

An Analytical Study on the Instability Phenomena During the Phase Transitions in a Thin Strip under Uniaxial Tension

Hui-Hui Dai ^a, Zongxi Cai ^b

^a*Department of Mathematics and Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, 83 Tat Chee Avenue, Hong Kong, P.R. China*

^b*Department of Mechanics, Tianjin University, Tianjin, 300072, P.R. China*

Abstract

In the experiments on stress-induced phase transitions in SMA strips, several interesting instability phenomena have been observed, including a necking-type instability (associated with the stress drop), a shear-type instability (associated with the inclination of the transformation front) and an orientation instability (associated with the switch of the inclination angle). In order to shed more light on these phenomena, in this paper we conduct an analytical study. We consider the problem in a three-dimensional setting, which implies that one needs to study the difficult problem of solution bifurcations of high-dimensional nonlinear partial differential equations. By using the smallness of the maximum strain, the thickness and width of the strip, we use a methodology, which combines series expansions and asymptotic expansions, to derive the asymptotic normal form equations, which can yield the leading-order behavior of the original three-dimensional field equations. An important feature of the second normal form equation is that it contains a turning point for the localization (necking) solution of the first equation. It is the presence of such a turning point which causes the inclination of the phase front. The WKB method is used to construct the asymptotic solutions, which can capture the shear instability and the orientation instability successfully. Our analytical results reveal that the inclination of the phase front is a phenomenon of localization-induced buckling (or phase-transition-induced buckling as the localization is caused by the phase transition). Due to the similarities between the development of the Luders band in a mild steel and the stress-induced transformations in a SMA, the present results give a strong analytical evidence that the former is also caused by macroscopic effects instead of microscopic effects. Our analytical results also reveal more explicitly the important roles played by the geometrical parameters.

Key words: phase transitions, instabilities, SMAs, thin strip, asymptotic analysis, bifurcations of PDE's

PACS: 64.70.Kd, 64.60.Bd, 62.20.mg, 62.20.fg

1 Introduction

Shape memory alloys (SMAs), due to their two important characteristics, shape memory effect and pseudoelasticity, have broad applications (see Duerig et al 1990, Otsuka and Wayman 1998). To understand the behaviour of this type of materials, systematic experiments have been carried out on uniaxial tension of superelastic NiTi (a kind of SMAs) wires, strips and tubes (see Shaw & Kyriakides 1995, 1997, 1998, Sun et al 2000, Tse & sun 2000, Feng & Sun 2006, Chang et al 2006). An experimental video on the tension of a NiTi strip by Q. P. Sun's group can be found in the website: www.me.ust.hk/~meqpsun/video/tension-strip.htm. Among many important observations in these experiments, some key features are the various instability phenomena associated with stress-induced phase transitions. For example, for the stress-induced phase transitions in a strip during a loading process, at least three instability phenomena were observed (see Shaw & Kyriakides 1998, Sun et al 2000, Tse & Sun 2000): (i) a stress drop after the nucleation of the martensite phase and associated with it there is a formation of two phase fronts, which manifest like a neck (a necking-type instability); (ii) the phase front inclines an angle with the strip axis (a shear-type instability); (iii) the front can switch the inclination to an opposite angle. Finite element simulations have been carried out (see Shaw & Kyriakides 1998, Shaw 2000) to capture the main features observed in experiments. The numerical results in these two papers revealed some important information about the stress-induced phase transitions in strips. For example, it was found that the evolution of phase transition events is strongly influenced by overall geometric (structural) effects. The results of these two papers strongly suggest that continuum level events remain dominant players in the SMAs considered by them.

Motivated by the experimental and numerical results by others mentioned above, in this paper we shall study instability phenomena during the phase transitions in a strip *analytically*. We model this problem in a continuum three-dimensional setting with a no-convex strain energy function, in view of the results of Shaw & Kyriakides (1998) (cf. the last sentence of the paragraph above). The difference between the stress-strain relation used in this paper and Shaw & Kyriakides (1998) is that the former is a cubic nonlinear curve while the latter is a trilinear one. The nonlinearity could play certain role (see, Fig. 8 of Shaw 2000). Since in the experiments, the maximum strain is less than 8%, keeping the nonlinearity up to the third order (see (2.3)) is accurate enough, at least not worse than a trilinear approximation. We also point out that here the intention is to study macroscopic instability phenomena in the loading process only and no attempt is made to consider the microscopic effects.

Email address: mahhdai@cityu.edu.hk (Hui-Hui Dai).

Analytical results, if achievable, have a number of advantages. One is that there is no need to introduce an artificial imperfection to capture the post-bifurcation mode. Secondly, mathematically, an instability is caused by the fact that there are multiple solutions, and analytical results can shed light on how this situation arises and help to understand the mechanism. Thirdly, from the analytical results one can see more clearly the roles played by various parameters (for the present problem, in particular, the geometric parameters, e.g. the thickness and the width). Indeed, the analytical results obtained in this paper reveal more explicitly the important role of the thickness of the strip and show that the width influences the instability phenomena through the thickness-width ratio rather its magnitude.

As pointed out in Shaw & Kyriakides (1998), in the macroscopic scale there are many similarities between stress-induced transformations in a SMA and the development of Luders bands in a mild steel. In the literature, there are different views whether the Luders band is caused by microscopic effects or macroscopic effects (see Estrin & Kubin 1995). Here, we have shown that the inclination of the transformation front is a phenomenon of localization-induced buckling. This offers a strong analytical evidence that this phenomenon in a SMA, and plausibly the phenomenon of Luders bands (due to the similarities), is due to macroscopic effects.

Since we formulate this problem in a three-dimensional setting with a nonlinear constitutive relation, the governing field equations are three coupled nonlinear partial differential equations (PDEs). It is extremely difficult to deduce the post-bifurcation solutions of nonlinear PDEs analytically. Fortunately, for the present problem, several quantities are small, e.g., the thickness, the width, and the maximum strain, which then permit us to use a methodology of coupled series and asymptotic expansions to deduce asymptotic solutions. This methodology was first introduced to study nonlinear waves in solids (see Dai & Huo 2002, Dai & Fan 2004). Recently, it has been successfully used to study various instability phenomena in solids (see Dai & Cai 2006, Cai & Dai 2006, Dai et al 2008, Dai & Wang 2008). However, all those problems studied before are essentially two-dimensional. Here, for the first time this methodology is used to study a three-dimensional problem.

The remaining of this paper is arranged as follows. In section 2, we give the general three-dimensional field equations for a plate. Then, in section 3, we non-dimensionalize the three-dimensional governing equations to identify the key small variables and small parameters for a thin plate. And then, by using the smallness of these variables and parameters and a methodology of coupled series and asymptotic expansions, in section 4 we derive the asymptotic two-dimensional equations. By considering the smallness of the width further and using a similar methodology in section 4, we obtain the quasi one-dimensional asymptotic normal form equations for a thin strip in section 5. These normal

form equations are then solved analytically in section 6 for an infinitely long strip and the solutions obtained seem to be able to describe many features observed in experiments. Finally, some conclusions are drawn.

2 Three-Dimensional Field Equations

We consider the deformation of a three-dimensional plate composed of a hyperelastic material. Its thickness is $2a$ and its width is $2b$. We use the Cartesian coordinates (x, y, z) (equivalently x_i) and (X, Y, Z) (equivalently X_i) to represent a material point in the current and reference configurations, respectively. The geometry of the object of study is shown in Figure 1.

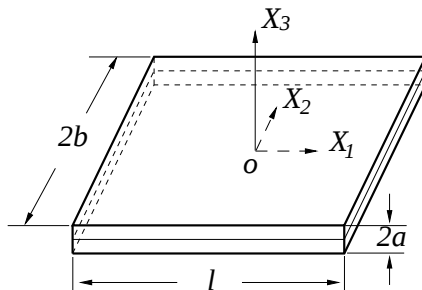


Fig. 1. The geometry of the object of study

To ensure phase transitions can take place, we suppose that the strain energy function Φ , which is a function of the invariants of the left Cauchy-Green strain tensor for a homogeneous isotropic hyperelastic material, is non-convex such that there is a local maximum and a local minimum in the uniaxial stress-strain curve under a homogenous constant strain state. The first Piola-Kirchhoff stress tensor Σ is given by

$$\Sigma = \frac{\partial \Phi}{\partial \mathbf{F}}, \quad (2.1)$$

where $\mathbf{F}$ is the deformation gradient and the components of $(\mathbf{F} - \mathbf{I})$ are

$$u_{i,j} = \frac{\partial u_i}{\partial X_j}, \quad (i, j = 1, 2, 3), \quad (2.2)$$

where u_i are the components of the displacement vector and X_j are coordinates in the undeformed configuration. If the strains are small, it is possible to

expand the first Piola-Kirchhoff stress components in term of the strains up to any order. Due to the complexity of the problem, we shall consider the material nonlinearity up to the third order. The formula containing terms up to the third-order material nonlinearity has been provided in Fu and Ogden (1999) as

$$\Sigma_{ji} = a_{jilk}^1 u_{k,l} + \frac{1}{2} a_{jilknm}^2 u_{k,l} u_{m,n} + \frac{1}{6} a_{jilknmqp}^3 u_{k,l} u_{m,n} u_{p,q}, \quad (2.3)$$

where a_{jilk}^1 , a_{jilknm}^2 and $a_{jilknmqp}^3$ are incremental elastic moduli, which can be calculated once a specific form of the strain energy function is given. Their expressions can be found in Appendix A, where it is also shown that a_{jilk}^1 has 4 non-zero members and only two are independent, a_{jilknm}^2 has 9 non-zero members and only three are independent and $a_{jilknmqp}^3$ has 22 non-zero members and only four are independent. For the convenience of the sequel analysis, we write out the index 3 explicitly. As a result, from Eqs. (2.3) we have

$$\begin{aligned} \Sigma_{ji} = & a_{ji33}^1 u_{3,3} + a_{ji\alpha 3}^1 u_{3,\alpha} + a_{ji3\alpha}^1 u_{\alpha,3} + a_{ji\beta\alpha}^1 u_{\alpha,\beta} \\ & + \frac{1}{2} (a_{ji3333}^2 u_{3,3}^2 + 2(a_{ji\alpha 333}^2 u_{3,\alpha} + a_{ji3\alpha 33}^2 u_{\alpha,3} + a_{ji\beta\alpha 33}^2 u_{\alpha,\beta}) u_{3,3} \\ & + a_{ji\alpha 3\beta 3}^2 u_{3,\alpha} u_{\beta,3} + a_{ji3\alpha\beta 3}^2 u_{\alpha,3} u_{\beta,3} + a_{ji\beta\alpha\sigma\gamma}^2 u_{\alpha,\beta} u_{\gamma,\sigma} \\ & + 2(a_{ji3\beta\alpha 3}^2 u_{3,\alpha} u_{\beta,3} + a_{ji\beta\alpha\gamma 3}^2 u_{\alpha,\beta} u_{\gamma,3} + a_{ji\beta\alpha 3\gamma}^2 u_{\alpha,\beta} u_{\gamma,3})) \\ & + \frac{1}{6} (3(a_{ji\beta\alpha 3333}^3 u_{\alpha,\beta} + a_{ji\alpha 33333}^3 u_{3,\alpha} + a_{ji3\alpha 3333}^3 u_{\alpha,3}) u_{3,3}^2 \\ & + 3(a_{ji\alpha 3\beta 333}^3 u_{3,\alpha} u_{\beta,3} + a_{ji3\alpha\beta 333}^3 u_{\alpha,3} u_{\beta,3} + a_{ji\beta\alpha\sigma\gamma 33}^3 u_{\alpha,\beta} u_{\gamma,\sigma}) u_{3,3} \\ & + 6(a_{ji3\beta\alpha 333}^3 u_{3,\alpha} u_{\beta,3} + a_{ji\beta\alpha 3\gamma 33}^3 u_{\alpha,\beta} u_{\gamma,3} + a_{ji\beta\alpha\sigma 333}^3 u_{\alpha,\beta} u_{\gamma,3}) u_{3,3} \\ & + a_{ji\alpha 3\beta 3\gamma 3}^3 u_{3,\alpha} u_{\beta,3} u_{\gamma,3} + (a_{ji3\alpha\beta 3\gamma 3}^3 u_{\alpha,3} u_{\beta,3} + 3a_{ji\alpha 3\beta 33\gamma}^3 u_{3,\alpha} u_{\beta,3} \\ & + 3a_{ji\alpha 33\beta 3\gamma}^3 u_{3,\alpha} u_{\beta,3}) u_{\gamma,3} + (3a_{ji\alpha 3\beta 3\sigma\gamma}^3 u_{3,\alpha} u_{\beta,3} + 3a_{ji3\alpha\beta 3\sigma\gamma}^3 u_{\alpha,3} u_{\beta,3} \\ & + 6a_{ji3\alpha\beta 3\sigma\gamma}^3 u_{\alpha,3} u_{\beta,3} + 3a_{ji3\tau\beta\alpha\sigma\gamma}^3 u_{\tau,3} u_{\alpha,\beta} + 3a_{ji\tau 3\beta\alpha\sigma\gamma}^3 u_{3,\tau} u_{\alpha,\beta}) u_{\gamma,\sigma} \\ & + a_{ji333333}^3 u_{3,3}^3 + a_{ji\tau\lambda\beta\alpha\sigma\gamma}^3 u_{\lambda,\tau} u_{\alpha,\beta} u_{\gamma,\sigma}), \quad (\alpha, \beta, \gamma, \sigma, \tau, \lambda = 1, 2). \end{aligned} \quad (2.4)$$

The equations of equilibrium are

$$\Sigma_{ji,j} = 0. \quad (2.5)$$

Substituting (2.4) into (2.5), we obtain

$$\begin{aligned} & b_{i333} u_{3,33} + (b_{i3\alpha 3} + b_{i\alpha 33}) u_{3,3\alpha} + b_{i\beta\alpha 3} u_{3,\alpha\beta} \\ & + b_{i33\alpha} u_{\alpha,33} + (b_{i\beta 3\alpha} + b_{i3\beta\alpha}) u_{\alpha,3\beta} + b_{i\gamma\beta\alpha} u_{\alpha,\beta\gamma} = 0, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned}
b_{ijkl} = & a_{jilk}^1 + a_{jilk33}^2 u_{3,3} + a_{jilk\beta\alpha}^2 u_{\alpha,\beta} + a_{jilk3\alpha}^2 u_{\alpha,3} + a_{jilk\alpha3}^2 u_{3,\alpha} \\
& + \frac{1}{2} a_{jilk3333}^3 u_{3,3}^2 + (a_{jilk\beta\alpha33}^3 u_{\alpha,\beta} + a_{jilk\alpha333}^3 u_{3,\alpha} + a_{jilk3\alpha33}^3 u_{\alpha,3}) u_{3,3} \\
& + \frac{1}{2} a_{jilk\beta\alpha\sigma\gamma}^3 u_{\alpha,\beta} u_{\gamma,\sigma} + (a_{jilk\alpha3\sigma\gamma}^3 u_{3,\alpha} + a_{jilk3\alpha\sigma\gamma}^3 u_{\alpha,3}) u_{\gamma,\sigma} \\
& + a_{jilk\beta33\alpha}^3 u_{\alpha,3} u_{3,\beta} + \frac{1}{2} (a_{jilk\beta3\alpha3}^3 u_{3,\alpha} u_{3,\beta} + a_{jilk3\beta3\alpha}^3 u_{\alpha,3} u_{\beta,3}). \quad (2.7)
\end{aligned}$$

It should be noted that, for an initially isotropic material these terms in the right hand side of (2.7) will vanish when the number of index 3 in their all subscripts is odd.

