (4.5) has one solution

$$p \neq 0, \quad q = p^2, \quad r = p^2, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 = 0, \quad l_3 = 0, \quad l_4 \neq 0, \quad l_5 = 0, \quad l_6 = 0$$

with $l_4^2 = p^8$ and $p^5 = -g_1$, which gives two algebras:

Algebra **B**:

$$\begin{aligned} f_1 &= x^3y + px^2yx + p^2xyx^2 + p^3yx^3, \\ f_2 &= x^2y^2 + p^4y^2x^2, \\ f_3 &= xy^3 + pyxy^2 + p^2y^2xy + p^3y^3x, \quad p \in k \setminus \{0\}; \end{aligned}$$

Algebra **C**:

$$\begin{aligned} f_1 &= x^3y + px^2yx + p^2xyx^2 + p^3yx^3, \\ f_2 &= x^2y^2 - p^4y^2x^2, \\ f_3 &= xy^3 + pyxy^2 + p^2y^2xy + p^3y^3x, \quad p \in k \setminus \{0\}. \end{aligned}$$

By the diamond lemma [Be], $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a $k$-linear basis of algebra **B** and algebra **C**. The algebra **C** has a normal regular element of degree 3, but the algebra **B** does not have any normal element of degree 3. Both algebra **B** and algebra **C** are strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorems 5.2, 5.5 and 5.9).

If $l_3 \neq 0$, then (4.5) has one solution

$$p \neq 0, \quad q \neq 0, \quad r = -p\big(2p^2 + q\big), \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 = 0, \quad l_3 = -p\big(p^2 + q\big), \quad l_4 = -q^2, \quad l_5 = q - p^2, \quad l_6 = q - p^2$$

where $p, q \in k \setminus \{0\}$ satisfy $2p^4 - p^2q + q^2 = 0$, which gives an algebra:

Algebra **D**:

$$\begin{aligned} f_1 &= x^3y + px^2yx + qxyx^2 - p\big(2p^2 + q\big)yx^3, \\ f_2 &= x^2y^2 - p\big(p^2 + q\big)yxyx - q^2y^2x^2 + \big(q - p^2\big)xy^2x + \big(q - p^2\big)yx^2y, \\ f_3 &= xy^3 + pyxy^2 + qy^2xy - p\big(2p^2 + q\big)y^3x, \end{aligned}$$

where $p, q \in k \setminus \{0\}$ satisfy $2p^4 - p^2q + q^2 = 0$.

By the diamond lemma [Be], $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a $k$-linear basis of **D**, and **D** does not have any normal element of degree 3. Algebra **D** is an iterated Ore extension of a polynomial ring, so it is strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorem 5.8).

*4.7.4. Sub-subcase $a_{23212} \neq 0$, $d = p$, $ql_2 \neq 0$*

By the first, second and fifth equations in (4.5), $q - l_6 = p(p - l_2)$. By the seventh, eighteenth and twenty-second equations in (4.5), $u - l_5 = d(d - l_2)$. It follows from the seventh and eighteenth equations in (4.5) that

$$a_{13123} + g_1^2 g_2^4 (d - l_2) a_{23221} = 0.$$

Since $a_{13123} \neq 0$, $d \neq l_2$. By the fourth and seventh equations in (4.5) and $d = p$, $(p - l_2)a_{22212} = 0$. So $a_{22212} = 0$. Then it is easy to see that $a_{23122} \neq 0$, $g_1 g_2 = 1$ and $l_5 = l_6$. Hence $v = r$ and $u = q$.

If $l_5 = l_6 = 0$, then (4.5) has one solution:

$$p \neq 0, \quad q = p(p - l_2), \quad r = (p - l_2)^3, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 \neq 0, \quad l_3 = -l_2(p - l_2)^2, \quad l_4 = -(p - l_2)^4, \quad l_5 = 0, \quad l_6 = 0$$

with $(p - l_2)^5 = -g_1$, which gives an algebra:

Algebra **E**:

$$f_1 = x^3 y + p x^2 y x + p t x y x^2 + t^3 y x^3,$$
$$f_2 = x^2 y^2 + (p - t) x y x y + t^2 (t - p) y x y x - t^4 y^2 x^2,$$
$$f_3 = x y^3 + p y x y^2 + p t y^2 x y + t^3 y^3 x, \quad p, t \in k \setminus \{0\}, \ p \neq t.$$

Algebra **E** is a normal extension of a 4-dimensional AS-regular algebra (see Theorem 5.9), so **E** is AS-regular of dimension 5, strongly noetherian, Auslander regular and Cohen–Macaulay.

If $l_5 = l_6 \neq 0$, and $q = p^2$ (which is equivalent to that $l_5 = pl_2$ or $a_{22221} = 0$), then (4.5) has one solution:

$$p \neq 0, \quad q = p^2, \quad r = p^3, \quad d = p, \quad u = q, \quad v = r;$$
$$l_1 = 1, \quad l_2 \neq 0, \quad l_3 = p^2 l_2, \quad l_4 = p^4, \quad l_5 = pl_2, \quad l_6 = pl_2,$$

which gives an algebra:

Algebra **F**:

$$f_1 = x^3 y + p x^2 y x + p^2 x y x^2 + p^3 y x^3,$$
$$f_2 = x^2 y^2 + l_2 x y x y + l_2 p^2 y x y x + p^4 y^2 x^2 + l_2 p x y^2 x + l_2 p y x^2 y,$$
$$f_3 = x y^3 + p y x y^2 + p^2 y^2 x y + p^3 y^3 x, \quad p, l_2 \in k \setminus \{0\}, \ p \neq l_2.$$

By the diamond lemma [Be], $\{y^i (xy^2)^j (xy)^k (x^2 y)^l x^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a $k$-linear basis of **F**. Algebra **F** is strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorem 5.5).

