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DESIGN AND APPROXIMATION CAPABILITIES ANALYSIS OF TIME-VARIANT FUZZY SYSTEMS

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Abstract. *In this paper, the design and analysis for a class of time-variant fuzzy systems are investigated. Firstly, a novel modeling method for time-variant fuzzy system, called variable weighted interpolating modeling (VWIM) method is proposed. It is pointed out that the time-variant fuzzy systems constructed by VWIM method can be represented by some interpolation functions. Then, VWIM method is applied to the nonlinear dynamic systems modeling. It is proved that time-variant fuzzy systems based on VWIM method are universal approximators to a class of nonlinear systems. And, the approximation error bounds for various classes of time-variant fuzzy systems are established. Finally, a simulation example is provided to demonstrate how to utilize a time-variant fuzzy system to approximate a given nonlinear system with arbitrary precision.*
Keywords: Time-variant fuzzy system, Variable weighted interpolating modeling method, Universal approximators, Nonlinear systems

1. **Introduction.** In face of some complex systems with fuzziness and linguistic variable, such as dynamic systems, control systems and economic systems, mastering the mathematical model is the prerequisite for the theoretical analysis of the system [1-8]. Fuzzy system theory, which is introduced by L. A. Zadeh in [9], is a useful tool for system modeling when the exact model is unknown. It is well known that fuzzy system consists of four principle components: fuzzifier, fuzzy rules, fuzzy inference engine and defuzzifier. From the viewpoint of mathematics, the main object to construct a fuzzy system is to approximate a desired model or function within a given level of accuracy.

The existing research on the approximation of fuzzy systems can be classified into two aspects: the qualitative aspect and the quantitative aspect. On the qualitative aspect, the authors mainly investigate the universal approximation properties of various classes of fuzzy systems [10-14] and the theoretical foundation of fuzzy system modeling [15-17]. On the quantitative aspect, the approximation error bounds of TS fuzzy systems and Mamdani fuzzy systems are established in [13,14,18]. Besides, the perturbation error bounds of various classes of fuzzy systems are deduced in [19].

So far, the majority of existing results about approximation theory of fuzzy systems are only suitable to the time-invariant fuzzy systems. They can guarantee their abilities for approximating a wide class of functions. It should be noted that many practical systems operate in environment with time-varying characteristics and fuzziness. Most of them are represented as differential equations. Naturally, an important question followed is "how to design a fuzzy system to approximate a given nonlinear system with arbitrary accuracy". The research on this question is essential to the stability analysis and controller design for fuzzy systems. Until now, only a few results related to this subject can be found in [2,20,21]. The question "given a nonlinear system, how many rules are needed to guarantee the desired approximation accuracy" still remains unanswered. Besides, the fuzzy rule bases obtained by data or expert's knowledge are usually represented as the discrete-time information, that is, the corresponding rule bases are sparse with respect to time universe. If an input occurs in an empty space between two times, then no rule will be fired, thus we can not compute the corresponding output by fuzzy modeling method. To tackle the question of fuzzy reasoning with sparse rule bases, interpolated reasoning methods are introduced in [22-25]. However, how to utilize discrete-time fuzzy rule bases to depict the dynamic behavior of the system is still a question which needs to be studied.

Above facts bring about the motivation of integrating interpolation method and fuzzy system theory to deal with the design and analysis of time-variant fuzzy systems. In this paper, we concentrate on the modeling method for time-variant fuzzy systems and the approximation of a class of nonlinear systems by them. The paper is organized as follows. In Section 2, we introduce some preliminaries knowledge. In Section 3, we propose a novel modeling method for time-variant fuzzy systems, called variable weighted interpolating modeling (VWIM) method. In Section 4, we analyze the approximation accuracy of time-variant fuzzy systems constructed by VWIM method to a class of nonlinear systems. In Section 5, we provide a simulation example to demonstrate the validity of the VWIM method. In Section 6, we give a few concluding remarks.