Eqs. (2.6) are the governing equations for the three unknowns $u_i (i = 1, 2, 3)$. In order to investigate the instability phenomena, one needs to study the solution bifurcations of the three-dimensional nonlinear partial differential equations (PDE's) (2.6) with the traction free conditions on the top/bottom surfaces and two side surfaces and under proper end conditions. Mathematically, this is a very challenging problem, since there is no available a general method for studying the bifurcations of three-dimensional nonlinear PDE's. Here, we shall use a novel approach involving coupled series and asymptotic expansions to derive the asymptotic normal form equations in order to carry out the analysis. For that purpose, we first non-dimensionalize the governing equations to identify the key small variables and small parameters.

3 Non-dimensional Equations

Suppose that the loads acting on the boundaries of the plate are symmetrical about the mid plane and therefore the deformation is also symmetrical about this plane. Then, we have

$$u_\alpha(X_\beta, -X_3) = u_\alpha(X_\beta, X_3), \quad u_3(X_\beta, -X_3) = -u_3(X_\beta, X_3). \quad (3.1)$$

Based on Eq. (3.1), we introduce a transformation

$$u_3 = X_3 w, \quad s = X_3^2. \quad (3.2)$$

The dimensionless quantities are defined through the following scalings:

$$s = l^2 \tilde{s}, \quad X_\alpha = l \tilde{x}_\alpha, \quad u_\alpha = h \tilde{u}_\alpha, \quad w = \frac{h}{l} \tilde{w}, \quad (3.3)$$

where l is the length of the plate and h is a characteristic displacement in the mid-plane. From (3.2) and (3.3), we obtain

$$\begin{aligned} u_{3,3} &= \epsilon(\tilde{w} + 2\tilde{s}\frac{\partial\tilde{w}}{\partial\tilde{s}}), & u_{\alpha,\beta} &= \epsilon\frac{\partial\tilde{u}_\alpha}{\partial\tilde{x}_\beta}, \\ u_{3,\alpha} &= \epsilon\sqrt{\tilde{s}}\frac{\partial\tilde{w}}{\partial\tilde{x}_\alpha}, & u_{\alpha,3} &= 2\epsilon\sqrt{\tilde{s}}\frac{\partial\tilde{u}_\alpha}{\partial\tilde{s}}, \end{aligned} \quad (3.4)$$

where $\epsilon = h/l$ is a small parameter (i.e., we are considering a weak nonlinearity). The second-order derivatives can be treated similarly.

Substituting (3.3) and (3.4) into (2.7) and (2.6), we obtain

$$\begin{aligned} & a_{\beta\tau\gamma\alpha}^1 u_{\alpha,\beta\gamma} + 2a_{3\tau3\alpha}^1 u_{\alpha,s} + (a_{3\tau\alpha3}^1 + a_{\alpha\tau33}^1) w_{,\alpha} \\ & + s(4a_{3\tau3\alpha}^1 u_{\alpha,ss} + 2(a_{3\tau\alpha3}^1 + a_{\alpha\tau33}^1) w_{,\alpha s}) \\ & + \epsilon(a_{\beta\tau\gamma\alpha\eta\xi}^2 u_{\alpha,\beta\gamma} u_{\xi,\eta} + 2a_{3\tau3\alpha\eta\xi}^2 u_{\alpha,s} u_{\xi,\eta} + a_{\beta\tau\gamma\alpha33}^2 u_{\alpha,\beta\gamma} w + 2a_{3\tau\alpha333}^2 u_{\alpha,s} w \\ & + (a_{3\tau\alpha3\eta\xi}^2 + a_{\alpha\tau33\eta\xi}^2) u_{\xi,\eta} w_{,\alpha} + (a_{3\tau\alpha333}^2 + a_{\alpha\tau3333}^2) w w_{,\alpha} \\ & + s(4(a_{3\tau\alpha\beta\eta3}^2 + a_{\beta\tau3\alpha\eta3}^2) u_{\alpha,\beta s} u_{\eta,s} + 4a_{3\tau\alpha3\beta\eta}^2 u_{\alpha,ss} u_{\eta,\beta} \\ & + 4a_{3\tau\alpha333}^2 (u_{\alpha,ss} w + u_{\alpha,s} w_{,s}) + 2a_{\alpha\tau\beta3\eta3}^2 u_{\eta,s} w_{,\alpha\beta} + 2a_{\beta\tau\gamma\alpha33}^2 u_{\alpha,\beta\gamma} w_{,s} \\ & + 12a_{3\tau33\alpha3}^2 u_{\alpha,s} w_{,s} + 2(a_{3\tau\alpha333}^2 + a_{\alpha\tau3333}^2 + 3a_{3\tau333\alpha}^2) w_{,\alpha} w_{,s} \\ & + 2(a_{3\tau3\gamma\beta\alpha}^2 + a_{\gamma\tau33\beta\alpha}^2) u_{\alpha,\beta} w_{,\gamma s} + 2(a_{\gamma\tau3333}^2 + a_{3\tau\gamma333}^2) w w_{,\gamma s} \\ & + 2(a_{3\tau\beta\alpha3\eta}^2 + a_{\beta\tau3\alpha3\eta}^2) u_{\alpha,\beta s} w_{,\eta} + a_{\alpha\tau\beta33\eta}^2 w_{,\alpha\beta} w_{,\eta}) \\ & + 4s^2(2a_{3\tau\alpha333}^2 u_{\alpha,ss} w_{,s} + (a_{\alpha\tau3333}^2 + a_{3\tau\alpha333}^2) w_{\alpha,s} w_{,s} \\ & + 2a_{3\tau33\alpha3}^2 u_{\alpha,s} w_{,ss} + a_{3\tau333\alpha}^2 w_{,\alpha} w_{,ss})) \\ & + \epsilon^2(H_1) = 0 \quad (\tau = 1, 2), \end{aligned} \quad (3.5)$$

$$\begin{aligned} & 2(a_{33\beta\alpha}^1 + a_{\beta33\alpha}^1) u_{\alpha,\beta s} + a_{\alpha3\beta3}^1 w_{,\alpha\beta} + 6a_{3333}^1 w_{,s} + 4sa_{3333}^1 w_{,ss} \\ & + \epsilon(2a_{\beta3\gamma\alpha\eta3}^2 u_{\alpha,\beta\gamma} u_{\eta,s} + 4a_{\alpha333\eta3}^2 u_{\alpha,s} u_{\eta,s} + 2(a_{33\beta\alpha\eta\xi}^2 + a_{\alpha3\beta3\eta\xi}^2) u_{\alpha,\beta s} u_{\xi,\eta} \\ & + 2(a_{33\beta\alpha33}^2 + a_{\alpha3\beta333}^2) u_{\alpha,\beta s} w + 2(a_{\eta3333\beta}^2 + a_{\beta333\eta3}^2) u_{\eta,s} w_{,\alpha} \\ & + a_{\alpha3\beta3\eta\xi}^2 u_{\xi,\eta} w_{,\alpha\beta} + a_{\alpha3\beta333}^2 w w_{,\alpha\beta} + 6a_{3333\beta\alpha}^2 u_{\alpha,\beta} w_{,s} + 6a_{333333}^2 w w_{,s} \\ & + a_{\beta3\gamma\alpha3\eta}^2 u_{\alpha,\beta\gamma} w_{,\eta} + 2a_{\alpha3333\eta}^2 u_{\alpha,s} w_{,\eta} + 2a_{\alpha3333\beta}^2 w_{,\alpha} w_{,\beta} \\ & + 2s(4a_{\alpha33333}^2 u_{\alpha,ss} u_{\beta,s} + 2(a_{33\beta\alpha33}^2 + a_{\alpha3\beta333}^2) u_{\alpha,\beta s} w_{,s} + a_{\alpha3\beta333}^2 w_{,\alpha\beta} w_{,s} \\ & + a_{333333}^2 (2w w_{,ss} + 6w_{,s}^2) + 4a_{\alpha333\beta3}^2 u_{\beta,s} w_{,\alpha s} + 2a_{3333\beta\alpha}^2 u_{\alpha,\beta} w_{,ss} \\ & + 2a_{\alpha3333\beta}^2 u_{\alpha,ss} w_{,\beta} + (a_{\alpha3333\beta}^2 + a_{\alpha3333\beta}^2) w_{,\beta} w_{,\alpha s}) \\ & + 8s^2 a_{333333}^2 w_{,s} w_{,ss}) \\ & + \epsilon^2(H_2) = 0, \end{aligned} \quad (3.6)$$

where and thereafter the tilde over non-dimensional variables has been dropped for convenience. The lengthy expressions for $H_i (i = 1, 2, \dots)$ are omitted although they are needed for the calculations (interested readers can contact the corresponding author for their expressions). We consider the case that the top and bottom surfaces of the plate are traction-free. By using (2.4), we have

$$\begin{aligned}
& 2a_{3\tau\alpha 3}^1 u_{\alpha,s} + a_{3\tau 3\alpha}^1 w_{,\alpha} \\
& + \epsilon(4a_{3\tau\alpha\beta\gamma 3}^2 u_{\beta,\alpha} u_{\gamma,s} + 4a_{3\tau\alpha 333}^2 u_{\alpha,s} w + 2a_{3\tau\alpha\beta 3\gamma}^2 u_{\beta,\alpha} w_{,\gamma} + 2a_{3\tau 3\alpha 33}^2 w w_{,\beta} \\
& + 2s w_{,s} (2a_{3\tau\alpha 333}^2 u_{\alpha,s} + a_{3\tau 3\alpha 33}^2 w_{,\alpha})) \\
& + \epsilon^2 (H_3)|_{s=\nu_1} = 0 \quad (\tau = 1, 2), \tag{3.7}
\end{aligned}$$

$$\begin{aligned}
& a_{33\alpha\beta}^1 u_{\beta,\alpha} + a_{3333}^1 (w + 2s w_{,s}) \\
& + \frac{\epsilon}{2} (a_{33\alpha\beta\gamma\delta}^2 u_{\beta,\alpha} u_{\delta,\gamma} + 2a_{33\alpha\beta 33}^2 u_{\beta,\alpha} w + a_{333333}^2 w^2 \\
& + s(4a_{33\alpha\beta 33}^2 u_{\alpha,s} u_{\beta,s} + 4a_{33\alpha\beta 33}^2 u_{\beta,\alpha} w_{,s} + 4a_{333333}^2 w w_{,s} \\
& + 4a_{33\alpha 33\beta}^2 u_{\alpha,s} w_{,\beta} + a_{333\alpha 3\beta}^2 w_{,\alpha} w_{,\beta}) + 4s^2 a_{333333}^2 w_{,s}^2) \\
& + \epsilon^2 (H_4)|_{s=\nu_1} = 0, \tag{3.8}
\end{aligned}$$

where $\nu_1 = a^2/l^2$. Please note that due to symmetry, the boundary conditions at the bottom surface are automatically satisfied.

Eqs. (3.5) and (3.6) provide the governing equations for three unknowns u_α and w and the boundary conditions are (3.7) and (3.8). However, they still comprise a formidable system of nonlinear PDE's to be analyzed directly. To go further, we assume that the plate is thin. Then ν_1 is a small parameter and $0 \leq s \leq \nu_1$ is a small variable. It is clear that the unknowns are functions of the spatial variables x_1 and x_2 , the small variable s and two small parameters ν_1 and ϵ . Next, we shall use the smallness of the variable s and two parameters ν_1 and ϵ to derive the asymptotic two-dimensional equations.

Remark: Since the current methodology to deduce the one-dimensional asymptotic normal equations from the three-dimensional nonlinear field equations has not been done before, in the next two sections we shall provide some detailed derivations.