If $l_5 = l_6 \neq 0$, $q \neq p^2$, then the solution gives the following:

Algebra **G**:

$$f_1 = x^3 y + p x^2 y x + q x y x^2 + r y x^3,$$
$$f_2 = x^2 y^2 + l_2 x y x y + l_3 y x y x + l_4 y^2 x^2 + l_5 x y^2 x + l_5 y x^2 y,$$
$$f_3 = x y^3 + p y x y^2 + q y^2 x y + r y^3 x,$$

where

$$p=-\frac{r^5+qrg^2+g^3}{r^3g},\qquad l_2=\frac{r^2(g-qr)}{g(g+qr)},\qquad l_3=r-\frac{pg(pr-q^2)}{q(qr+g)},$$
$$l_4=-\frac{g^2}{r^2},\qquad l_5=\frac{pr^2+qg}{qr+g},$$

$g\neq 0$, $q$ satisfies the equation $q^3r^8g^3+(r^5+qrg^2+g^3)^3=0$, $q^2r^2\neq g^2$, $r^5+qrg^2+g^3+q^2r^2g\neq 0$ and $r^5+g^3\neq 0$.

Algebra **G** is an iterated Ore extension of a polynomial ring, so it is strongly noetherian, AS-regular, Auslander regular and Cohen–Macaulay (see Theorem 5.8).

*4.7.5. Sub-subcase $a_{23212}\neq 0$, $d\neq p$*

If $l_2=0$, then it follows from $d\neq p$ that $a_{22212}\neq 0$ and $l_3=0$. Since $a_{23212}\neq 0$, both $l_5$ and $l_6$ are not zero.

Suppose $l_2=0$ and $l_5\neq 0$, then $l_6=0$ and (4.5) has one solution

$$p\neq 0,\qquad q=p^2,\qquad r=p^3,\qquad d=ip,\qquad u=-iq,\qquad v=r;$$
$$l_1=1,\qquad l_2=0,\qquad l_3=0,\qquad l_4=-ip^4,\qquad l_5=p^2(1-i),\qquad l_6=0$$

where $p\in k\setminus\{0\}$ and $i\in k$ satisfies $i^2+1=0$ which gives an algebra:

Algebra **H**:

$$f_1=x^3y+px^2yx+p^2xyx^2+p^3yx^3,$$
$$f_2=x^2y^2-ip^4y^2x^2+p^2(1-i)xy^2x,$$
$$f_3=xy^3+ipyxy^2-ip^2y^2xy+p^3y^3x,\quad p\in k\setminus\{0\},\ i^2+1=0.$$

Algebra **H** is a normal extension of a 4-dimensional AS-regular algebra (see Theorem 5.9), so **H** is AS-regular of dimension 5, strongly noetherian, Auslander regular and Cohen–Macaulay.

Suppose $l_2=0$ and $l_5=0$, then $l_6\neq 0$ and (4.5) has one solution

$$p=di,\qquad q=-iu,\qquad r=v,\qquad d\neq 0,\qquad u=d^2,\qquad v=d^3;$$
$$l_1=1,\qquad l_2=0,\qquad l_3=0,\qquad l_4=-d^4i,\qquad l_5=0,\qquad l_6=d^2(1-i),$$

where $d\in k\setminus\{0\}$ and $i\in k$ satisfies $i^2+1=0$ which gives an algebra:

Algebra **H**$'$:

$$f_1=x^3y+dix^2yx-d^2ixyx^2+d^3yx^3,$$
$$f_2=x^2y^2-d^4iy^2x^2+d^2(1-i)yx^2y,$$
$$f_3=xy^3+dyxy^2+d^2y^2xy+d^3y^3x,\quad d\in k\setminus\{0\},\ i^2+1=0.$$

After changing $x$ and $y$, algebra **H**$'$ is in fact isomorphic to algebra **H** with $p=d^{-1}$.

Suppose $l_2\neq 0$. If neither $l_5$ nor $l_6$ is zero, then (4.5) has no solution.

If $l_2\neq 0$ and $l_6=0$, then $l_5\neq 0$ and (4.5) has one solution:

$$p=-c\big(1+g^3\big),\qquad q=-c^2g^2\big(1+g^2\big),\qquad r=c^3g^2,$$
$$d=-cg^3(1+g),\qquad u=-c^2g^4(1+g),\qquad v=c^3g^3;$$

$$l_1 = 1, \qquad l_2 = cg(1+g), \qquad l_3 = -c^3 g(1+g),$$
$$l_4 = -c^4 g^3, \qquad l_5 = -c^2\bigl(1-g^3\bigr), \qquad l_6 = 0,$$

where $c \in k \setminus \{0\}$ and $g = g_1 g_2$, which gives an algebra:

Algebra **I**:

$$f_1 = x^3 y - c\bigl(1+g^3\bigr)x^2 yx - c^2 g^2\bigl(1+g^2\bigr)xyx^2 + c^3 g^2 yx^3,$$
$$f_2 = x^2 y^2 + cg(1+g)xyxy - c^3 g(1+g)yxyx - c^4 g^3 y^2 x^2 - c^2\bigl(1-g^3\bigr)xy^2 x,$$
$$f_3 = xy^3 - cg^3(1+g)yxy^2 - c^2 g^4(1+g)y^2 xy + c^3 g^3 y^3 x,$$

where $c \in k \setminus \{0\}$ and $g \in k$ satisfies the equation $1 + g + g^2 + g^3 + g^4 = 0$.