2. **Problem Statement and Preliminaries.**

In this section, we will introduce several concepts which are used in this paper and two famous fuzzy systems: Mamdani fuzzy systems and TS fuzzy systems.

Definition 2.1. *Given a universe X, let $\mathcal{A} \triangleq \{A_i\}_{1\le i\le n}$ be a family of normal fuzzy sets on X and let x_i be the peak point of A_i. $\mathcal{A}$ is called a fuzzy partition of X, if it satisfies the conditions $(\forall(i,j))$ $(i \neq j \Rightarrow x_i \neq x_j)$ and $(\forall x \in X)(\sum_{i=1}^{n} A_i(x) = 1)$.*

In [26], if $\{A_i\}_{1\le i\le n}$ is the fuzzy partition of X, then the set of functions $\alpha = \{A_i(x), i = 1, \cdots, n\}$ is said to be a normal basis set.

Definition 2.2. [26] *For a given function $f\colon X \longrightarrow R$ and a normal basis set $\alpha = \{A_i(x), i = 0, \cdots, n\}$, the function $\hat{f}\colon X \longrightarrow R$ defined as $y = \hat{f}(x) = \sum_{i=0}^{n} f(x_i^*)A_i(x)$, is called a fuzzy interpolation of f over α, and x_i^* is the peak point for $A_i(x)$.*

Definition 2.3. *The mappings ω_r $(r = 1, 2)$ from $[a, b]$ to $[0, 1]$ are variable weights if the following conditions hold: (a) for any $t \in [a, b]$, $\sum_{r=1}^{2} \omega_r(t) = 1$; (b) $\omega_1(a) = 1$, $\omega_2(b) = 1$.*

Example 2.1. *Let $U = \bigcup_{k=1}^{p-1}[t_k, t_{k+1}]$, then the following mappings are respectively variable weights on $[t_k, t_{k+1}]$ $(k = 1, \cdots, p-1)$:*

(i) $\omega_{1k}(t) = \dfrac{t_{k+1} - t}{t_{k+1} - t_k}$, $\omega_{2k}(t) = \dfrac{t - t_k}{t_{k+1} - t_k}$;

(ii) $\omega_{1k}(t) = \sin^2 \dfrac{\pi(t_{k+1} - t)}{2(t_{k+1} - t_k)}$, $\omega_{2k}(t) = \cos^2 \dfrac{\pi(t_{k+1} - t)}{2(t_{k+1} - t_k)}$.

In the following, we take multi-input-single-output (MISO) fuzzy system as an example to introduce the general expressions of Mamdani fuzzy system and TS fuzzy system. From the viewpoint of mathematics, fuzzy system can be regarded as a mapping from input universe $X_1 \times \cdots \times X_n$ to output universe Y, i.e., $f\colon X_1 \times \cdots \times X_n \longrightarrow Y$.

The fuzzy rules of Mamdani fuzzy systems are formed as follows:

$$\text{If } x_1 \text{ is } A_{1j_1} \text{ and } x_2 \text{ is } A_{2j_2} \text{ and } \cdots \text{ and } x_n \text{ is } A_{nj_n} \text{ then } y \text{ is } B_{j_1\cdots j_n}, \tag{1}$$

where x_i $(i = 1, \cdots, n)$ are the input variables of fuzzy systems, y is the output variable and $A_{ij_i} \in \mathcal{F}(X_i)$, $B_{j_1\cdots j_n} \in \mathcal{F}(Y)$ $(j_i = 1, \cdots, p_i;\ i = 1, \cdots, n)$.