4 Two-dimensional Asymptotic Equations

As discussed in the previous section, we can write

$$u_\alpha = u_\alpha(x_\beta, s; \epsilon, \nu_1), \quad w = w(x_\beta, s; \epsilon, \nu_1). \tag{4.1}$$

As the variable s is small, as long as we assume that the unknowns are sufficiently smooth in s , we can take the series expansions in s for the unknowns, i.e.,

$$\begin{aligned} u_\alpha &= U_{0\alpha}(x_\beta; \epsilon, \nu_1) + sU_{1\alpha}(x_\beta; \epsilon, \nu_1) + s^2U_{2\alpha}(x_\beta; \epsilon, \nu_1) + \cdots, \\ w &= W_0(x_\beta; \epsilon, \nu_1) + sW_1(x_\beta; \epsilon, \nu_1) + s^2W_2(x_\beta; \epsilon, \nu_1) + \cdots. \end{aligned} \quad (4.2)$$

Substituting Eq. (4.2) into the boundary conditions (3.7) and (3.8), and noting that $s = \nu_1$, we obtain

$$\begin{aligned} & 2a_{3\tau\alpha 3}^1 U_{1\alpha} + a_{3\tau 3\alpha}^1 W_{0,\alpha} + \nu_1(4a_{3\tau\alpha 3}^1 U_{2\alpha} + a_{3\tau 3\alpha}^1 W_{1,\alpha}) \\ & + \frac{\epsilon}{2}(4a_{3\tau\alpha 3\eta\xi}^2 U_{0\xi,\eta} U_{1\alpha} + 4a_{3\tau\alpha 333}^2 W_0 U_{1\alpha} \\ & + 2a_{3\tau 3\alpha\eta\xi}^2 U_{0\xi,\eta} W_{0,\alpha} + 2a_{3\tau 333\alpha}^2 W_0 W_{0,\alpha}) \\ & + \frac{\epsilon^2}{6}(6a_{3\tau\alpha 3\eta\xi\lambda\kappa}^3 U_{1\alpha} U_{0\xi,\eta} U_{0\kappa,\lambda} + 12a_{3\tau\alpha 333\lambda\kappa}^3 U_{1\alpha} W_0 U_{0\kappa,\lambda} + 6a_{3\tau\alpha 33333}^3 U_{1\alpha} W_0^2 \\ & + 3a_{3\tau 3\alpha\eta\xi\lambda\kappa}^3 W_{0,\alpha} U_{0\xi,\eta} U_{0\kappa,\lambda} + 6a_{3\tau 3\alpha\eta\xi 33}^3 W_{0,\alpha} U_{0\xi,\eta} W_0 + 3a_{3\tau 3\alpha 33333}^3 W_{0,\alpha} W_0^2) \\ & + \mathcal{O}(\epsilon^3, \nu_1\epsilon) = 0 \quad (\tau = 1, 2), \end{aligned} \quad (4.3)$$

$$\begin{aligned} & a_{33\beta\alpha}^1 U_{0\alpha,\beta} + a_{3333}^1 W_0 + \nu_1(a_{33\beta\alpha}^1 U_{1\alpha,\beta} + 3a_{3333}^1 W_1) \\ & + \frac{\epsilon}{2}(a_{33\alpha\beta\eta\xi}^2 U_{0\xi,\eta} U_{0\beta,\alpha} + 2a_{33\alpha\beta 33}^2 W_0 U_{0\beta,\alpha} + a_{333333}^2 W_0^2) \\ & + \frac{\epsilon^2}{6}(a_{33\alpha\beta\eta\xi\lambda\kappa}^3 U_{0\beta,\alpha} U_{0\xi,\eta} U_{0\kappa,\lambda} + 3a_{33\alpha\beta\eta\xi 33}^3 U_{0\beta,\alpha} U_{0\xi,\eta} W_0 \\ & + 3a_{33\alpha\beta 33333}^3 U_{0\beta,\alpha} W_0^2 + a_{33333333}^3 W_0^3) \\ & + \mathcal{O}(\epsilon^3, \nu_1\epsilon) = 0. \end{aligned} \quad (4.4)$$

It should be noted that boundary conditions (4.3) and (4.4) can be expanded to any needed order. But, for the problem we consider here, we omit $\mathcal{O}(\epsilon^3, \nu_1\epsilon)$ and higher-order terms, since the purpose is to deduce the leading-order behavior. We note that the above three equations contain eight unknowns $U_{0\alpha}$, W_0 , $U_{1\alpha}$, W_1 and $U_{2\alpha}$. Thus we need another five equations to have a closed system.

Substituting Eq. (4.2) into (3.5), the left-hand side becomes a series in s , and all the coefficients of $s^n (n = 0, 1, 2, \dots)$ should be zero. As a result, we have two sets ($\tau = 1, 2$) of infinitely many equations. Among them, we only consider those contain the eight unknowns as in (4.3) and (4.4). Actually, from the coefficients of s^0 and s^1 , we obtain

$$a_{\alpha\tau\gamma\beta}^1 U_{0\beta,\alpha\gamma} + 2a_{3\tau 3\alpha}^1 U_{1\alpha} + (a_{3\tau\alpha 3}^1 + a_{\alpha\tau 33}^1) W_{0,\alpha}$$

$$\begin{aligned}
& + \epsilon(a_{\alpha\tau\gamma\beta\eta\xi}^2 U_{0\beta,\alpha\gamma} U_{0\xi,\eta} + 2a_{3\tau\alpha3\eta\xi}^2 U_{0\xi,\eta} U_{1\alpha} + a_{\alpha\tau\gamma\beta33}^2 U_{0\beta,\alpha\gamma} W_0 \\
& + (a_{3\tau3\beta\eta\xi}^2 + a_{\beta\tau33\eta\xi}^2) U_{0\xi,\eta} W_{0,\beta} + 2a_{3\tau\alpha333}^2 U_{1\alpha} W_0 \\
& + (a_{3\tau3\beta33}^2 + a_{\beta\tau3333}^2) W_0 W_{0,\beta}) \\
& + \epsilon^2(H_5) = 0 \quad (\tau = 1, 2), \tag{4.5}
\end{aligned}$$

$$\begin{aligned}
& a_{\alpha\tau\gamma\beta}^1 U_{1\beta,\alpha\gamma} + 12a_{3\tau3\alpha}^1 U_{2\alpha} + 3(a_{3\tau\alpha3}^1 + a_{\alpha\tau33}^1) W_{1,\alpha} \\
& + \epsilon(a_{\beta\tau\gamma\alpha\eta\xi}^2 (U_{1\alpha,\beta\gamma} U_{0\xi,\eta} + U_{0\alpha,\beta\gamma} U_{1\xi,\eta}) + 4(a_{\beta\tau3\alpha\eta3}^2 + a_{3\tau\beta\alpha\eta3}^2) U_{1\alpha,\beta} U_{1\eta} \\
& + 12a_{3\tau\alpha3\eta\xi}^2 U_{0\xi,\eta} U_{2\alpha} + a_{\beta\tau\gamma\alpha33}^2 U_{1\alpha,\beta\gamma} W_0 + (3a_{3\tau3\beta\eta\xi}^2 + a_{\beta\tau33\eta\xi}^2) U_{1\xi,\eta} W_{0,\beta} \\
& + 12a_{\alpha\tau3333}^2 U_{2\alpha} W_0 + 3(a_{3\tau3\beta33}^2 + a_{\beta\tau3333}^2) (W_0 W_{1,\beta} + W_1 W_{0,\beta}) \\
& + 2a_{3\tau\alpha3\eta\xi}^2 U_{1\xi,\eta} U_{1\alpha} + 2a_{\alpha\tau\beta3\eta3}^2 U_{1\eta} W_{0,\alpha\beta} + a_{\alpha\tau3\beta3\eta}^2 W_{0,\alpha\beta} W_{0,\eta} \\
& + 3a_{\alpha\tau\eta\xi33}^2 U_{0\xi,\alpha\eta} W_1 + 18a_{3\tau\alpha333}^2 U_{1\alpha} W_1 + (9a_{3\tau3\alpha33}^2 + 3a_{\alpha\tau3333}^2) W_{0,\alpha} W_1 \\
& + 3(a_{3\tau3\alpha\eta\xi}^2 + a_{\alpha\tau33\eta\xi}^2) U_{0\xi,\eta} W_{1,\alpha}) \\
& + \epsilon^2(H_6) = 0 \quad (\tau = 1, 2). \tag{4.6}
\end{aligned}$$

Similarly, substituting Eq. (4.2) into (3.6), we have a set of infinitely-many equations. We only use the equation coming from the coefficient of s^0 since only it contains the eight unknowns mentioned before. The equation takes the form:

$$\begin{aligned}
& 2(a_{33\beta\alpha}^1 + a_{\alpha3\beta3}^1) U_{1\alpha,\beta} + a_{\alpha3\beta3}^1 W_{0,\alpha\beta} + 6a_{3333}^1 W_1 \\
& + \epsilon(2(a_{\alpha3\beta3\eta\xi}^2 + a_{33\beta\alpha\eta\xi}^2) U_{1\alpha,\beta} U_{0\xi,\eta} + 2(a_{\alpha3\xi333}^2 + a_{33\xi\alpha33}^2) U_{1\alpha,\xi} W_0 \\
& + 2a_{\beta3\alpha\eta\xi3}^2 U_{0\alpha,\eta\beta} U_{1\xi} + 4a_{333\alpha\xi3}^2 W_{0,\alpha} U_{1\xi} + 4a_{\alpha333\xi3}^2 U_{1\alpha} U_{1\xi} \\
& + a_{\alpha3\beta3\eta\xi}^2 W_{0,\alpha\beta} U_{0\xi,\eta} + a_{\beta3\gamma\alpha3\xi}^2 U_{0\alpha,\beta\gamma} W_{0,\xi} + a_{\alpha3\beta333}^2 W_{0,\alpha\beta} W_0 \\
& + 2a_{\alpha3333\xi}^2 (U_{1\alpha} + W_{0,\alpha}) W_{0,\xi} + 6a_{3333\alpha\xi}^2 U_{0\xi,\alpha} W_1 + 6a_{333333}^2 W_0 W_1) \\
& + \epsilon^2(H_7) = 0. \tag{4.7}
\end{aligned}$$

Now, the equations (4.3) to (4.7) provide the eight governing equations for the eight unknowns $U_{0\alpha}$, W_0 , $U_{1\alpha}$, W_1 and $U_{2\alpha}$ and we have a closed system to work with.

To further simplify the two-dimensional system of Eqs. (4.3) to (4.7), we shall further use the smallness of the parameter ϵ through asymptotic expansions. By a perturbation method, from (4.5) we obtain

$$\begin{aligned}
U_{1\alpha} = & -\frac{1}{2A_3} (a_{\beta\alpha\lambda\gamma}^1 U_{0\lambda,\beta\gamma} + (A_2 + A_3) W_{0,\alpha}) \\
& - \frac{\epsilon}{2A_3^2} [(A_3(B_2 + B_7) - (A_2 + A_3)B_4) W_0 W_{0,\alpha}
\end{aligned}$$

$$\begin{aligned}
& + (A_3 a_{\beta\alpha\gamma\delta 33}^2 - B_4 a_{\beta\alpha\gamma\delta}^1) U_{0\delta,\beta\gamma} W_0 \\
& + (A_3 (a_{\delta\alpha 33\beta\lambda}^2 + a_{3\alpha\delta 3\beta\lambda}^2) - (a_{3\gamma\delta 3}^1 + a_{\delta\gamma 33}^1) a_{3\alpha\gamma 3\beta\lambda}^2) U_{0\lambda,\beta} W_{0,\delta} \\
& + (A_3 a_{\beta\alpha\delta\gamma\xi\eta}^2 - a_{\beta\zeta\gamma\delta}^1 a_{3\alpha 3\zeta\xi\eta}^2) U_{0\delta,\beta\gamma} U_{0\eta,\xi} \\
& + \epsilon^2(H_8),
\end{aligned} \tag{4.8}$$

where A_i and B_i ($i = 1, 2, \dots$) are defined in Appendix A. Substituting (4.8) into (4.7), we obtain

$$W_1 = \frac{A_2}{6A_3} W_{0,\alpha\alpha} + \frac{A_2 + A_3}{6A_1 A_3} a_{\beta\gamma\delta\alpha}^1 U_{0\alpha,\beta\gamma\delta} + \mathcal{O}(\epsilon), \tag{4.9}$$

where we only give the expression of the leading-order term of W_1 since the higher-order terms have no influence on the final asymptotic equations. Substituting (4.8) and (4.9) into (4.6), we obtain

$$\begin{aligned}
U_{2\alpha} = & \frac{1}{24A_1 A_3^2} (A_1 a_{\delta\zeta\gamma\beta}^1 a_{\zeta\alpha\eta\xi}^1 U_{0\beta,\gamma\delta\eta\zeta} - (A_2 + A_3)^2 a_{\zeta\beta\eta\xi}^1 U_{0\xi,\eta\zeta\beta\alpha} \\
& - A_1 (A_2 + A_3) (A_2 W_{0,\alpha\beta\beta} - a_{\zeta\alpha\eta\xi}^1 W_{0,\zeta\eta\xi})) + \mathcal{O}(\epsilon).
\end{aligned} \tag{4.10}$$

Substituting (4.8) to (4.10) into (4.3) and (4.4), we obtain

$$\begin{aligned}
& A_3 U_{0\tau,\alpha\alpha} + A_2 W_{0,\tau} + (A_2 + A_3) U_{0\alpha,\alpha\tau} \\
& - \frac{\nu_1}{6} (A_3 U_{0\tau,\alpha\alpha\beta\beta} + (3A_2 + 2A_3) W_{0,\alpha\alpha\tau} + 3(A_2 + A_3) U_{0\alpha,\alpha\beta\beta\tau}) \\
& + \frac{\epsilon}{A_3} ((A_3 a_{\beta\tau\gamma\delta 33}^2 - B_4 a_{\beta\tau\gamma\delta}^1) U_{0\delta,\beta\gamma} W_0 + (A_1 (B_7 - B_4) + A_3 B_2) W_0 W_{0,\tau} \\
& + (a_{3\tau\alpha 3\zeta\xi\eta}^2 a_{\beta\alpha\lambda\gamma}^1 + A_3 a_{\beta\tau\lambda\gamma\xi\eta}^2 - a_{\beta\zeta\gamma\lambda}^1 a_{3\tau\zeta 3\zeta\xi\eta}^2) U_{0\lambda,\beta\gamma} U_{0\eta,\xi} \\
& + (A_1 (a_{3\tau\alpha 3\beta\lambda}^2 - a_{3\tau 3\alpha\beta\lambda}^2) - A_3 a_{\alpha\tau 33\beta\lambda}^2) U_{0\lambda,\beta} W_{0,\alpha}) \\
& + \epsilon^2(H_9) = 0,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
& A_2 U_{0\alpha,\alpha} + A_1 W_0 + \frac{\nu_1}{2} (A_1 U_{0\alpha,\alpha\beta\beta} + A_2 W_{0,\alpha\alpha}) \\
& + \frac{\epsilon}{2} (a_{33\alpha\beta\eta\xi}^2 U_{0\xi,\eta} U_{0\beta,\alpha} + 2B_2 W_0 U_{0\alpha,\alpha} + B_1 W_0^2) \\
& + \frac{\epsilon^2}{6} (a_{33\alpha\beta\eta\xi\lambda\kappa}^3 U_{0\beta,\alpha} U_{0\xi,\eta} U_{0\kappa,\lambda} + 3a_{33\alpha\beta\eta\xi 33}^3 U_{0\beta,\alpha} U_{0\xi,\eta} W_0 \\
& + 3C_2 U_{0\alpha,\alpha} W_0^2 + C_1 W_0^3) = 0,
\end{aligned} \tag{4.12}$$

where we have omitted the terms higher than $\mathcal{O}(\epsilon^2, \nu_1)$.