Algebra **I** is a normal extension of a 4-dimensional AS-regular algebra (see Theorem 5.9), so **I** is AS-regular of dimension 5, strongly noetherian, Auslander regular and Cohen–Macaulay.

If $l_2 \neq 0$ and $l_5 = 0$, then $l_6 \neq 0$ and (4.5) has one solution:

$$p = -c(1+g), \qquad q = -c^2 g^3(1+g), \qquad r = c^3 g^4,$$
$$d = -cg\bigl(1+g^3\bigr), \qquad u = -c^2 g\bigl(1+g^3\bigr), \qquad v = c^3;$$
$$l_1 = 1, \qquad l_2 = cg^2(1+g), \qquad l_3 = -c^3(1+g),$$
$$l_4 = -c^4 g^3, \qquad l_5 = 0, \qquad l_6 = c^2\bigl(1-g^3\bigr)$$

where $c \in k \setminus \{0\}$ and $g = g_1 g_2$, which gives an algebra:

Algebra **I**$'$:

$$f_1 = x^3 y - c(1+g)x^2 yx - c^2 g^3(1+g)xyx^2 + c^3 g^4 yx^3,$$
$$f_2 = x^2 y^2 + cg^2(1+g)xyxy - c^3(1+g)yxyx - c^4 g^3 y^2 x^2 + c^2\bigl(1-g^3\bigr)yx^2 y,$$
$$f_3 = xy^3 - cg\bigl(1+g^3\bigr)yxy^2 - c^2 g\bigl(1+g^3\bigr)y^2 xy + c^3 y^3 x,$$

where $c \in k \setminus \{0\}$ and $g \in k$ satisfies the equation $1 + g + g^2 + g^3 + g^4 = 0$.

Algebra **I**$'$ is isomorphic to algebra **I** by exchanging $x$ and $y$.

## 5. Proof of the AS-regularity and other properties

In this section, we study homological properties of the algebras given in the previous section.

### 5.1. Algebras **A**, **B** and **F**

Let $\mathcal{A}$ be the quotient algebra $k\langle x, y\rangle/(f_1, f_2, f_3)$, where the generating relations $f_1$, $f_2$ and $f_3$ are

$$f_1 = x^3 y + px^2 yx + qxyx^2 + ryx^3,$$
$$f_2 = x^2 y^2 + l_2 xyxy + l_3 yxyx + l_4 y^2 x^2 + l_5 xy^2 x + l_5 yx^2 y,$$
$$f_3 = xy^3 + pyxy^2 + qy^2 xy + ry^3 x,$$

with the parameters $p, q, r, l_2, l_3, l_4, l_5 \in k$, $p \neq l_2$ and $r \neq 0$.

The algebras **A**, **B** and **F** are of this type, and $\{y^i (xy^2)^j (xy)^k (x^2 y)^l x^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a $k$-linear basis for each of them, as we have already seen by using the diamond lemma.

**Lemma 5.1.** *Suppose that* $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ *is a k-linear basis of* $\mathcal{A}$, *and that there is a complex of right* $\mathcal{A}$*-modules of the form*:

$$0 \to \mathcal{A}(-10) \xrightarrow{d_5} \mathcal{A}(-9)^{\oplus 2} \xrightarrow{d_4} \mathcal{A}(-6)^{\oplus 3} \xrightarrow{d_3} \mathcal{A}(-4)^{\oplus 3} \xrightarrow{d_2} \mathcal{A}(-1)^{\oplus 2} \xrightarrow{d_1} \mathcal{A} \xrightarrow{\epsilon} k_{\mathcal{A}} \to 0, \quad (5.1)$$

*where* $\epsilon$ *is the augmented map and each* $d_i$ *is the left multiplication of a matrix given by*

$$d_1 = (x \quad y),$$

$$d_2 = \begin{pmatrix} x^2y + pxyx + qyx^2 & xy^2 + l_2yxy + l_5y^2x & y^3 \\ rx^3 & l_3xyx + l_4yx^2 + l_5x^2y & pxy^2 + qyxy + ry^2x \end{pmatrix},$$

$$d_3 = \begin{pmatrix} 0 & Dy^2 & Hxy + Kyx \\ Ay^2 & Exy + Fyx & Lx^2 \\ Bxy + Cyx & Gx^2 & 0 \end{pmatrix},$$

$$d_4 = \begin{pmatrix} px^2y + qxyx + ryx^2 & x^3 \\ l_3yxy + l_4y^2x + l_5xy^2 & x^2y + l_2xyx + l_5yx^2 \\ ry^3 & xy^2 + pyxy + qy^2x \end{pmatrix},$$

$$d_5 = \begin{pmatrix} x \\ y \end{pmatrix},$$

*for some* $A, B, C, D, E, F, G, H, K, L \in k$ *such that* $ADGL \neq 0$ *and* $K \neq pH$. *Then the complex* (5.1) *is exact and* $\mathcal{A}$ *is an AS-regular algebra of dimension* 5.