For a given input $\boldsymbol{x} = (x_1, \cdots, x_n) \in X_1 \times \cdots \times X_n$, the Mamdani fuzzy system which is determined by rules (1), singleton fuzzifier, CRI method and centroid defuzzifier technology can be expressed as

$$f(\boldsymbol{x}) = \frac{\sum\limits_{j_1=1}^{p_1} \cdots \sum\limits_{j_n=1}^{p_n} \left(\bigvee\limits_{j_1=1}^{p_1} \cdots \bigvee\limits_{j_n=1}^{p_n} \theta \left(\bigotimes\limits_{i=1}^{n} A_{il_i}(x_i), B_{l_1\cdots l_n}(y_{j_1\cdots j_n}) \right) \right) \cdot y_{j_1\cdots j_n}}{\sum\limits_{j_1=1}^{p_1} \cdots \sum\limits_{j_n=1}^{p_n} \left(\bigvee\limits_{j_1=1}^{p_1} \cdots \bigvee\limits_{j_n=1}^{p_n} \theta \left(\bigotimes\limits_{i=1}^{n} A_{il_i}(x_i), B_{l_1\cdots l_n}(y_{j_1\cdots j_n}) \right) \right)}.$$

In [18] when both t-norm $\otimes$ and implication operator θ are the product operator, CRI method is called product inference engine. When they are the min operator, CRI method is called min inference engine.

Different from Mamdani fuzzy systems, the fuzzy rule bases of TS fuzzy systems with linear consequent are given by:

$$\text{If } x_1 \text{ is } A_{1j_1} \text{ and } \cdots \text{ and } x_n \text{ is } A_{nj_n} \text{ then } y_{j_1\cdots j_n} = \sum_{i=0}^{n} c_{ij_1\cdots j_n} x_i, \tag{2}$$

where $j_1 = 1, \cdots, p_1;\ \cdots;\ j_n = 1, \cdots, p_n$, $x_0 \equiv 1$.

The TS fuzzy systems with singleton fuzzifier, rules (2), product inference engine and centroid defuzzifiier can be expressed as following,

$$f(\boldsymbol{x}) = \frac{\sum\limits_{j_1=1}^{p_1} \cdots \sum\limits_{j_n=1}^{p_n} \left(\prod\limits_{i=1}^{n} A_{ij_i}(x_i) \cdot \left(\sum\limits_{i=0}^{n} c_{ij_1\cdots j_n} x_i \right) \right)}{\sum\limits_{j_1=1}^{p_1} \cdots \sum\limits_{j_n=1}^{p_n} \left(\prod\limits_{i=1}^{n} A_{ij_i}(x_i) \right)}.$$

3. **VWIM Method for Time-Variant Fuzzy Systems.** In this section, we also take MISO time-variant system as an example to introduce a novel modeling method for time-variant fuzzy system. Figure 1 shows an n-input-single-output time-variant fuzzy system.

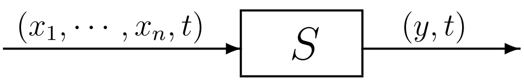

FIGURE 1. n-input-single-output time-variant fuzzy system

The time-variant system S can be regarded as a mapping from input universe $X_1 \times \cdots \times X_n \times U$ to output universe Y, which is denoted by F, i.e., F: $X_1 \times \cdots \times X_n \times U \to Y$, $(\boldsymbol{x}, t) = (x_1, \cdots, x_n, t) \mapsto F(\boldsymbol{x}, t)$, where $(\boldsymbol{x}, t)$ and $F(\boldsymbol{x}, t)$ are the input value and output value of system at time t respectively. For convenience, we take X_i, Y and U as real number intervals respectively, where $X_i = [a_i, b_i]$ $(i = 1, \cdots, n)$, $Y = [c, d]$ and $U = [0, T]$.

Now, we introduce some basic steps in time-variant fuzzy system modeling.

Step 1. Construct fuzzy rule bases. Firstly, we divide time universe $U = [0, T]$ as: $0 = t_1 < \cdots < t_p = T$. With respect to each index k $(k = 1, \cdots, p)$, we make a partition of interval $[a_i, b_i]$: $a_i = x_{ki1} < \cdots < x_{kip_{i_k}} = b_i$, $i = 1, \cdots, n$. Then, we construct normal fuzzy set $A_{kij_i^{(k)}} \in \mathcal{F}([a_i, b_i])$ such that $x_{kij_i^{(k)}}$ is the peak point of