For the two side surfaces, we suppose that they are traction-free. Also, since for a thin plate they are much smaller than the top and bottom surfaces, for

the latter we need the boundary conditions to be satisfied at every point (cf. (4.3) and (4.4)) while for the former we only require the boundary conditions to be satisfied in an average sense along the thickness. By integrating the traction-free boundary conditions $\Sigma_{2i} = 0$ at $x_2 = \pm\sqrt{\nu_2}$ (where $\nu_2 = b^2/l^2$) along the thickness from 0 to a , we obtain

$$\begin{aligned}
& A_3(U_{01,2} + U_{02,1}) \\
& - \frac{\nu_1}{6}(2(A_2 + A_3)(W_0 + U_{01,1} + U_{02,2})_{,12} + A_3(U_{01,2} + U_{02,1})_{,\alpha\alpha}) \\
& + \frac{\epsilon}{2}(a_{21\alpha\beta\gamma\delta}^2 U_{0\beta,\alpha} U_{0\delta,\gamma} + 2a_{2133\alpha\beta}^2 U_{0\beta,\alpha} W_0) \\
& + \frac{\epsilon^2}{6}(a_{21\alpha\beta\gamma\delta\kappa\lambda}^3 U_{0\beta,\alpha} U_{0\delta,\gamma} U_{0\lambda,\kappa} + 3a_{2133\alpha\beta\gamma\delta}^3 U_{0\alpha,\alpha} U_{0\delta,\gamma} W_0 \\
& + 3a_{213333\alpha\beta}^3 U_{0\alpha,\alpha} W_0^2)|_{x_2=\pm\sqrt{\nu_2}} = 0,
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
& 2A_3U_{02,2} + A_2(W_0 + U_{0\alpha,\alpha}) \\
& - \frac{\nu_1}{6}(2(A_2 + A_3)(W_0 + U_{01,1} + U_{02,2})_{,22} + (A_1U_{02,2} + A_2(W_0 + U_{01,1}))_{,\alpha\alpha}) \\
& + \frac{\epsilon}{2}(a_{22\alpha\beta\gamma\delta}^2 U_{0\beta,\alpha} U_{0\delta,\gamma} + 2a_{22\alpha\beta33}^2 U_{0\beta,\alpha} W_0 + B_2W_0^2) \\
& + \frac{\epsilon^2}{6}(a_{22\alpha\beta\gamma\delta\kappa\lambda}^3 U_{0\beta,\alpha} U_{0\delta,\gamma} U_{0\lambda,\kappa} + 3a_{22\alpha\beta\gamma\delta33}^3 U_{0\beta,\alpha} U_{0\delta,\gamma} W_0 \\
& + 3a_{22\alpha\beta3333}^3 U_{0\beta,\alpha} W_0^2 + 3C_2W_0^3)|_{x_2=\pm\sqrt{\nu_2}} = 0.
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
& A_2W_{0,2} + (A_2 + A_3)U_{0\alpha,\alpha 2} + A_3U_{02,\alpha\alpha} \\
& - \frac{\nu_1}{12}((3A_2 + 2A_3)W_{0,\alpha\alpha 2} + 3(A_2 + A_3)U_{0\alpha,\alpha\beta 2} + A_3U_{02,\alpha\alpha\beta\beta}) \\
& + \epsilon(H_{10}) + \epsilon^2(H_{11})|_{x_2=\pm\sqrt{\nu_2}} = 0.
\end{aligned} \tag{4.15}$$

Eqs. (4.11) and (4.12) are the three asymptotically-valid governing equations for the three unknowns $U_{0\alpha}$ and W_0 , among which $U_{0\alpha}$ are the two displacement components of a point in the middle plane and W_0 is the normal strain (along the thickness direction) of that point. Since the two-dimensional system of Eqs. (4.11) and (4.12) together with the six boundary conditions (4.13) to (4.15) are derived from the three-dimensional field equations, once the solution of this system is obtained, the three-dimensional displacement field (thus, also the strain and stress fields) can be easily calculated.

Remark: It is easy to see that these governing equations and boundary conditions have two significant features which are different from the equations for a

standard plane-stress problem. Firstly, the out-plane normal strain W_0 is coupled with the in-plane displacement components $U_{0\alpha}$. And secondly, there are some ν_1 terms and the orders of the derivatives of these terms are two-order higher than the other terms, which indicate the influence of the thickness of the plate. We shall see later that the thickness has an important influence on the bifurcations. Thus, a model based on a plane-stress problem may be defective for capturing the instability phenomena in a thin plate.

5 Asymptotic Normal Form Equations for a Thin Strip

Now we consider the case that the plate is a thin strip in terms that both the thickness and the width of the plate are much smaller than the length (this is in agreement with of the experimental setting of Shaw and Kyriakides 1998). Thus, besides ν_1 being small, ν_2 is also small. As a result, $-\sqrt{\nu_2} \leq x_2 \leq \sqrt{\nu_2}$ is a small variable. From Eqs. (4.11)-(4.15), it is clear that the unknowns are functions of the variable $x_1(=: x)$, the small variable $x_2(=: y)$ and the three small parameters ϵ, ν_1, ν_2 , i.e.,

$$U_{0\alpha} = U_{0\alpha}(x, y; \epsilon, \nu_1, \nu_2), \quad W_0 = W_0(x, y; \epsilon, \nu_1, \nu_2). \quad (5.1)$$

We assume that the unknowns are sufficiently smooth in y and seek the series expansions in the small variable y :

$$\begin{aligned} U_{01} &= u_0(x) + y^2 u_2(x) + y^4 u_4(x) + y^6 u_6(x) + \cdots, \\ &\quad + \sqrt{\nu_2} y \cdot (u_1(x) + y^2 u_3(x) + y^4 u_5(x) + y^6 u_7(x) + \cdots), \\ U_{02} &= \sqrt{\nu_2} \cdot (v_0(x) + y^2 v_2(x) + y^4 v_4(x) + y^6 v_6(x) + \cdots) \\ &\quad + y \cdot (v_1(x) + y^2 v_3(x) + y^4 v_5(x) + y^6 v_7(x) + \cdots), \\ W_0 &= w_0(x) + y^2 w_2(x) + y^4 w_4(x) + y^6 w_6(x) + \cdots \\ &\quad + \sqrt{\nu_2} y \cdot (w_1(x) + y^2 w_3(x) + y^4 w_5(x) + y^6 w_7(x) + \cdots), \end{aligned} \quad (5.2)$$

where $\sqrt{\nu_2}$ is introduced into the expansions based on the assumption that the maximum non-dimensional lateral displacement of a point in the center line of the middle plane is $\mathcal{O}(\sqrt{\nu_2})$ (i.e., $U_{02}|_{y=0} = \mathcal{O}(\sqrt{\nu_2})$), since in the experiment there was only a small bending (see Shaw and Kyriakides 1998).

Substituting (5.2) into (4.13) and (4.14) and omitting terms higher than $\mathcal{O}(\epsilon^2, \nu_2)$ and with some manipulations, we obtain

$$A_3(u_1 + v_{0x} + \nu_2(3u_3 + v_{2x}))$$

$$\begin{aligned}
& -\frac{\nu_1}{6}(2A_2(2v_{2x} + w_{1x} + u_{1xx}) + A_3(6u_3 + 6v_{2x} + 2w_{1x} + 3u_{1xx} + v_{0xxx})) \\
& + \epsilon(u_1(B_4(u_{0x} + v_1) + B_5w_0) + v_{0x}(B_7(u_{0x} + v_1) + B_8w_0)) \\
& + \frac{\epsilon^2}{6}(3(v_{0x}(C_{12}u_{0x}^2 + C_{12}v_1^2 + 2C_{15}v_1w_0 + C_{13}w_0^2 + 2u_{0x}(C_{14}v_1 + C_{15}w_0)) \\
& + u_1(C_5u_{0x}^2 + C_5v_1^2 + 2C_8v_1w_0 + C_6w_0^2 + 2u_{0x}(C_7v_1 + C_8w_0)))) = 0,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
& A_3(2u_2 + v_{1x} + \nu_2(4u_4 + v_{3x})) \\
& -\frac{\nu_1}{6}(4A_2(3v_{3x} + w_{2x} + u_{2xx}) + A_3(24u_4 + 18v_{3x} + 4w_{2x} + 6u_{2xx} + v_{1xxx})) \\
& + \epsilon(2u_2(B_4(u_{0x} + v_1) + B_5w_0) + v_{1x}(B_7(u_{0x} + v_1) + B_8w_0)) \\
& + \frac{\epsilon^2}{6}(3(v_{1x}(C_{12}u_{0x}^2 + C_{12}v_1^2 + 2C_{15}v_1w_0 + C_{13}w_0^2 + 2u_{0x}(C_{14}v_1 + C_{15}w_0)) \\
& + 2u_2(C_5u_{0x}^2 + C_5v_1^2 + 2C_8v_1w_0 + C_6w_0^2 + 2u_{0x}(C_7v_1 + C_8w_0)))) = 0,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
& A_2(u_{0x} + w_0 + v_1) + 2A_3v_1 + \nu_2(A_2(u_{2x} + w_2) + 3(A_2 + 2A_3)v_3) \\
& -\frac{\nu_1}{6}(6(3A_2 + 4A_3)v_3 + 2(3A_2 + 2A_3)(w_2 + u_{2x}) \\
& + A_1v_{1xx} + A_2(w_{0xx} + u_{0xxx})) \\
& + \frac{\epsilon}{2}(B_2u_{0x}^2 + B_1v_1^2 + 2B_2v_1w_0 + B_2w_0^2 + 2u_{0x}(B_2v_1 + B_3w_0)) \\
& + \frac{\epsilon^2}{6}(C_2u_{0x}^3 + C_1v_1^3 + 3C_2v_1^2w_0 + 3C_3v_1w_0^2 + C_2w_0^3 \\
& + 3u_{0x}^2(C_3v_1 + C_4w_0) + 3u_{0x}(C_2v_1^2 + 2C_4v_1w_0 + C_4w_0^2)) = 0,
\end{aligned} \tag{5.5}$$

$$\begin{aligned}
& A_2(u_{1x} + w_1 + 2v_2) + 4A_3v_2 + \nu_2(A_2(u_{3x} + 4v_4 + w_3) + 8A_3v_4) \\
& -\frac{\nu_1}{6}(24(3A_2 + 4A_3)v_4 + 6(3A_2 + 2A_3)(w_3 + u_{3x}) \\
& + 2A_1v_{2xx} + A_2(w_{1xx} + u_{1xxx})) \\
& + \epsilon(2u_2(B_4u_1 + B_7v_{0x}) + (B_7u_1 + B_4v_{0x})v_{1x} + 2B_2v_2(u_{0x} + w_0) \\
& + u_{1x}(B_2u_{0x} + B_3w_0) + (B_3u_{0x} + B_2w_0)w_1 \\
& + v_1(2B_1v_2 + B_2(u_{1x} + w_1))) \\
& + \frac{\epsilon^2}{6}(3(C_2u_{0x}^2u_{1x} + 4C_{14}u_{0x}u_2v_{0x} + 2C_3u_{0x}^2v_2 + 2C_4u_{0x}u_{1x}w_0 \\
& + 4C_{15}u_2v_{0x}w_0 + 4C_4u_{0x}v_2w_0 + C_4u_{1x}w_0^2 + 2C_3v_2w_0^2 \\
& + 2v_{0x}v_{1x}(C_7u_{0x} + C_8w_0) + 2u_1(v_{1x}(C_{14}u_{0x} + C_{12}v_1 + C_{15}w_0) \\
& + 2u_2(C_7u_{0x} + C_5v_1 + C_8w_0)) + C_4u_{0x}^2w_1 \\
& + 2C_4u_{0x}w_0w_1 + C_2w_0^2w_1 + 2v_1(v_{0x}(2C_{12}u_2 + C_5v_{1x}))
\end{aligned}$$

$$\begin{aligned}
& +2C_2v_2(u_{0x} + w_0) + u_{1x}(C_3u_{0x} + C_4w_0) + (C_4u_{0x} + C_3w_0)w_1 \\
& +v_1^2(2C_1v_2 + C_2(u_{1x} + w_1))) = 0.
\end{aligned} \tag{5.6}$$

We note that the above four equations contain fourteen unknowns $u_0 - u_4$, $v_0 - v_4$, $w_0 - w_3$ and to have a closed system, we need another ten equations.

Substituting (5.2) into Eq. (4.11) for $\tau = 1$, all the coefficients of y^n ($n = 1, 2, \dots$) should be zero. So, we have a set of infinitely-many equations. However, only the coefficients of y^0 , y^1 and y^2 contain the fourteen unknowns mentioned above, which yield the following three equations:

$$\begin{aligned}
& A_2(u_{0xx} + w_{0x} + v_{1x}) + A_3(2u_{0xx} + v_{1x} + 2u_2) \\
& -\frac{\nu_1}{6}(A_2(3u_{0xxx} + 6u_{2xx} + 3w_{0xxx} + 6w_{2x} + 18v_{3x} + 3v_{1xxx}) \\
& +A_3(4u_{0xxx} + 10u_{2xx} + 24u_4 + 2w_{0xxx} + 4w_{2x} + 18v_{3x} + 3v_{1xxx})) \\
& +\epsilon(B_2u_{0x}v_{1x} + B_7u_{0x}v_{1x} + B_3v_{1x}w_0 + B_8v_{1x}w_0 + u_{0xx}(B_1u_{0x} + B_2w_0) \\
& +2u_2(B_4(u_{0x} + v_1) + B_5w_0) + B_2u_{0x}w_{0x} + B_2w_0w_{0x} \\
& +v_1(B_2u_{0xx} + (B_2 + B_7)v_{1x} + B_3w_{0x})) \\
& +\frac{\epsilon^2}{2}(C_{12}u_{0x}^2v_{1x} + C_2u_{0x}^2v_{1x} + 2C_{15}u_{0x}v_{1x}w_0 \\
& +2C_4u_{0x}v_{1x}w_0 + C_{13}v_{1x}w_0^2 + C_4v_{1x}w_0^2 \\
& +u_{0xx}(C_1u_{0x}^2 + 2C_2u_{0x}w_0 + C_3w_0^2) + 2u_2(C_5u_{0x}^2 + C_5v_1^2 + 2C_8v_1w_0 \\
& +C_6w_0^2 + 2u_{0x}(C_7v_1 + C_8w_0)) + C_2u_{0x}^2w_{0x} + 2C_3u_{0x}w_0w_{0x} \\
& +C_2w_0^2w_{0x} + v_1^2(C_3u_{0xx} + (C_{12} + C_2)v_{1x} + C_4w_{0x}) + 2v_1(u_{0xx}(C_2u_{0xx} \\
& +(C_{14} + C_3)v_{1x} + C_4w_{0x}) + w_0((C_{15} + C_4)v_{1x} + C_4(u_{0xx} + w_{0x})))) = 0,
\end{aligned} \tag{5.7}$$

$$A_2(u_{1xx} + w_{1x} + 2v_{2x}) + 2A_3(u_{1xx} + v_{2x} + 3u_3) + \mathcal{O}(\epsilon) = 0, \tag{5.8}$$

$$A_2(u_{2xx} + 3v_{3x} + w_{2x}) + A_3(2u_{2xx} + 3v_{3x} + 12u_4) + \mathcal{O}(\epsilon) = 0. \tag{5.9}$$