**Proof.** Since $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid i, j, k, l, m \in \mathbb{N}\}$ is a $k$-linear basis of $\mathcal{A}$, the Hilbert series of $\mathcal{A}$ is $(1-t)^{-2}(1-t^2)^{-1}(1-t^3)^{-2}$ and $\text{GK-dim}\,\mathcal{A} = 5$. Since $y$ is not a left zero-divisor, the complex (5.1) is exact at $\mathcal{A}(-10)$.

The composition $d_1 \circ d_2$ is exactly the generating relations of $\mathcal{A}$. The complex (5.1) is exact at $\mathcal{A}(-1)$, $\mathcal{A}$ and $k$ by [AS, (1.4)].

To show (5.1) is exact, it suffices to check the exactness at $\mathcal{A}(-9)^{\oplus 2}$ and $\mathcal{A}(-6)^{\oplus 3}$ by using the Hilbert series.

Suppose that $(f, g)^T \in \operatorname{Ker} d_4$. Writing $g$ in the standard form, by modulo $\operatorname{Im} d_5$ we may assume that no monomial appearing in $g$ starts with $y$. Since $ry^3f + xy^2g + pyxyg + qy^2xg = 0$, then $(xy^2)g = -y(ry^2f + pxyg + qyxg)$. It follows that $g = 0$. Hence $f = 0$ (mod $\operatorname{Im} d_5$), and $\operatorname{Ker} d_4 = \operatorname{Im} d_5$, that is, (5.1) is exact at $\mathcal{A}(-9)^{\oplus 2}$.

Notice that $H \neq 0$. In fact, if $H = 0$, then $Dy^2(x^2y + l_2xyx + l_5yx^2) + Kyx(xy^2 + pyxy + qy^2x) = 0$, i.e., $(D - Kl_5)y^2(x^2y) + (Dl_2 - Kl_3)y^2(xy)x + (Dl_5 - Kl_4)y^3x^2 + K(p - l_2)y(xy)^2 + K(q - l_5)y(xy^2)x = 0$. It follows from $p - l_2 \neq 0$ that $K = 0$. Then $Dy^2(x^2y + l_2xyx + l_5yx^2) = 0$. This contradicts $D \neq 0$.

Suppose that $(f, g, h)^T \in \operatorname{Ker} d_3$. Writing $h$ in the standard form, by modulo $\operatorname{Im} d_4$ we may assume that no monomial appearing in $h$ starts with $xy^2$ or $y^3$. Then $h = yh_1 + y^2h_2 + (xy)^lh_3$ ($l \geqslant 0$), with no monomial appearing in $h_1$ or $h_2$ starts with $y$, and no monomial appearing in $h_3$ starts with $y$ or $xy^2$. Since $Dy^2g + (Hxy + Kyx)h = 0$, $Hxy^2h_1 + H(xy)^{l+1}h_3 = yz$ for some $z \in \mathcal{A}$. It follows that $h_1 = h_3 = 0$. So $h = y^2h_2$. Then

$$0 = Dy^2g + (Hxy + Kyx)h = Dy^2g + Hxy^3h_2 + Kyxy^2h_2,$$

which implies

$$Dy^2g + (K - pH)yxy^2h_2 = Hqy^2xyh_2 + Hry^3xh_2.$$

Writing the terms in the above equation in the standard form, it follows from $K - pH \neq 0$ that $h_2 = 0$. So $h = 0$. It follows from $Dy^2g = 0$ and $D \neq 0$ that $g = 0$. Then $Ay^2f = 0$, which implies that $f = 0$ as $A \neq 0$. So $(f, g, h)^T \in \operatorname{Im} d_4$ and $\operatorname{Ker} d_3 = \operatorname{Im} d_4$, i.e., (5.1) is exact at $\mathcal{A}(-6)^{\oplus 3}$.

So the complex (5.1) is a minimal projective resolution of the trivial module $k$.

Applying $\mathrm{Hom}_{\mathcal{A}}(-,\mathcal{A})$ to this projective resolution, we get a complex of left $\mathcal{A}$-modules

$$0 \to \mathcal{A} \xrightarrow{d_1^*} \mathcal{A}(1)^{\oplus 2} \xrightarrow{d_2^*} \mathcal{A}(4)^{\oplus 3} \xrightarrow{d_3^*} \mathcal{A}(6)^{\oplus 3} \xrightarrow{d_4^*} \mathcal{A}(9)^{\oplus 2} \xrightarrow{d_5^*} \mathcal{A}(10) \to 0, \tag{5.2}$$

where each $d_i^*$ is given by the right multiplication of the corresponding matrix. The complex (5.2) is exact at $\mathcal{A}$ since $x$ is not a right zero-divisor. It is also exact at $\mathcal{A}(9)^{\oplus 2}$ again by [AS, (1.4)] and the dimension of the homology group at $\mathcal{A}(10)$ is 1. Similarly, to show the exactness of (5.2) at all other positions, it suffices to check the exactness of (5.2) at $\mathcal{A}(1)^{\oplus 2}$ and $\mathcal{A}(4)^{\oplus 3}$.

Suppose $(f,g) \in \mathrm{Ker}\, d_2^*$. By modulo $\mathrm{Im}\, d_1^*$ we may assume that no monomial appearing in $f$ ends with $x$. Since $f(x^2y+pxyx+qyx^2)+rgx^3=0$, which implies that the monomials in $fx^2y$ would end with $x$, then $f=0$. Since $r \neq 0$, $g=0$. So $\mathrm{Ker}\, d_2^* = \mathrm{Im}\, d_1^*$, i.e., (5.2) is exact at $\mathcal{A}(1)^{\oplus 2}$.