$A_{kij_i^{(k)}}$ and $\{A_{kij_i^{(k)}}\}_{1\le j_i^{(k)}\le p_{i_k}}$ is a fuzzy partition of $[a_i,b_i]$. Furthermore, through experiment or observation method, we can determine the corresponding element $y_{kj_1^{(k)}\cdots j_n^{(k)}}\in Y$ with respect to the data $\left(\boldsymbol{x}_{kj_1^{(k)}\cdots j_n^{(k)}},t_k\right)\triangleq\left(x_{k1j_1^{(k)}},\cdots,x_{knj_n^{(k)}},t_k\right)$. Similarly, we take $y_{kj_1^{(k)}\cdots j_n^{(k)}}$ as the peak point to determine the normal fuzzy set $B_{kj_1^{(k)}\cdots j_n^{(k)}}\in\mathcal{F}(Y)$ such that $\left\{B_{kj_1^{(k)}\cdots j_n^{(k)}}\right\}_{1\le j_1^{(k)}\le p_{1_k};\cdots;1\le j_n^{(k)}\le p_{n_k}}$ is a fuzzy partition of Y.

Accordingly, a group of discrete-time fuzzy rule bases can be formed as follows:

$$\text{If } x_1 \text{ is } A_{k1j_1^{(k)}} \text{ and } \cdots \text{ and } x_n \text{ is } A_{knj_n^{(k)}} \text{ then } y \text{ is } B_{kj_1^{(k)}\cdots j_n^{(k)}}, \tag{3}$$

$$j_i^{(k)}=1,\cdots,p_{i_k};\quad i=1,\cdots,n;\quad k=1,\cdots,p.$$

Step 2. For any given input $(\boldsymbol{x},t)\in X_1\times\cdots X_n\times U$. Assume that $t\in[t_k,t_{k+1})$, then by fuzzy system modeling method, for instance Mamdani fuzzy systems and TS fuzzy systems, we can compute the output values at time t_k and time t_{k+1}, which are denoted as $F_k(\boldsymbol{x},t_k)$ and $F_{k+1}(\boldsymbol{x},t_{k+1})$ respectively.

Step 3. Take weighed sum of $F_k(\boldsymbol{x},t_k)$ and $F_{k+1}(\boldsymbol{x},t_{k+1})$ to determine the output value of system at time t, i.e., $F(\boldsymbol{x},t)=\omega_{1k}(t)\cdot F_k(\boldsymbol{x},t_k)+\omega_{2k}(t)\cdot F_{k+1}(\boldsymbol{x},t_{k+1})$.

Step 4. Using the characteristic function, the time-variant fuzzy system can be expressed as

$$F(\boldsymbol{x},t)=\sum_{k=1}^{p}\left(\omega_{1k}(t)\cdot F_k(\boldsymbol{x},t_k)+\omega_{2k}(t)\cdot F_{k+1}(\boldsymbol{x},t_{k+1})\right)\cdot\chi_{[t_k,t_{k+1})}(t), \tag{4}$$

where F_k denotes the fuzzy system at time t_k $(k=1,\cdots,p)$ and $[t_p,t_{p+1})\triangleq\{T\}$.

In this way, we use the variable weight interpolation technology and fuzzy system modeling method to obtain the mathematical representation of time-variant fuzzy system. In this paper, such a modeling method is called variable weight interpolation modeling (VWIM) method.

Remark 3.1. *By Steps 1 – 4, we find that VWIM method can transfer a group of discrete-time rule bases into a continuous dynamic model. And, the properties of the corresponding time-variant fuzzy systems depend on two factors: variable weights and the fuzzy systems at every sample time.*

Naturally, we may propose such a question: how to choose variable weight and time-invariant fuzzy system in practical application? Next, we will answer this question.