The $\mathcal{O}(\epsilon)$ terms in (5.8) and (5.9) will not make contributions to the final results and are thus not written out. Similarly, substituting (5.2) into Eq. (4.11) for $\tau = 2$ and from the coefficients of y^0 , y^1 and y^2 , we obtain

$$\begin{aligned}
& A_2(u_{1x} + w_1 + 2v_2) + A_3(u_{1x} + v_{0xx} + 4v_2) \\
& -\frac{\nu_1}{6}(A_2(3u_{1xxx} + 18u_{3x} + 3w_{1xx} + 18w_3 + 72v_4 + 6v_{2xx}) \\
& +A_3(3u_{1xxx} + 18u_{3x} + 2w_{1xx} + 12w_3 + 96v_4 + 10v_{2xx} + v_{0xxx})) \\
& +\epsilon(2B_2u_{0x}v_2 + 2B_1v_1v_2 + 2B_2v_2w_0 + v_{0xx}(B_4(u_{0x} + v_1) + B_5w_0) \\
& +u_{1x}((B_2 + B_7)(u_{0x} + v_1) + (B_3 + B_8)w_0) \\
& +v_{0x}(B_4u_{0xx} + 2B_7u_2 + 2B_4v_{1x} + B_5w_{0x}) + B_3u_{0x}w_1
\end{aligned}$$

$$\begin{aligned}
& +u_1(B_7u_{0xx} + 2B_4u_2 + 2B_7v_{1x} + B_8w_{0x}) + B_2v_1w_1 + B_2w_0w_1) \\
& +\frac{\epsilon^2}{2}(C_{12}u_{0x}^2u_{1x} + C_2u_{0x}^2u_{1x} + 2C_5u_{0x}u_{0xx}v_{0x} + 4C_{14}u_{0x}u_2v_{0x} \\
& +4C_7u_{0x}v_{0x}v_{1x} + 2C_3u_{0x}^2v_2 + 2C_{15}u_{0x}u_{1x}w_0 + 2C_4u_{0x}u_{1x}w_0 \\
& +2C_8u_{0xx}v_{0x}w_0 + 4C_{15}u_2v_{0x}w_0 + 4C_8v_{0x}v_{1x}w_0 + 4C_4u_{0x}v_2w_0 \\
& +C_{13}u_{1x}w_0^2 + C_4u_{1x}w_0^2 + 2C_3v_2w_0^2 + C_4u_{0x}^2w_1 \\
& +v_{0xx}(C_5u_{0x}^2 + 2C_8u_{0x}w_0 + C_6w_0^2) + 2C_8u_{0x}v_{0x}w_{0x} + 2C_6v_{0x}w_0w_{0x} \\
& +2u_1(2((C_{14}u_{0x} + C_{12}v_1)v_{1x} + C_{15}v_{1x}w_0) + u_{0xx}(C_{12}u_{0x} + C_{14}v_1 + C_{15}w_0) \\
& +2u_2(C_7u_{0x} + C_5v_1 + C_8w_0) + C_{15}u_{0x}w_{0x} + C_{15}v_1w_{0x} + C_{13}w_0w_{0x}) \\
& +2C_4u_{0x}w_0w_1 + C_2w_0^2w_1 + v_1^2((C_{12} + C_2)u_{1x} + C_5v_{0xx} + 2C_1v_2 + C_2w_1) \\
& +2v_1(2C_2v_2(u_{0x} + w_0) + v_{0x}(C_7u_{0xx} + 2C_{12}u_2 + 2C_5v_{1x} + C_8w_{0x}) \\
& +w_0((C_{15} + C_4)u_{1x} + C_8v_{0xx} + C_3w_1) \\
& +u_{0x}((C_{14} + C_3)u_{1x} + C_7v_{0xx} + C_4w_1))) = 0, \tag{5.10}
\end{aligned}$$

$$2A_2(u_{2x} + w_2 + 3v_3) + A_3(2u_{2x} + v_{1xx} + 12v_3) + \mathcal{O}(\epsilon) = 0, \tag{5.11}$$

$$3A_2(u_{3x} + w_3 + 4v_4) + A_3(3u_{3x} + v_{2xx} + 24v_4) + \mathcal{O}(\epsilon) = 0. \tag{5.12}$$

Substituting (5.2) into (4.12) and from the coefficients of y^0 , y^1 , y^2 and y^3 , we obtain

$$\begin{aligned}
& A_2(u_{0x} + w_0 + v_1) + 2A_3w_0 \\
& -\frac{\nu_1}{6}(A_2(3u_{0xxx} + 6u_{2x} + 3w_{0xx} + 6w_2 + 18v_3 + 3v_{1xx}) \\
& +A_3(6u_{0xxx} + 12u_{2x} + 18v_3 + 6v_{1xx})) \\
& +\frac{\epsilon}{2}(B_2u_{0x}^2 + B_2v_1^2 + 2B_2v_1w_0 + B_1w_0^2 + 2u_{0x}(B_3v_1 + B_2w_0)) \\
& +\frac{\epsilon^2}{6}(C_2u_{0x}^3 + C_2v_1^3 + 3C_3v_1^2w_0 + 3C_2v_1w_0^2 + C_1w_0^3 \\
& +3u_{0x}^2(C_4v_1 + C_3w_0) + 3u_{0x}(C_4v_1^2 + 2C_4v_1w_0 + C_2w_0^2)) = 0, \tag{5.13}
\end{aligned}$$

$$\begin{aligned}
& A_2(u_{1x} + w_1 + 2v_2) + 2A_3w_1 \\
& -\frac{\nu_1}{6}(A_2(3u_{1xxx} + 18u_{3x} + 3w_{1xx} + 18w_3 + 72v_4 + 6v_{2xx}) \\
& +A_3(6u_{1xxx} + 36u_{3x} + 144v_4 + 12v_{2xx})) \\
& +\epsilon(2u_2(B_5u_1 + B_8v_{0x}) + (B_8u_1 + B_5v_{0x})v_{1x} + B_2u_{1x}(u_{0x} + w_0) \\
& +2v_2(B_3u_{0x} + B_2w_0) + (B_2u_{0x} + B_1w_0)w_1 + v_1(B_3u_{1x} + 2B_2v_2 + B_2w_1)) \\
& +\frac{\epsilon^2}{6}(3(C_2u_{0x}^2u_{1x} + 4C_{15}u_{0x}u_2v_{0x} + 2C_3u_{0x}u_{1x}w_0 + 4C_{13}u_2v_{0x}w_0 \\
& +2C_4u_{0x}^2v_2 + 4C_4u_{0x}v_2w_0 + C_2(u_{1x} + 2v_2)w_0^2 + 2v_{0x}v_{1x}(C_8u_{0x} + C_6w_0) \\
& +2u_1(v_{1x}(C_{15}(u_{0x} + v_1) + C_{13}w_0) + 2u_2(C_8(u_{0x} + v_1) + C_6w_0)) \\
& +C_3u_{0x}^2w_1 + 2C_2u_{0x}w_0w_1 + C_1w_0^2w_1 + v_1^2(C_4u_{1x} + 2C_2v_2 + C_3w_1)
\end{aligned}$$

$$+2v_1(v_{0x}(2C_{15}u_2 + C_8v_{1x}) + C_4u_{1x}(u_{0x} + w_0) + 2v_2(C_4u_{0x} + C_3w_0) + (C_4u_{0x} + C_2w_0)w_1))) = 0, \quad (5.14)$$

$$A_2(u_{2x} + w_2 + 3v_3) + 2A_3w_2 + \mathcal{O}(\epsilon) = 0, \quad (5.15)$$

$$A_2(u_{3x} + w_3 + 4v_4) + 2A_3w_3 + \mathcal{O}(\epsilon) = 0. \quad (5.16)$$

Now, we have a closed system: the fourteen equations (5.3) to (5.16) provide the governing equations for fourteen unknowns $u_0 - u_4$, $v_0 - v_4$ and $w_0 - w_3$. Please note that the boundary conditions (4.15) can be satisfied automatically, and so it will not be used. It is still formidable to analyze the solutions of this system directly. Next, we shall use the smallness of the three parameters ϵ , ν_1 and ν_2 to proceed further.

By a regular perturbation expansion, from (5.3) we can obtain the expression of u_1 as a function of u_0 and other unknowns. By substituting this expression of u_1 into the remain equations of (5.3) to (5.16), we can eliminate u_1 from all these equations. Similarly, by solving the resulting equation of $(5.7) \times A_2 + (5.4) \times (7A_2 + 4A_3)$, we can obtain u_2 as a function of u_0 , v_0 , $\dots$. And by substituting this expression of u_2 into the remain equations of (5.3) to (5.16), we can eliminate u_2 from all these equations. Similarly, we can express $u_1 - u_4$, $w_0 - w_3$ and $v_1 - v_4$ in terms of u_0 and v_0 . The concrete forms are

$$u_1 = -v_{0x} + \frac{A_2\nu_1 - 3(3A_2 + 2A_3)\nu_2}{6(A_2 + A_3)}v_{0xxx} + \epsilon \frac{3A_2 + 2A_3}{2(A_2 + A_3)}u_{0x}v_{0x} + \epsilon^2(H_{12})u_{0x}^2v_{0x}, \quad (5.17)$$

$$u_2 = \frac{A_2^2}{8(A_2 + A_3)(2A_2 + A_3)}u_{0xx} - \frac{A_2(3A_2 + 2A_3)}{192(A_2 + A_3)^2(2A_2 + A_3)}((15A_2 + 8A_3)\nu_1 - 3(3A_2 + 2A_3)\nu_2)u_{0xxxx} + \frac{\epsilon(3A_2 + 2A_3)}{32A_3(A_2 + A_3)^3(2A_2 + A_3)}(4A_3^3B_2 - 4A_2^3(A_3 + B_6) - 2A_2A_3^2(4B_6 - B_2 + B_3) - A_2^2A_3(5A_3 + 4(B_6 - B_2 + B_3)))u_{0x}u_{0xx} + \epsilon^2(H_{13})u_{0x}^2u_{0xx}, \quad (5.18)$$

$$u_3 = \frac{5A_2 + 4A_3}{12(A_2 + A_3)}v_{0xxx} + \mathcal{O}(\epsilon), \quad (5.19)$$

$$u_4 = -\frac{A_2(19A_2^2 + 20A_2A_3 + 4A_3^2)}{384(A_2 + A_3)^2(2A_2 + A_3)}u_{0xxxx} + \mathcal{O}(\epsilon), \quad (5.20)$$

$$w_3 = -\frac{A_2}{12(A_2 + A_3)}v_{0xxxx} + \mathcal{O}(\epsilon), \quad (5.21)$$

$$w_2 = -\frac{A_2^2(5A_2 + 2A_3)}{32(A_2 + 2A_3)^2(2A_2 + A_3)}u_{0xxx} + \mathcal{O}(\epsilon), \quad (5.22)$$

$$\begin{aligned} w_1 = & \frac{A_2}{2(A_2 + A_3)}v_{0xx} + \frac{A_2(5(3A_2 + 2A_3)\nu_2 - A_2\nu_1)}{24(A_2 + A_3)^2}v_{0xxxx} \\ & - \epsilon \frac{A_2(19A_2^2 + 26A_2A_3 + 82A_3^2)}{16(A_2 + A_3)^2(2A_2 + A_3)}u_{0xx}v_{0x} \\ & + \epsilon \left(\frac{4A_3^2B_2 + A_2^2(B_1 + 3B_2 - 4B_3) + 4A_2A_3(B_2 - B_3)}{8(A_2 + A_3)^3} \right. \\ & \quad \left. - \frac{A_2(3A_2 + 2A_3)}{4(A_2 + A_3)^2} \right) u_{0x}v_{0xx} \\ & + \epsilon^2(H_{14}u_{0xx}v_{0x} + H_{15}u_{0x}v_{0xx})u_{0x}, \end{aligned} \quad (5.23)$$

$$\begin{aligned} w_0 = & -\frac{A_2}{2(A_2 + A_3)}u_{0x} \\ & + \frac{3A_2^2(3A_2 + 2A_3)\nu_2 - (71A_2^3 + 208A_2^2A_3 + 180A_2A_3^2 + 48A_3^3)\nu_1}{96(A_2 + A_3)^2(2A_2 + A_3)}u_{0xxx} \\ & - \epsilon \frac{4A_3^2B_2 + A_2^2(B_1 + 3B_2 - 4B_3) + 4A_2A_3(B_2 - B_3)}{16(A_2 + A_3)^3}u_{0x}^2 \\ & + \epsilon^2(H_{16})u_{0x}^3, \end{aligned} \quad (5.24)$$

$$v_4 = -\frac{2A_2 + A_3}{24(A_2 + A_3)}v_{0xxxx} + \mathcal{O}(\epsilon), \quad (5.25)$$

$$v_3 = \frac{A_2(A_2^2 + 8A_2A_3 + 4A_3^2)}{96(A_2 + A_3)^2(2A_2 + A_3)}u_{0xxx} + \mathcal{O}(\epsilon), \quad (5.26)$$

$$\begin{aligned} v_2 = & \frac{A_2}{4(A_2 + A_3)}v_{0xx} + \frac{(3A_2 + 2A_3)((7A_2 + 2A_3)\nu_2 - A_2\nu_1)}{48(A_2 + A_3)^2}v_{0xxxx} \\ & - \epsilon \frac{A_2(3A_2 + 2A_3)^2}{32(A_2 + A_3)^2(2A_2 + A_3)}u_{0xx}v_{0x} \\ & + \epsilon \left(\frac{4A_3^2B_2 + A_2^2(B_1 + 3B_2 - 4B_3) + 4A_2A_3(B_2 - B_3)}{16(A_2 + A_3)^3} \right. \\ & \quad \left. - \frac{A_2(3A_2 + 2A_3)}{8(A_2 + A_3)^2} \right) u_{0x}v_{0xx} \\ & + \epsilon^2(H_{17}u_{0xx}v_{0x} + H_{18}u_{0x}v_{0xx})u_{0x}, \end{aligned} \quad (5.27)$$

$$\begin{aligned} v_1 = & -\frac{A_2}{2(A_2 + A_3)}u_{0x} \\ & + \frac{A_2((71A_2^2 + 98A_2A_3 + 32A_3^2)\nu_1 - 3(A_2 + 2A_3)(3A_2 + 2A_3)\nu_2)}{96(A_2 + A_3)^2(2A_2 + A_3)}u_{0xxx} \end{aligned}$$

$$\begin{aligned}
& -\epsilon \frac{4A_3^2 B_2 + A_2^2 (B_1 + 3B_2 - 4B_3) + 4A_2 A_3 (B_2 - B_3)}{16(A_2 + A_3)^3} u_{0x}^2 \\
& + \epsilon^2 (H_{19}) u_{0x}^3.
\end{aligned} \tag{5.28}$$