Suppose $(f,g,h) \in \mathrm{Ker}\, d_3^*$. By modulo $\mathrm{Im}\, d_2^*$ we may assume that no monomial appearing in $f$ ends with $x^2y$ or $x^3$. Writing $f$ as $f = f_1x + f_2x^2 + f_3(xy)^s$ $(s \geqslant 0)$ with that no monomial appearing in $f_1$ or $f_2$ ends with $x$, and no monomial appearing in $f_3$ ends with $x$ or $x^2y$. Since $f(Hxy+Kyx)+Lgx^2 = (f_1x+f_2x^2+f_3(xy)^s)(Hxy+Kyx)+Lgx^2=0$,

$$Hf_1x^2y + Hf_3(xy)^{s+1} = Hf_2\bigl(px^2yx+qxyx^2+ryx^3\bigr) - K\bigl(f_1x+f_2x^2+f_3(xy)^s\bigr)yx - Lgx^2.$$

Writing the right-hand side in standard form, it follows from $H \neq 0$ that $f_1 = f_3 = 0$. So $f = f_2x^2$ and

$$0 = f_2x^2(Hxy+Kyx)+Lgx^2 = (K-pH)f_2x^2yx - Hf_2\bigl(qxyx^2+ryx^3\bigr)+Lgx^2.$$

Since $K-pH \neq 0$, $f_2=0$. So $f=0$. Then $Lgx^2=0$, which implies $g=0$ as $L \neq 0$. It follows from $Ghx^2=0$ that $h=0$ as $G \neq 0$. Hence $\mathrm{Ker}\, d_3^* = \mathrm{Im}\, d_2^*$, i.e. (5.2) is exact at $\mathcal{A}(4)^{\oplus 3}$.

Therefore $\mathcal{A}$ satisfies the Gorenstein condition with $\mathrm{gldim}\,\mathcal{A} = \mathrm{GK\text{-}dim}\,\mathcal{A} = 5$, i.e., $\mathcal{A}$ is a 5-dimensional AS-regular algebra. □

Now we can prove the regularity for the algebras **A**, **B** and **F**.

**Theorem 5.2.** *Algebras* **A**, **B** *and* **F** *are all AS-regular.*

**Proof.** It suffices to list the suitable parameters satisfying the conditions of Lemma 5.1.

For algebra **A**, take

$$A=-t^6,\quad B=-t^9,\quad C=-t^9l_2,\quad D=1,\quad E=F=0,$$
$$G=-t^6,\quad H=-l_2t^{-4},\quad K=-t^{-2},\quad L=1.$$

For algebra **B**, take

$$A=p^6,\quad B=0,\quad C=-p^{10},\quad D=1,\quad E=p^2,$$
$$F=p^3,\quad G=p^6,\quad H=-p^{-3},\quad K=0,\quad L=1.$$

For algebra **F**, take

$$A=p^6,\quad B=l_2p^8,\quad C=(l_2-p)p^9,\quad D=1,$$
$$E=p^2,\quad F=p^3,\quad G=p^6,\quad H=(l_2-p)p^{-4}$$

and $K=l_2p^{-3}$, $L=1$. □

To prove other homological properties, let $A(l,t) = \mathbf{A} = k\langle x, y\rangle/(f_1, f_2, f_3)$, where

$$\begin{aligned} f_1 &= x^3y + t^3yx^3, \\ f_2 &= x^2y^2 + lxyxy - t^2lyxyx - t^4y^2x^2, \\ f_3 &= xy^3 + t^3y^3x, \quad t, l \in k \text{ and } tl \neq 0. \end{aligned}$$

**Lemma 5.3.** *The algebra $A(l,t)$ is graded twist-equivalent* [Zh1] *to $A(l^2/t^2, l/t)$.*

**Proof.** Let $\sigma : A(l,t) \to A(l,t)$, $\sigma(x) = t^2x$, $\sigma(y) = ly$. Then $A^\sigma \cong A(l^2/t^2, l/t)$. $\square$

**Theorem 5.4.** *Algebra **A** is strongly noetherian, Auslander regular and Cohen–Macaulay.*

**Proof.** It suffices to prove the properties for $A(t^2,t)$ for some $t \neq 0$ by [Zh1, Theorem 1.3] under the condition that $A(t^2,t)$ is noetherian. Now

$$A\big(t^2,t\big) = k\langle x,y\rangle/\big(x^3y + t^3yx^3, x^2y^2 + t^2xyxy - t^4yxyx - t^4y^2x^2, xy^3 + t^3y^3x\big).$$

Note that $\{x^3, y^3, x^2y^2 - t^4yxyx\}$ is a sequence of normal regular elements of $A(t^2,t)$. By [ASZ, Proposition 4.9] and [Le, Theorem 5.10] it is enough to show that $A(t^2,t)/(x^3, y^3, x^2y^2 - t^4yxyx)$ is strongly noetherian, Auslander–Gorenstein and Cohen–Macaulay.

Let $A_1 = A(t^2,t)/(x^3, y^3, x^2y^2 - t^4yxyx) \cong k\langle x,y\rangle/(x^3, y^3, x^2y^2 - t^4yxyx, y^2x^2 - t^{-2}xyxy)$.

Now twisting $A_1$ by the graded automorphism

$$\sigma : A_1 \to A_1, \qquad \sigma(x) = x, \qquad \sigma(y) = t^{-1}y,$$

we get a new algebra

$$A_2 = (A_1)^\sigma = k\langle x,y\rangle/\big(x^3, y^3, x^2y^2 - tyxyx, xyxy - t^{-1}y^2x^2\big).$$

By [Zh1, Theorem 1.3] it suffices to show that $A_2$ is strongly noetherian, Auslander–Gorenstein and Cohen–Macaulay.

Let

$$\Omega_1 = xy^2xyx + yxyx^2y + ty^2xyx^2 \quad \text{and} \quad \Omega_2 = xy^2x^2y + t^{-1}yxy^2x^2 + t^{-1}y^2x^2yx.$$

Then $\Omega_1$ and $\Omega_2$ are normal elements of $A_2$ such that $\Omega_1\Omega_2 = \Omega_2\Omega_1 = 0$.

Let $A_3 = A_2/(\Omega_1, \Omega_2)$, then

$$A_3 \cong k\langle x,y\rangle/\big(x^3, y^3, \Omega_1, \Omega_2, x^2y^2 - tyxyx, xyxy - t^{-1}y^2x^2\big).$$

Similarly, we can find two normal elements

$$\omega_1 = \big(xy^2\big)^3 + y\big(xy^2\big)^2xy + y^2\big(xy^2\big)^2x \quad \text{and} \quad \omega_2 = \big(x^2y\big)^3 + xy\big(x^2y\big)^2x + y\big(x^2y\big)^2x^2$$

of $A_3$ such that $\omega_1\omega_2 = \omega_2\omega_1 = 0$.

Let $A_4 = A_3/(\omega_1, \omega_2)$. Then

$$A_4 \cong k\langle x,y\rangle/\big(x^3, y^3, \Omega_1, \Omega_2, \omega_1, \omega_2, x^2y^2 - tyxyx, xyxy - t^{-1}y^2x^2\big)$$

is a finite-dimensional algebra. So $A_4$ is strongly noetherian. It follows that $A_2$ is also strongly noetherian by [ASZ, Proposition 4.9].

Since $\{\Omega_1, \Omega_2, \omega_1, \omega_2\}$ is a sequence of normal elements of $A_2$, $A_2$ has enough normal elements. So it is Auslander–Gorenstein and Cohen–Macaulay by [Zh, Theorem 1] which ends the proof. □

**Theorem 5.5.** *The algebras* **B** *and* **F** *are strongly noetherian, Auslander regular and Cohen–Macaulay.*

**Proof.** If we set $l_2 = 0$ in algebra **F**, then **F** reduces to **B**. By [Zh1, Theorem 1.3], it suffices to prove the conclusion for the twisted algebra $\mathbf{F}^\sigma$ where $\sigma$ is the automorphism defined by $\sigma(x) = x$ and $\sigma(y) = p^{-1}y$. Or equivalently, we may assume $p = 1$ in **F**. Then $x^4$, $y^4$, $\Omega_1 = (x^2y - yx^2)^2$, $\Omega_2 = (xy^2 - y^2x)^2$ and $\Omega_3 = (xy + yx)^4$ are central regular elements of **F**.

Let $F' = \mathbf{F}/(x^4, y^4, \Omega_1, \Omega_2, \Omega_3)$ be the quotient algebra. Then $F'$ is a finite-dimensional algebra with a basis $\{y^i(xy^2)^j(xy)^k(x^2y)^lx^m \mid 0 \leqslant i, k, m \leqslant 3,\ 0 \leqslant j, l \leqslant 1\}$. Since $F'$ is strongly noetherian, Cohen–Macaulay and has an Auslander dualizing complex, by [YZ, Theorem 5.1] **F** is strongly noetherian, Auslander regular and Cohen–Macaulay. □

*5.2. Algebras* **D** *and* **G**

Recall that a ring $B$ is an Ore extension $A[z; \sigma, \delta]$ of a ring $A$, for some endomorphism $\sigma$ of $A$ and $\sigma$-derivation $\delta$, if and only if that $B = \bigoplus_{i\geqslant 0} Az^i$ as a free $A$-module with $zA \subseteq Az + A$ [GW,MR]. Graded version of Ore extensions is defined accordingly. We show in this subsection that the algebras **D** and **G** are given by iterated Ore extensions.

Let $A$ be the graded polynomial ring $k[y]$ over $k$ with $\deg y = 1$. We proceed to construct an algebra $A_4$ from $A$ by an iterated Ore extension in the following four steps.

Step 1: Let $z_1$ be a new variable of degree 3 and $A_1 = A[z_1; \sigma_1]$ be the graded Ore extension of $A$, where $\sigma_1$ is the automorphism of $A$ given by

$$\sigma_1(y) = ay$$

for a fixed $0 \neq a \in k$.

Step 2: Let $z_2$ be a new variable of degree 2 and let $0 \neq b \in k$ and

$$A_2 = k\langle y, z_1, z_2\rangle/(z_1y = ayz_1, z_2y = byz_2 + z_1, z_2z_1 = az_1z_2).$$

It follows from the diamond lemma [Be] that $A_2 = \bigoplus_{i\geqslant 0} A_1z_2^i$ as a free $A_1$-module. Obviously, $z_2A_1 \subseteq A_1z_2 + A_1$. So $A_2 = A_1[z_2; \sigma_2, \delta_2]$ is a graded Ore extension of $A_1$, with $\sigma_2$ and $\delta_2$ defined by

$$\sigma_2(y) = by, \qquad \sigma_2(z_1) = az_1;$$
$$\delta_2(y) = z_1, \qquad \delta_2(z_1) = 0.$$