By inserting (5.17) to (5.28) into (5.7) and (5.10), we obtain two governing equations for the two basic unknowns u_0 and v_0 :

$$u_{0xx} + 2D_1 \epsilon u_{0x} u_{0xx} + 3D_2 \epsilon^2 u_{0x}^2 u_{0xx} - \frac{\nu}{4} u_{0xxxx} = 0, \tag{5.29}$$

$$E((\epsilon u_{0x} + (D_1 - 1)\epsilon^2 u_{0x}^2) v_{0x})_x - \frac{E\nu_2}{3} v_{0xxxx} = 0, \tag{5.30}$$

where

$$E = \frac{A_3(3A_2 + 2A_3)}{A_2 + A_3} \tag{5.31}$$

is the Young's modulus, and

$$\begin{aligned}
D_1 = & \frac{1}{8A_3(A_2 + A_3)^2(3A_2 + 2A_3)} (4A_3^3 B_1 + 12A_2 A_3^2 (B_1 - B_2) \\
& + 6A_2^2 A_3 (2B_1 - 3B_2 + B_3) + 3A_2^3 (B_1 - 3B_2 + 2B_3)),
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
D_2 = & \frac{1}{96A_3(A_2 + A_3)^4(3A_2 + 2A_3)} \cdot \\
& \cdot ((16A_3^4(A_3 C_1 - 3B_2^2) + 16A_2 A_3^3(-6B_2^2 + 6B_2 B_3 + 5A_3 C_1 - 4A_3 C_2) \\
& + 18A_2^5(C_1 - 4C_2 + 3C_3) + 8A_2^2 A_3^2(-3B_1 B_2 - 15B_2^2 + 24B_2 B_3 - 6B_3^2 \\
& + 20A_3 C_1 - 32A_3 C_2 + 6A_3 C_3 + 6A_3 C_4) + 8A_2^3 A_3(-3B_1 B_2 - 9B_2^2 \\
& + 3B_1 B_3 + 21B_2 B_3 - 12B_3^2 + 20A_3 C_1 - 50A_3 C_2 + 18A_3 C_3 + 12A_3 C_4) \\
& + A_2^4(-3B_1^2 - 18B_1 B_2 - 27B_2^2 + 24B_1 B_3 + 72B_2 B_3 - 48B_3^2 + 82A_3 C_1 \\
& - 280A_3 C_2 + 150A_3 C_3 + 48A_3 C_4)),
\end{aligned} \tag{5.33}$$

$$\nu = \frac{2(3A_2 + 2A_3)\nu_1}{3(A_2 + A_3)}. \tag{5.34}$$

It can be seen that D_1 and D_2 are constants which depend on material constants and ν is proportional to the small parameter ν_1 . As $u_1 - u_4$, $v_1 - v_4$, $w_0 - w_3$ are expressed in terms of u_{0x} and v_{0x} , once u_{0x} and v_{0x} are found and all these quantities can also be found.

Integrating (5.29) with respect to x once, we obtain

$$u_{0x} + D_1 \epsilon u_{0x}^2 + D_2 \epsilon^2 u_{0x}^3 - \frac{\nu}{4} u_{0xxx} = C, \tag{5.35}$$

where C is an integration constant. It is important to find the physical meaning of C , since we aim to investigate the instability phenomena as the physical parameters vary. For that purpose, we consider the resultant axial force T acting on the material cross section that is planar and perpendicular to the strip axis in the reference configuration, and the formula is

$$T = \int_{-b}^b \int_{-a}^a \Sigma_{11} dX_3 dX_2. \quad (5.36)$$

By using Eqs. (2.4), (4.2), (4.8)-(4.10) and (5.17)-(5.28) in (5.36), it is possible to express Σ_{11} in terms of u_{0x} and v_{0x} . Then, carrying out the integration in (5.36), we find that

$$T = 4abE\epsilon(u_{0x} + D_1\epsilon u_{0x}^2 + D_2\epsilon^2 u_{0x}^3 - \frac{\nu}{4}u_{0xxx}). \quad (5.37)$$

Comparing Eqs. (5.35) and (5.37), we have $C = \frac{T}{4abE\epsilon}$. Thus, we can rewrite (5.35) as

$$\epsilon u_{0x} + D_1(\epsilon u_{0x})^2 + D_2(\epsilon u_{0x})^3 - \frac{\nu}{4}\epsilon u_{0xxx} = \frac{T}{4abE}. \quad (5.38)$$

If we retain the original dimensional variable and let $U = u_{0X} = \epsilon u_{0x}$ (where $X = lx = X_1$), we have

$$U + D_1U^2 + D_2U^3 - \frac{a_1^2}{4}U_{XX} = \gamma, \quad (5.39)$$

where

$$\gamma = \frac{T}{4abE} \quad (5.40)$$

is the engineering stress and $a_1^2 = \frac{2(3A_2+2A_3)a^2}{3(A_2+A_3)}$.

Similarly, integrating (5.30) with respect to x once, we obtain

$$E[\epsilon u_{0x} + (D_1 - 1)\epsilon^2 u_{0x}^2]v_{0x} - \frac{E\nu_2}{3}v_{0xxx} = D, \quad (5.41)$$

where D is an integration constant. To find out the physical meaning of D , we consider the resultant shear force Q acting on the material cross section that

is planar and perpendicular to the strip axis in the reference configuration, and the formula is

$$Q = \int_{-b-a}^b \int_{-a}^a \Sigma_{12} dX_3 dX_2. \quad (5.42)$$

By using Eqs. (2.4), (4.2), (4.8)-(4.10) and (5.17)-(5.28) in (5.42), it is possible to express Σ_{12} in terms of u_{0x} and v_{0x} . Then, carrying out the integration in (5.42), we find that

$$Q = 4abE\sqrt{\nu_2}\epsilon((\epsilon u_{0x} + (D_1 - 1)\epsilon^2 u_{0x}^2)v_{0x} - \frac{\nu_2}{3}v_{0xxx}). \quad (5.43)$$

Comparing Eqs. (5.41) and (5.43), we have $D = \frac{Q}{4ab\sqrt{\nu_2}\epsilon}$. Thus, we can rewrite (5.41) as

$$4abE((\epsilon u_{0x} + (D_1 - 1)\epsilon^2 u_{0x}^2)\epsilon\sqrt{\nu_2}v_{0x} - \frac{\nu_2}{3}\epsilon\sqrt{\nu_2}v_{0xxx}) = Q. \quad (5.44)$$

If we retain the original dimensional variables and let $V = \epsilon\sqrt{\nu_2}v_{0x}$ ($= u_{2,1}|_{(X_2, X_3)=(0,0)}$, where $u_{2,1}$ is defined in (2.2)), we have

$$EA(U + (D_1 - 1)U^2)V - EJV_{XX} = Q, \quad (5.45)$$

where A is the area of the cross section and J is the moment inertia around the X_3 -axis of the cross section.

Once U (i.e., u_{0x}) and V (i.e., v_{0x}) are found from equations (5.39) and (5.45), all the other physical quantities can be calculated immediately. Also, since these two equations are derived in a mathematically consistent manner and contain all the required terms to yield the leading-term behavior of the original three-dimensional problem, we call them to be the asymptotic normal form equations of the governing three-dimensional nonlinear PDE's (2.5) with the traction-free boundary conditions on the top/bottom and two side surfaces under the given end axial resultant T and end shear resultant Q .

Remarks: The results established by us can also provide some useful information on some classical results obtained before in an ad hoc manner. The details are discussed below.

In the case that U is constant, and the nonlinear terms in (5.39) are neglected, we have $U = \gamma$. If the shear force Q is zero and the nonlinear term

U^2 is neglected, then Eq. (5.45) equation becomes the linearized Euler bucking equation, which was obtained based on the *hypothesis* that the bending moment is proportional to the curvature. Here, we have derived this classical equation based on a three-dimensional setting in a mathematically consistent manner without such a hypothesis.

We can also obtain the moment M acting on the X_3 -axis of the material cross section. The formula is

$$M = - \int_{-b}^b \int_{-a}^a \Sigma_{11}(X_2 + u_2) dX_3 dX_2. \quad (5.46)$$

where u_2 is the displacement component along the X_2 -axis direction. By the same procedure as above, carrying out the integration in (5.46), we find that

$$M = -4ablE\epsilon\sqrt{\nu_2}((\epsilon u_{0x} + D_1\epsilon^2 u_{0x}^2)v_0 - \frac{\nu_2}{3}v_{0xx}). \quad (5.47)$$

If we retain the original dimensional variable and let $\mathcal{V} = l\epsilon\sqrt{\nu_2}v_0$ ($= u_2|_{(X_2, X_3)=(0,0)}$), we have

$$M = EJ\mathcal{V}_{XX} - EA(U + D_1U^2)\mathcal{V}. \quad (5.48)$$

Clearly, we have

$$V = \mathcal{V}_X. \quad (5.49)$$

Note that if we let M be zero and do not consider the effect of the nonlinear term U^2 , equation (5.48) also becomes the classical Euler bucking equation.

It should also be noted that in general that $M_X \neq -Q$ as can be seen from (5.47) and (5.43). But in the case that u_{0x} is constant, denoting $\hat{x}$ the deformed coordinate of the x axis (namely, $\hat{x} = X + u_0$), then by neglecting the $\mathcal{O}(\epsilon^4\nu_2^{1/2}, \epsilon^2\nu_2^{3/2})$ terms, we have

$$\frac{dM}{d\hat{x}} = \frac{M_X}{1 + \epsilon u_{0x}} = -Q. \quad (5.50)$$

Thus, here we have deduced the restriction under which the above classical result for a beam is valid.

6 Solutions for an Infinitely-long Strip

According to the experiments (Shaw and Kyriakides 1998, Sun et al 2000, Tse and Sun 2000), for the phase transitions in a thin strip due to tension/extension, there are at least three instability phenomena: (i) there is a formation of the transformation fronts (manifested as a neck); (ii) the neighboring two transformation fronts incline a same angle with the axial axis; (iii) the transformation front can switch to an orientation with the opposite angle. We refer (i) as a necking-type instability, (ii) as a shear instability and (iii) as a front-orientation instability.

Now, we analyze the asymptotic normal form equations (5.39) and (5.45) in order to shed insight into the instability phenomena observed in the experiments.

As mentioned in Section 2, we consider this class of non-convex strain energy functions such that in a one-dimensional stress setting with a homogeneous strain state the engineering stress-strain curve has a local maximum and a local minimum, which is same as that considered in the classical paper by Ericksen (1975). To the third-order material nonlinearity, the stress-strain relation in this setting is provided by

$$U + D_1 U^2 + D_2 U^3 = \gamma$$

(i.e., in (5.39) by setting $U_{XX} = 0$, since U is independent of X in a homogeneous strain state). The requirement that the $\gamma - U$ curve has a local maximum and minimum is equivalent to

$$D_1 < 0, \quad D_2 > 0, \quad 3D_2 < D_1^2 < 4D_2. \quad (6.1)$$

The peak stress value γ_2 , the valley stress value γ_1 and the Maxwell stress value γ_m can be expressed in terms of D_1 and D_2 (cf. Dai & Cai (2006)):

$$\begin{aligned} \gamma_1 &= \frac{2D_1^3 - 2(D_1^2 - 3D_2)^{3/2} - 9D_1D_2}{27D_2^2}, \\ \gamma_2 &= \frac{2D_1^3 + 2(D_1^2 - 3D_2)^{3/2} - 9D_1D_2}{27D_2^2}, \\ \gamma_m &= \frac{2D_1^3 - 9D_1D_2}{27D_2^2}. \end{aligned} \quad (6.2)$$

The first normal form equation (5.39) has the same form as that derived for a slender cylinder composed of an incompressible hyperelastic material (see Cai

and Dai 2006 and Dai and Cai 2006). In those two papers, it has been shown that this equation can be used to describe the necking-type instability and to capture the main features of the structure response (engineering stress-strain) curve. Thus, here we shall not study this equation and discuss the necking-type instability further. Instead, we concentrate on the other two instabilities: the shear instability and the front-orientation instability. For that purpose, we shall conduct a detailed study on the second normal form equation (5.45).

We focus on the case that there are two transformation fronts. If they are some distance away from the two ends of the strip and any other transformation front (if present), without loss of generality, we can take the strip to be infinitely-long. Also we consider the case that the resultant shear force is zero (for an infinitely-long strip, it has to be zero otherwise the moment is infinite). Then to determine the solutions of the normal form equation (5.45) becomes an eigenvalue problem with the eigenvalue equation

$$EA(U + (D_1 - 1)U^2)V - EJV_{XX} = 0,$$

or

$$c^2(-U + (1 - D_1)U^2)V + V_{XX} = 0, \quad (6.3)$$

where $c = \sqrt{EA/EJ} = \frac{\sqrt{3}}{b}$ is a large parameter for a strip, and the boundary conditions are

$$V = 0, \quad \text{at} \quad X = \pm\infty. \quad (6.4)$$

To solve (6.3) under (6.4), one needs to solve the first normal form equation (5.39) to get $U(X; \gamma)$. It can be viewed that γ is the eigenvalue for (6.3). Denote $f(X; \gamma) = -U + (1 - D_1)U^2$, then we can write (6.3) as

$$V_{XX} + c^2 f(X; \gamma)V = 0. \quad (6.5)$$

Since c is a large parameter, it is possible to use the WKB method to construct the leading-order asymptotic solution (see Holmes 1998). There are three cases. If $f(X; \gamma) > 0$ for $-\infty < X < +\infty$, the general solution (to the leading order) is

$$V = E_1 \frac{e^{ic \int \sqrt{f(X; \gamma)} dX}}{[f(X; \gamma)]^{1/4}} + E_2 \frac{e^{-ic \int \sqrt{f(X; \gamma)} dX}}{[f(X; \gamma)]^{1/4}}. \quad (6.6)$$

If $f(X; \gamma) < 0$ for $-\infty < X < +\infty$, the general solution is

$$V = E_3 \frac{e^{c \int \sqrt{-f(X;\gamma)} dX}}{[-f(X;\gamma)]^{1/4}} + E_4 \frac{e^{-c \int \sqrt{-f(X;\gamma)} dX}}{[-f(X;\gamma)]^{1/4}}. \quad (6.7)$$

If there are two turning points at $X_0(> 0)$ and $-X_0$, i.e., $f(\pm X_0; \gamma) = 0$, $f(X; \gamma) < 0$ for $X_0 < X < +\infty$ and $-\infty < X < -X_0$ and $f(X; \gamma) > 0$ for $-X_0 < X < X_0$, the general solution is

$$V = \begin{cases} (-f(X; \gamma))^{-1/4} (C_1 e^{-c \int_{X_0}^X \sqrt{-f(t; \gamma)} dt} + C_1' e^{c \int_{X_0}^X \sqrt{-f(t; \gamma)} dt}) & \text{for } X > X_0 + \delta_1, \\ C_2 \text{Ai}(C_0(X - X_0)) + C_3 \text{Bi}(C_0(X - X_0)) & \text{for } X \in (X_0 - \delta_1, X_0 + \delta_1], \\ (f(X; \gamma))^{-1/4} (C_4 \sin(c \int_0^X \sqrt{f(t; \gamma)} dt) + C_5 \cos(c \int_0^X \sqrt{f(t; \gamma)} dt)) & \text{for } X \in [-X_0 + \delta_2, X_0 - \delta_1], \\ C_6 \text{Ai}(C_0(X + X_0)) + C_7 \text{Bi}(C_0(X + X_0)) & \text{for } X \in [-X_0 - \delta_2, -X_0 + \delta_2], \\ (-f(X; \gamma))^{-1/4} (C_8' e^{-c \int_{-X_0}^X \sqrt{-f(t; \gamma)} dt} + C_8 e^{c \int_{-X_0}^X \sqrt{-f(t; \gamma)} dt}) & \text{for } X < -X_0 - \delta_2, \end{cases} \quad (6.8)$$

where $C_1, C_1', \dots, C_8$ are arbitrary constants, $\text{Ai}(\cdot)$ and $\text{Bi}(\cdot)$ are the Airy functions of the first and second kinds respectively, $C_0 = |c^2 f_X(X_0; \gamma)|^{1/3}$ and δ_1, δ_2 are quantities of $\mathcal{O}(c^{-2/3})$.