Step 3: Let $z_3$ be a new variable of degree 3 and let

$$A_3 = k\langle y, z_1, z_2, z_3\rangle \Big/ \begin{pmatrix} z_1y = ayz_1, & z_2y = byz_2 + z_1, & z_3z_1 = b^3z_1z_3 + (a-b)z_2^3, \\ z_2z_1 = az_1z_2, & z_3z_2 = az_2z_3, & z_3y = b^3a^{-1}yz_3 + z_2^2 \end{pmatrix}.$$

Again by the diamond lemma, $A_3 = \bigoplus_{i\geqslant 0} A_2z_3^i$ as a free $A_2$-module. It follows from $z_3A_2 \subseteq A_2z_3 + A_2$ that $A_3 = A_2[z_3; \sigma_3, \delta_3]$ is a graded Ore extension of $A_2$, with $\sigma_3$ and $\delta_3$ defined by

$$\sigma_3(y) = b^3a^{-1}y, \qquad \sigma_3(z_1) = b^3z_1, \qquad \sigma_3(z_2) = az_2;$$
$$\delta_3(y) = z_2^2, \qquad \delta_3(z_1) = (a-b)z_2^3, \qquad \delta_3(z_2) = 0.$$

Step 4: Let $x$ be a new variable of degree 1. Suppose that $a \neq -b$. Let

$$A_4 = k\langle y, z_1, z_2, z_3, x\rangle \Big/ \begin{pmatrix} z_1 y = a y z_1, & z_2 z_1 = a z_1 z_2, & z_3 z_2 = a z_2 z_3, \\ z_2 y = b y z_2 + z_1, & z_3 z_1 = b^3 z_1 z_3 + (a-b) z_2^3, & x z_3 = a z_3 x, \\ z_3 y = b^3 a^{-1} y z_3 + z_2^2, & x z_1 = b^3 a^{-1} z_1 x + (a^3 - b^3)(a^2+ab)^{-1} z_2^2, & \\ xy = b^2 a^{-1} yx + z_2, & x z_2 = b z_2 x + (a^3-b^3)(a^2+ab)^{-1} z_3 & \end{pmatrix}.$$

Similarly, $A_4 = \bigoplus_{i \geqslant 0} A_3 x^i$ as a free $A_3$-module and $xA_3 \subseteq A_3 x + A_3$ which implies that $A_4 = A_3[x; \sigma_4, \delta_4]$ is a graded Ore extension of $A_3$, with $\sigma_4$ and $\delta_4$ defined by

$$\sigma_4(y) = b^2 a^{-1} y, \qquad \sigma_4(z_1) = b^3 a^{-1} z_1, \qquad \sigma_4(z_2) = b z_2, \qquad \sigma_4(z_3) = a z_3;$$
$$\delta_4(y) = z_2, \qquad \delta_4(z_1) = \frac{a^3 - b^3}{a(a+b)} z_2^2, \qquad \delta_4(z_2) = \frac{a^3 - b^3}{a(a+b)} z_3, \qquad \delta_4(z_3) = 0.$$

**Lemma 5.6.** *Given $a, b \in k$ such that $ab(a+b) \neq 0$. Then the algebra $A_4$ is an AS-regular algebra of dimension* 5 *with Hilbert series* $(1-t)^{-2}(1-t^2)^{-1}(1-t^3)^{-2}$.

**Proof.** By [ZZ2, Lemma 5.3], $A_4$ is 5-dimensional AS-regular. By the definition of graded Ore extensions, $A_4$ is a free left $A$-module, and

$$\begin{aligned} H_{A_4}(t) &= H_A(t) \cdot \frac{1}{(1 - t^{\deg z_1})(1 - t^{\deg z_2})(1 - t^{\deg z_3})(1 - t^{\deg x})} \\ &= \frac{1}{(1-t)^2(1-t^2)(1-t^3)^2}. \qquad \square \end{aligned}$$

Now, let $\mathcal{A}(a, b) = k\langle x, y\rangle/(f_1, f_2, f_3)$, with the generating relations $f_1$, $f_2$ and $f_3$ as follows:

$$\begin{aligned} f_1 &= x^3 y + p x^2 y x + q x y x^2 + r y x^3, \\ f_2 &= x^2 y^2 + l_2 xyxy + l_3 yxyx + l_4 y^2 x^2 + l_5 x y^2 x + l_5 y x^2 y, \\ f_3 &= x y^3 + p yxy^2 + q y^2 xy + r y^3 x \end{aligned}$$

where

$$p = -\frac{ab + b^2 + a^2}{a}, \qquad q = \frac{b(ab + b^2 + a^2)}{a}, \qquad r = -b^3,$$
$$l_2 = -\frac{a^2 + ab + 2b^2}{a+b}, \qquad l_3 = \frac{b^5(2a^2 + ab + b^2)}{a^3(a+b)}, \qquad l_4 = -\frac{b^6}{a^2}, \qquad l_5 = \frac{b^2(a^3 - b^3)}{a^2(a+b)}. \tag{5.3}$$

Then we have the following proposition.