According to the experiments, the shear instability happens immediately after the formation of the two transformation fronts. Thus, we consider the anti-solitary wave solution of the first normal form equation (5.39) (which corresponding to the profile with two transformation fronts; cf. Figure 5 of Dai and Cai 2006). According to (6.9) of Dai and Cai (2006), the solution expression is

$$U = \frac{g_{2max} - \alpha_2 H_2 \tanh^2(\frac{X}{ga_1})}{1 - H_2 \tanh^2(\frac{X}{ga_1})}, \quad (6.9)$$

where

$$H_2 = \frac{g_{2max} - g_1}{\alpha_2 - g_1}, \quad g = \frac{\sqrt{2}}{\sqrt{(\alpha_2 - g_1)(g_{2max} - g_1)D_2}}, \quad (6.10)$$

and g_1 is a double root of and g_{2max} and α_2 are simple roots of

$$\frac{1}{2}U^2 + \frac{1}{3}D_1U^3 + \frac{1}{4}D_2U^4 - \gamma U - H = 0. \quad (6.11)$$

For this solution, it is easy to deduce that there are two turning points at $\pm X_0$ given by

$$X_0 = ga_1 \operatorname{arctanh} \sqrt{\frac{g_{2max} - U_0}{H_2(\alpha_2 - U_0)}}, \quad (6.12)$$

where

$$U_0 = U(X_0; \gamma) = \frac{1}{1 - D_1}. \quad (6.13)$$

Thus, in this case the general solution is given by (6.8). Upon using (6.4), it can be seen that $C'_1 = C'_8 = 0$. Also, simple calculations show that $f_X(X_0; \gamma) = U_X(X_0; \gamma)$, and thus $C_0 = |c^2 f_X(X_0; \gamma)|^{1/3} = |c^2 U_X(X_0; \gamma)|^{1/3}$.

Remark: Since a_1 is proportional to the strip thickness a (see the relation below (5.40)), thus the position of the turning point is also proportional to a . Thus, the thickness of the strip plays an important role in the instability phenomena.

By using the matching conditions at the neighborhood of $X = X_0$, we can obtain relationships between constants C_1, C_2, C_3, C_4 and C_5 in (6.8). At the neighborhood of $X = X_0$, we have $f(X; \gamma) \sim U_X(X_0; \gamma)(X - X_0)$. The integral in (6.8)₃ can be written as

$$\begin{aligned} c \int_0^X \sqrt{f(t; \gamma)} dt &= c \int_0^{X_0} \sqrt{f(t; \gamma)} dt - c \int_X^{X_0} \sqrt{f(t; \gamma)} dt \\ &= f_1(\gamma) - \frac{2}{3} |C_0(X_0 - X)|^{3/2}, \end{aligned} \quad (6.14)$$

where

$$\begin{aligned} f_1(\gamma) &= c \int_0^{X_0} \sqrt{f(t; \gamma)} dt = a_1 c \sqrt{\frac{1 - D_1}{2D_2}} \int_{U_0}^{g_{2max}} \sqrt{\frac{U(U - U_0)}{(\alpha_2 - U)(g_{2max} - U)}} \frac{dU}{U - g_1} \\ &= \frac{a_1}{b} (e_1 \Pi(\beta_1^2, \tilde{\kappa}) + \tilde{g} \Pi(\beta_2^2, \tilde{\kappa}) + e_2 K(\tilde{\kappa})), \end{aligned} \quad (6.15)$$

$$\begin{aligned}
e_1 &= \frac{g_1(U_0 - g_1)}{(\alpha_2 - g_1)(g_{2max} - g_1)} \tilde{g}, & e_2 &= -\frac{\alpha_2(\alpha_2 - U_0)}{(\alpha_2 - g_{2max})(\alpha_2 - g_1)} \tilde{g}, \\
\tilde{g} &= -\frac{\sqrt{6}(\alpha_2 - g_{2max})}{\sqrt{g_2 U_0(\alpha_2 - U_0) D_2}}, & \tilde{\kappa} &= \sqrt{\frac{\alpha_2(g_{2max} - U_0)}{g_{2max}(\alpha_2 - U_0)}}, \\
\beta_1^2 &= \tanh^2\left(\frac{X_0}{ga_1}\right), & \beta_2^2 &= H_2 \beta_1^2,
\end{aligned} \tag{6.16}$$

$\Pi(\cdot, \cdot)$ and $K(\cdot)$ are the complete elliptic integrals of the third and the first kind respectively. By comparing the asymptotic expansions of the second and the third forms in (6.8), we obtain

$$\begin{aligned}
C_2 &= \sqrt{\frac{\pi}{2}} C_0^{1/4} |U_X(X_0; \gamma)|^{-1/4} [C_4(\sin(f_1(\gamma)) - \cos(f_1(\gamma))), \\
&\quad + C_5(\cos(f_1(\gamma)) + \sin(f_1(\gamma)))], \\
C_3 &= \sqrt{\frac{\pi}{2}} C_0^{1/4} |U_X(X_0; \gamma)|^{-1/4} [C_5(\cos(f_1(\gamma)) - \sin(f_1(\gamma))), \\
&\quad + C_4(\cos(f_1(\gamma)) + \sin(f_1(\gamma)))].
\end{aligned} \tag{6.17}$$

By comparing the asymptotic expansions of the second and the first forms in (6.8), we have

$$\begin{aligned}
C_1 &= \frac{1}{2} \pi^{-1/2} C_0^{-1/4} |U_X(X_0; \gamma)|^{1/4} C_2, \\
C_3 &= 0.
\end{aligned} \tag{6.18}$$

Similarly, by comparing the asymptotic expansions of the fourth and the third forms in (6.8), we obtain

$$\begin{aligned}
C_6 &= \sqrt{\frac{\pi}{2}} C_0^{1/4} |U_X(X_0; \gamma)|^{-1/4} [-C_4(\sin(f_1(\gamma)) - \cos(f_1(\gamma))), \\
&\quad + C_5(\cos(f_1(\gamma)) + \sin(f_1(\gamma)))], \\
C_7 &= \sqrt{\frac{\pi}{2}} C_0^{1/4} |U_X(X_0; \gamma)|^{-1/4} [C_5(\cos(f_1(\gamma)) - \sin(f_1(\gamma))), \\
&\quad - C_4(\cos(f_1(\gamma)) + \sin(f_1(\gamma)))].
\end{aligned} \tag{6.19}$$

By comparing the asymptotic expansions of the fifth and the fourth forms in (6.8), we obtain

$$\begin{aligned}
C_8 &= \frac{1}{2} \pi^{-1/2} C_0^{-1/4} |U_X(X_0; \gamma)|^{1/4} C_6, \\
C_7 &= 0.
\end{aligned} \tag{6.20}$$

Substituting (6.17)₂ into (6.18)₂, we have

$$C_5 \cos(f_1(\gamma) + \frac{\pi}{4}) + C_4 \cos(f_1(\gamma) - \frac{\pi}{4}) = 0. \quad (6.21)$$

Substituting (6.19)₂ into (6.20)₂, we have

$$C_5 \cos(f_1(\gamma) + \frac{\pi}{4}) - C_4 \cos(f_1(\gamma) - \frac{\pi}{4}) = 0. \quad (6.22)$$

For Eqs. (6.21) and (6.22) to have nontrivial solutions, there are two cases. One case is that

$$C_4 = 0, \quad f_1(\gamma) = n\pi + \pi/4, \quad n = 0, 1, 2, 3, \dots \quad (6.23)$$

It is easy to see from (6.8) that in this case V is symmetric. Another case is that

$$C_5 = 0, \quad f_1(\gamma) = n\pi - \pi/4, \quad n = 1, 2, 3, \dots, \quad (6.24)$$

and in this case V is anti-symmetric.

Eqs. (6.23) and (6.24) are the eigenvalue equations for determining γ . These two eigenvalue equations can be rewritten as a uniform expression

$$f_1(\gamma) = \frac{(2N-1)\pi}{4}, \quad N = 1, 2, 3, \dots, \quad (6.25)$$

where odd N represents the symmetric solution for V and the even N represents the anti-symmetric solution. By substituting (6.15) into (6.25), we obtain

$$e_1 \Pi(\beta_1^2, \tilde{\kappa}) + \tilde{g} \Pi(\beta_2^2, \tilde{\kappa}) + e_2 K(\tilde{\kappa}) = \frac{b}{a_1} \frac{(2N-1)\pi}{4}. \quad (6.26)$$

It can be seen that the left hand side of equation (6.26) implicitly depends on γ and the right hand side only depends on the width-thickness ratio and the wave number N . Thus, the width-thickness ratio is a key factor for determining the stress eigenvalues.

Denote the eigenvalue corresponding to N by γ_{eN} . In the case that $D_1 = -18$, $D_2 = 100$ and the Poisson's ratio ($= \frac{A_2}{2(A_2+A_3)}$) $= 1/3$, $a = 0.01$ and $\frac{b}{a_1} = 10$ (which means $b \approx 0.1333$), we obtain the first six eigenvalues as follows:

$$\gamma_{e1} = \gamma_m + 6.045 \times 10^{-16}, \quad \gamma_{e2} = \gamma_m + 1.029 \times 10^{-41},$$

$$\begin{aligned}\gamma_{e3} &= \gamma_m + 1.751 \times 10^{-67}, & \gamma_{e4} &= \gamma_m + 2.980 \times 10^{-93}, \\ \gamma_{e5} &= \gamma_m + 5.071 \times 10^{-119}, & \gamma_{e6} &= \gamma_m + 6.830 \times 10^{-145},\end{aligned}\quad (6.27)$$

where the Maxwell stress $\gamma_m = 0.0168$. We note that these eigenvalues are very close.

Denoting $\mathcal{U}$ and $\mathcal{V}$ the axial and lateral displacements of a point in the center line respectively, and without loss of generality letting the displacement at $(0, 0, 0)$ be zero, we obtain

$$\mathcal{U} = \int_0^X U dX, \quad \mathcal{V} = \int_0^X V dX. \quad (6.28)$$

Curves of U , $\mathcal{U}$, V , $\mathcal{V}$ for $N = 1$ and $N = 2$ are plotted in Figure 1. It should be noted that the amplitude of the eigenfunction V (and then $\mathcal{V}$) is undetermined. We have scaled all the amplitudes to unit for all variable in Figure 2.

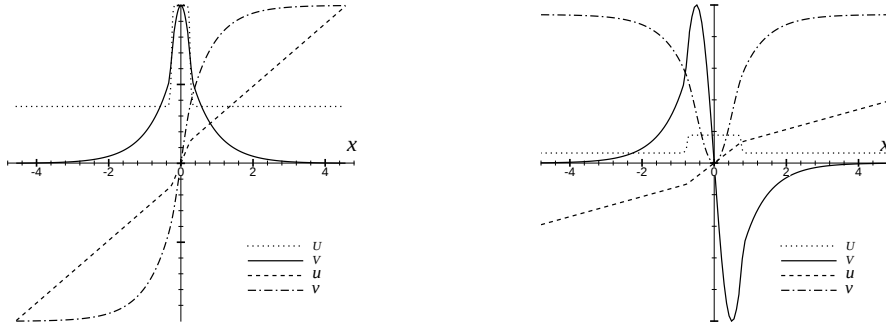


Fig. 2. Curves of U , $\mathcal{U}$, V , $\mathcal{V}$ as X varies. Left: $N=1$; Right: $N=2$.

The coordinates in the current configuration are (up to the leading order)

$$\begin{aligned}x &= X + \mathcal{U} - VY, \\ y &= Y + \mathcal{V} - \nu UY, \\ z &= Z - \nu UZ.\end{aligned}\quad (6.29)$$

The shapes of the thin strip corresponding to γ_{e1} to γ_{e6} are plotted in Figure 3. The thickness of the strip in the current configuration is illustrated by flood contours. It should be noted that they represent six different modes and do not need to appear consecutively.

From Figure 3, it can be seen that the transformation fronts are inclined with the strip axis. For $N = 1$, the inclined directions are the same and

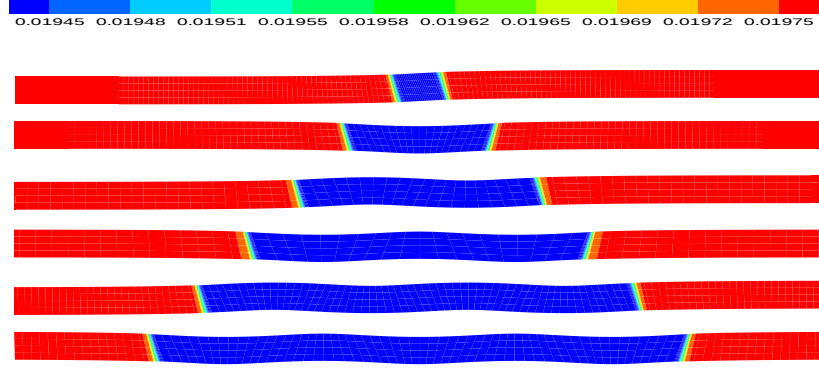


Fig. 3. Shapes of the thin strip corresponding to γ_{e1} to γ_{e6} (from top to bottom). The thickness is illustrated by flood contours.

the position of the turning point X_0 is about 0.2675 (roughly speaking, X_0 represents the position of the transformation front). For $N = 2$, the inclined directions are opposite and the position of the turning point X_0 is about 0.7619. In experiments it is found that, accompanying the formation of two phases, there is a stress drop ($\gamma > \gamma_m$) and the phase fronts become inclined in the same direction (a shear instability). And then with the developing of the high-strain phase, the propagating phase front can become inclined in the opposite direction (orientation instability). Our analytical results seem to capture these features.

The results obtained here shows that the inclination of the transformation front (shear instability) is a phenomenon of phase-transition-induced buckling. When the phase transition happens, there is a localized deformation (a neck). Due to that, the eigenvalue problem (6.5) has a turning point at X_0 , which in turn causes the buckling modes. Usually, for a buckling problem, one often only observes the first mode ($N = 1$; the two transformation fronts are parallel). However, in the present problem, the second eigenvalue ($N = 2$) is very close to the first eigenvalue. Thus, if there is a slight disturbance, the second mode may appear. This may explain why the front can switch to an opposite inclination direction.

We also point out that the solutions obtained above are also valid for a semi-infinite strip with the boundary conditions:

$$V = 0, \quad V_X = 0, \quad \text{at} \quad X = +\infty, \quad (6.30)$$

if we restrict the spatial interval to $0 < X < +\infty$. Then, the right-half part of Figure 3 describes the different modes of a single transformation front, which is initially located near the left end. This corresponds to one experimental

situation in Shaw & Kyriakides (1998) (see Fig. 1 of that paper).

7 Conclusions

In order to better understand several instability phenomena observed in experiments in a thin SMA strip during the process of stress-induced phase transitions, we carry out an analytical study. Mathematically, it is a very challenging problem as one needs to study the solution bifurcations of nonlinear partial differential equations in order to capture the instability phenomena. We start from the formulation of the three-dimensional field equations. By using the smallness of the thickness and the maximum strain, through a methodology which combines series expansions and asymptotic expansions, we derive the two-dimensional asymptotic equations, which take into account the lateral deformation and satisfy the traction-free boundary conditions up to the right order. Then, by further using the smallness of the width, we derive two one-dimensional asymptotic normal form equations for the phase transition problem in a thin strip. These two equations are analyzed and we manage to obtain some interesting analytic solutions for an infinite long strip under free-end boundary conditions through the WKB method. Our analytical results capture several instability phenomena observed in experiments successfully. It is shown analytically that the inclination of the transformation front is a phenomenon of localization-induced buckling (or phase-transition-induced buckling as the strain localization appears due to the phase transition). Also, it is demonstrated that there exists a second mode with a stress eigenvalue very close to that of the first one. Thus, a slight disturbance could cause a switch from the first mode to the second mode. This, in turn, implies a switch of the inclination of the transformation front to an opposite direction, which explains the orientation instability. Our results also reveal more explicitly the important role played by the thickness of the strip and show that the width influences the instability phenomena through the thickness-width ratio rather than its magnitude. In literature, whether the well-known phenomenon of the Luders band in a mild steel is caused by microscopic effects or macroscopic effects is still not completely settled issue. Due to the similarities between stress-induced transformations in a SMA and the development of Luders band in a mild steel, the present results also provide a strong mathematical evidence that the formation of the Luders band is a phenomenon due to macroscopic effects.

Finally, we point out that as a by-product of the present study we have also provided a mathematically consistent derivation of the classical Euler buckling equation, without using the ad hoc hypothesis that the bending moment is proportional to the curvature.

Appendix: Incremental elastic moduli

For initially isotropic material, in the case that there are no prestresses, Φ should be a function of the principle stretches λ_1 , λ_2 and λ_3 , namely $\Phi = \Phi(\lambda_1, \lambda_2, \lambda_3)$. Denote by $\Phi_j = \frac{\partial \Phi}{\partial \lambda_j}|_{\lambda_1=\lambda_2=\lambda_3=1}$, $\Phi_1 = \Phi_2 = \Phi_3$ should vanish since there are no prestresses. The non-zero first order incremental elastic moduli can be written as

$$\begin{aligned} A_1 &= a_{1111}^1 = \Phi_{11}, \\ A_2 &= a_{1122}^1 = \Phi_{12}, \\ A_3 &= a_{1212}^1 = \frac{1}{2}(A_1 - A_2), \\ A_4 &= a_{1221}^1 = A_3, \end{aligned}$$

where A_2 and A_3 are exactly the Lamé's constants for infinitesimal strain. There are only two independent constants among $A_i, i = 1, 2, 3, 4$.

The non-zero second order incremental elastic moduli can be written as

$$\begin{aligned} B_1 &= a_{111111}^2 = \Phi_{111}, \\ B_2 &= a_{111122}^2 = \Phi_{112}, \\ B_3 &= a_{112233}^2 = \Phi_{123}, \\ B_4 &= a_{111212}^2 = \frac{1}{4}(2A_2 + 2A_3 + B_1 - B_2), \\ B_5 &= a_{331212}^2 = \frac{1}{2}(A_2 + B_2 - B_3), \\ B_6 &= a_{121323}^2 = \frac{1}{2}(B_4 - B_5), \\ B_7 &= a_{111221}^2 = B_4 - A_2 - A_3, \\ B_8 &= a_{331221}^2 = B_5 - A_2, \\ B_9 &= a_{123123}^2 = B_6 - A_3. \end{aligned}$$

There are only three additional independent constants among $B_i, i = 1 \sim 9$.

The non-zero third order incremental elastic moduli can be written as

$$\begin{aligned} C_1 &= a_{11111111}^3 = \Phi_{1111}, \\ C_2 &= a_{11111122}^3 = \Phi_{1112}, \\ C_3 &= a_{11112222}^3 = \Phi_{1122}, \\ C_4 &= a_{11112233}^3 = \Phi_{1123}, \\ C_5 &= a_{11111212}^3 = -\frac{1}{12}(6A_2 + 6A_3 - 3B_1 - 3B_2 - 2C_1 + 2C_2), \end{aligned}$$

$$\begin{aligned}
C_6 &= a_{11112323}^3 = \frac{1}{2}(B_2 + C_3 - C_4), \\
C_7 &= a_{11221212}^3 = -\frac{1}{12}(6A_2 + 6A_3 - 6B_2 - C_1 - 2C_2 + 3C_3), \\
C_8 &= a_{11221313}^3 = -\frac{1}{4}(A_2 - B_2 - B_3 - C_2 + C_4), \\
C_9 &= a_{12121212}^3 = \frac{1}{8}(6A_2 + 6A_3 + 6B_1 - 6B_2 + C_1 - 4C_2 + 3C_3), \\
C_{10} &= a_{12121313}^3 = \frac{1}{3}C_9, \\
C_{11} &= a_{11121323}^3 = -\frac{1}{24}(3A_1 - 3B_2 + 3B_3 - C_1 + C_2 + 3C_3 - 3C_4), \\
C_{12} &= a_{11111221}^3 = \frac{1}{12}(6A_2 + 6A_3 - 3B_1 - 3B_2 + 2C_1 - 2C_2), \\
C_{13} &= a_{11112332}^3 = -\frac{1}{2}(B_2 - C_3 + C_4), \\
C_{14} &= a_{11221221}^3 = \frac{1}{12}(6A_2 + 6A_3 - 6B_2 + C_1 + 2C_2 - 3C_3), \\
C_{15} &= a_{11221331}^3 = \frac{1}{4}(A_2 - B_2 - B_3 + C_2 - C_4), \\
C_{16} &= a_{12121221}^3 = -\frac{1}{8}(6A_2 + 6A_3 - C_1 + 4C_2 - 3C_3), \\
C_{17} &= a_{12211221}^3 = \frac{1}{8}(6A_2 + 6A_3 - 2B_1 + 2B_2 + C_1 - 4C_2 + 3C_3), \\
C_{18} &= a_{12121331}^3 = \frac{1}{3}C_{16}, \\
C_{19} &= a_{12211331}^3 = \frac{1}{24}(6A_1 - 3B_1 - 3B_2 + 6B_3 + C_1 - 4C_2 + 3C_3), \\
C_{20} &= a_{11123123}^3 = \frac{1}{24}(6A_1 + 3A_2 - 3B_1 + 3B_3 + C_1 - C_2 - 3C_3 + 3C_4), \\
C_{21} &= a_{11123132}^3 = C_{20} - \frac{1}{4}(A_1 + A_2 + B_1 - B_2), \\
C_{22} &= a_{12122323}^3 = \frac{1}{24}(6A_2 - 6A_3 + 12B_2 - 12B_3 + C_1 - 4C_2 + 3C_3).
\end{aligned}$$

There are only four additional independent constants among $C_i, i = 1 \sim 22$.

Acknowledgement

The work described in this paper is supported by a grant from the Research Grants Council of the HKSAR, China (Project number: CityU 100807) and a grant from the National Natural Science Foundation of China (Project number: 10672117).

References

- [1] Cai, Z. X. and Dai, H.-H., PHASE TRANSITIONS IN A SLENDER CYLINDER COMPOSED OF AN INCOMPRESSIBLE ELASTIC MATERIAL II. ANALYTICAL SOLUTIONS FOR TWO BOUNDARY-VALUE PROBLEMS. *Proc R Soc London A* 462(2006) 419-438.
- [2] Chang, B. C., Shaw, J. A., and Iadicola, M. A., THERMODYNAMICS OF SHAPE MEMORY ALLOYS WIRE: MODELING, EXPERIMENTS, AND APPLICATION. *Continuum Mech. Thermodyn.* 18(2006) 83-118.
- [3] Dai, H.-H. and Cai, Z. X., PHASE TRANSITIONS IN A SLENDER CYLINDER COMPOSED OF AN INCOMPRESSIBLE ELASTIC MATERIAL I. ASYMPTOTIC MODEL EQUATION. *Proc R Soc London A* 462(2006), 75-95.
- [4] Dai, H.-H. & Fan, X. ASYMPTOTICALLY APPROXIMATE MODEL EQUATIONS FOR WEAKLY NONLINEAR LONG WAVES IN COMPRESSIBLE ELASTIC RODS AND THEIR COMPARISONS WITH OTHER SIMPLIFIED MODEL EQUATIONS. *Math. Mech. Solids* 9(2004), 61-79.
- [5] Dai, H.-H. & Huo, Y. ASYMPTOTICALLY APPROXIMATE MODEL EQUATIONS FOR NONLINEAR DISPERSIVE WAVES IN INCOMPRESSIBLE ELASTIC RODS. *Acta. Mech.* 157(2002), 97-112.
- [6] Dai, H.-H., Hao, Y. H. & Zhen, C. ON CONSTRUCTING THE ANALYTICAL SOLUTIONS FOR LOCALIZATIONS IN A SLENDER CYLINDER COMPOSED OF AN INCOMPRESSIBLE HYPERELASTIC MATERIAL. *Int J Solid Struct* 45(2008), 2613-2628.
- [7] Dai, H.-H. & Wang, F. F. BIFURCATION TO A CORNER-LIKE FORMATION IN A SLENDER NONLINEARLY ELASTIC CYLINDER: ASYMPTOTIC SOLUTION AND MECHANISM. *Proc R Soc London A* 464(2008), 1587-1613.
- [8] Duerig, T. W. Melton, K.N. Stoeckel, D. & Wayman C.M. ENGINEERING ASPECTS OF SHAPE MEMORY ALLOYS . London : Butterworth-Heinemann, 1990.
- [9] Ericksen, J. L. EQUILIBRIUM BARS. *J. Elasticity* 5(1975), 191-201.
- [10] Estrin, Y. & Kubin, L. P. SPATIAL COUPLING AND PROPAGATIVE PLASTIC INSTABILITIES, in: CONTINUUM MODELS FOR MATERIALS WITH MICROSTRUCTURES (H. B. Muhlhaus(ed.)), John Wiley & Sons, Chichester, U.K., 1995.
- [11] Feng, X. Q. & Sun, Q. EXPERIMENTAL INVESTIGATION ON MACROSCOPIC DOMAIN FORMATION AND EVOLUTION IN POLYCRYSTALLINE NiTi MICROTUBING UNDER MECHANICAL FORCE. *J Mech Phys Solids* 54(2006) 1568-1603.
- [12] Fu, Y. B. & Ogden, R.W. NONLINEAR STABILITY ANALYSIS OF PRE-STRESSED ELASTIC BODIES *Continuum Mech. Thermodyn.* 11(1999), 141-172.

- [13] Holmes, M. H. INTRODUCTION TO PERTURBATION METHODS. New York : Springer-Verlag, 1998.
- [14] Otsuka, K. & Wayman, C. M. SHAPE MEMORY MATERIALS. Cambridge : Cambridge University Press, 1998.
- [15] Shaw, J. A., SIMULATIONS OF LOCALIZED THERMO-MECHANICAL BEHAVIOR IN A NiTi SHAPE MEMORY ALLOY. Inter J Plasticity 16(2000) 541-562.
- [16] Shaw, J. A. and Kyriakides, S., THERMOMECHANICAL ASPECTS OF NiTi. J Mech Phys Solids 47(1995) 1243-1281.
- [17] Shaw, J. A. and Kyriakides, S., ON THE NUCLEATION AND PROPAGATION OF PHASE TRANSFORMATION FRONTS IN A NiTi ALLOY. Acta Mater 45(1997) 683-700.
- [18] Shaw, J. A. and Kyriakides, S., INITIATION AND PROPAGATION OF LOCALIZED DEFORMATION IN ELASTO-PLASTIC STRIPS UNDER UNIAXIAL TENSION. Inter J Plasticity 13(1998) 837-871.
- [19] Sun, Q. P., Li, Z. Q. and Tse, K. K., ON SUPERELASTIC DEFORMATION OF NiTi SHAPE MEMORY ALLOY MICRO-TUBES AND WIRES—BAND NUCLEATION AND PROPAGATION. in: PROCEEDINGS OF THE IUTAM SYMPOSIUM ON SMART STRUCTURES AND STRUCTURE SYSTEMS (U. Gabbert and H. S. Tzou (eds.)), pp. 1-8, Kluwer Academic Publishers, Magdeburg, Germany, 2000.
- [20] Tse, K. K. and Sun, Q. P., SOME DEFORMATION FEATURES OF POLYCRYSTALLINE SUPERELASTIC NiTi SHAPE MEMORY ALLOY THIN STRIPS AND WIRES UNDER TENSION. Key Engineering Materials 177-180(2000) 455-460.