**Proposition 5.7.** *Given $a, b \in k$ such that $ab(a+b)(a^2+b^2)(a^3-b^3) \neq 0$. Then $\mathcal{A}(a, b)$ is isomorphic to $A_4$ as a graded algebra. So, $\mathcal{A}(a, b)$ is strongly noetherian, Auslander regular, AS-regular of dimension* 5 *and Cohen–Macaulay.*

**Proof.** By the construction, $A_4 = k\langle y, z_1, z_2, z_3, x\rangle/I$, where $I$ is generated by the following ten relations:

$$z_1 y = ayz_1, \tag{5.4}$$

$$z_2 y = byz_2 + z_1, \tag{5.5}$$

$$z_2 z_1 = az_1 z_2, \tag{5.6}$$

$$z_3 y = b^3 a^{-1} yz_3 + z_2^2, \tag{5.7}$$

$$z_3 z_1 = b^3 z_1 z_3 + (a-b) z_2^3, \tag{5.8}$$

$$z_3 z_2 = az_2 z_3, \tag{5.9}$$

$$xy = b^2 a^{-1} yx + z_2, \tag{5.10}$$

$$xz_1 = \frac{b^3}{a} z_1 x + \frac{a^3 - b^3}{a(a+b)} z_2^2, \tag{5.11}$$

$$xz_2 = bz_2 x + \frac{a^3 - b^3}{a(a+b)} z_3, \tag{5.12}$$

$$xz_3 = az_3 x. \tag{5.13}$$

By (5.10) $z_2 = xy - b^2 a^{-1} yx$, by (5.5) $z_1 = xy^2 - (b^2 a^{-1} + b) yxy + b^3 a^{-1} y^2 x$, and by (5.12)

$$z_3 = \frac{a(a+b)x^2 y - (a+b)^2 bxyx + (a+b)b^3 yx^2}{a^3 - b^3}.$$

So $A_4$ is generated by $x$ and $y$ as a $k$-algebra. Moreover, replacing $z_1$, $z_2$ and $z_3$ with these expressions, the relations (5.4), (5.11) and (5.13) turn out to be the following three relations:

$$xy^3 + pyxy^2 + qy^2 xy + ry^3 x = 0,$$
$$x^2 y^2 + l_2 xyxy + l_3 yxyx + l_4 y^2 x^2 + l_5 xy^2 x + l_5 yx^2 y = 0,$$
$$x^3 y + px^2 yx + qxyx^2 + ryx^3 = 0,$$

where the parameters are given in (5.3). The relations (5.6), (5.7), (5.8) and (5.9) can be derived from the above three relations by using $a^2 + b^2 \neq 0$. So, $A_4 = \mathcal{A}(a,b)$.

It follows from [ASZ, Proposition 4.1] and [YZ, Theorem 5.1, Corollary 6.8] that $\mathcal{A}(a,b)$ is strongly noetherian, Auslander regular, AS-regular of dimension 5 and Cohen–Macaulay. □

**Theorem 5.8.** *Algebras* $\mathbf{D}$ *and* $\mathbf{G}$ *are strongly noetherian, Auslander regular, AS-regular of dimension* 5 *and Cohen–Macaulay.*

**Proof.** It is easy to check that

$$\mathbf{D} \cong \mathcal{A}(a,b) \quad \text{with } a = q^2/p^3,\ b = -q/p;$$
$$\mathbf{G} \cong \mathcal{A}(a,b) \quad \text{with } a = r^2/g,\ b = qr^3 g/\big(r^5 + qrg^2 + g^3\big).$$

The conclusions follow from Proposition 5.7. □

### 5.3. *Algebras* **C**, **E**, **H** *and* **I**

In this subsection, we show the algebras **C**, **E**, **H** and **I** are normal extensions of some 4-dimensional AS-regular algebras given in [LPWZ2].

**Theorem 5.9.** *Algebras* **C**, **E**, **H** *and* **I** *are all AS-regular algebras of dimension* 5, *which are strongly noetherian, Auslander regular and Cohen–Macaulay.*

**Proof.** By the diamond lemma [Be], $xy^2+p^2y^2x$ is a normal regular element of **C** and $\mathbf{C}/(xy^2+p^2y^2x)$ is isomorphic to $D(0,p)$ [LPWZ2, Theorem A]. So **C** is a normal extension of $D(0,p)$.

Algebra **E** is a normal extension of $D(p-t,t)$ since $xy^2+(p-t)yxy+t^2y^2x$ is a normal regular element of **E** and $\mathbf{E}/(xy^2+(p-t)yxy+t^2y^2x)$ is isomorphic to $D(p-t,t)$ [LPWZ2, Theorem A].

Algebra **H** is a normal extension of $B(p)$ since $xy^2-ip^2y^2x$ is a normal regular element of **H** and $\mathbf{H}/(xy^2-ip^2y^2x)$ is isomorphic to $B(p)$ [LPWZ2, Theorem A].

Algebra **I** is a normal extension of $D(cg(1+g),cg^4)$ since $xy^2+cg(1+g)yxy+c^2g^3y^2x$ is a normal regular element of **I** and $\mathbf{I}/(xy^2+cg(1+g)yxy+c^2g^3y^2x)$ is isomorphic to $D(cg(1+g),cg^4)$.

These algebras $B(p)$, $D(0,p)$, $D(p-t,t)$ and $D(cg(1+g),cg^4)$ are strongly noetherian, Auslander regular, Cohen–Macaulay and AS-regular of dimension 4 as given in [LPWZ2, Theorem A].

So, all the algebras considered here are normal extensions of AS-regular algebras of dimension 4. They are all noetherian by [ATV, Lemma 8.2]. By [Le, Theorem 5.10] and [LPWZ2, Lemma 7.6], they are AS-regular of dimension 5.

It follows from [ASZ, Proposition 4.9] and [YZ, Theorem 5.1] that all these algebras are strongly noetherian, Auslander regular and Cohen–Macaulay. □

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