Proposition 3.1. *Suppose that the time-invariant fuzzy systems are Mamdani fuzzy systems with min inference engine, then the time-variant fuzzy system determined by rules (3) and VWIM method can be expressed as*

$$\begin{aligned}F(\boldsymbol{x},t)=\sum_{k=1}^{p}&\left(\omega_{1k}(t)\cdot\frac{\sum\limits_{j_1^{(k)}=1}^{p_{1_k}}\cdots\sum\limits_{j_n^{(k)}=1}^{p_{n_k}}\left(\bigwedge\limits_{i=1}^{n}A_{kij_i^{(k)}}(x_i)\right)\cdot y_{kj_1^{(k)}\cdots j_n^{(k)}}}{\sum\limits_{j_1^{(k)}=1}^{p_{1_k}}\cdots\sum\limits_{j_n^{(k)}=1}^{p_{n_k}}\left(\bigwedge\limits_{i=1}^{n}A_{kij_i^{(k)}}(x_i)\right)}\right.\\&\left.+\omega_{2k}(t)\cdot\frac{\sum\limits_{j_1^{(k+1)}=1}^{p_{1_{k+1}}}\cdots\sum\limits_{j_n^{(k+1)}=1}^{p_{n_{k+1}}}\left(\bigwedge\limits_{i=1}^{n}A_{(k+1)ij_i^{(k+1)}}(x_i)\right)\cdot y_{(k+1)j_1^{(k+1)}\cdots j_n^{(k+1)}}}{\sum\limits_{j_1^{(k+1)}=1}^{p_{1_{k+1}}}\cdots\sum\limits_{j_n^{(k+1)}=1}^{p_{n_{k+1}}}\left(\bigwedge\limits_{i=1}^{n}A_{(k+1)ij_i^{(k+1)}}(x_i)\right)}\right)\cdot\chi_{[t_k,t_{k+1})}(t).\end{aligned}\tag{5}$$

Proposition 3.2. *Suppose that the time-invariant fuzzy systems are Mamdani fuzzy systems with product inference engine, then the time-variant fuzzy system determined by rules (3) and VWIM method can be expressed as*

$$\begin{aligned} F(\boldsymbol{x},t) = \sum_{k=1}^{p} \Bigg(\omega_{1k}(t) \cdot \sum_{j_1^{(k)}=1}^{p_{1_k}} \cdots \sum_{j_n^{(k)}=1}^{p_{n_k}} \left(\prod_{i=1}^{n} A_{kij_i^{(k)}}(x_i) \right) \cdot y_{kj_1^{(k)}\cdots j_n^{(k)}} + \omega_{2k}(t) \\ \cdot \sum_{j_1^{(k+1)}=1}^{p_{1_{k+1}}} \cdots \sum_{j_n^{(k+1)}=1}^{p_{n_{k+1}}} \left(\prod_{i=1}^{n} A_{(k+1)ij_i^{(k+1)}}(x_i) \right) \cdot y_{(k+1)j_1^{(k+1)}\cdots j_n^{(k+1)}} \Bigg) \cdot \chi_{[t_k,t_{k+1})}(t). \end{aligned} \tag{6}$$

Proposition 3.3. *Suppose that the time-invariant fuzzy systems are TS fuzzy systems determined by product inference engine, then the time-variant fuzzy system based on rules (3) and VWIM method can be expressed as*

$$\begin{aligned} F(\boldsymbol{x},t) = \sum_{k=1}^{p} \Bigg[\omega_{1k}(t) \cdot \sum_{j_1^{(k)}=1}^{p_{1_k}} \cdots \sum_{j_n^{(k)}=1}^{p_{n_k}} \left(\prod_{i=1}^{n} A_{kij_i^{(k)}}(x_i) \right) \cdot \left(\sum_{i=0}^{n} c_{kij_1^{(k)}\cdots j_n^{(k)}} \cdot x_i \right) \\ + \omega_{2k}(t) \cdot \sum_{j_1^{(k+1)}=1}^{p_{1_{k+1}}} \cdots \sum_{j_n^{(k+1)}=1}^{p_{n_{k+1}}} \left(\prod_{i=1}^{n} A_{(k+1)ij_i^{(k+1)}}(x_i) \right) \\ \cdot \left(\sum_{i=0}^{n} c_{(k+1)ij_1^{(k+1)}\cdots j_n^{(k+1)}} \cdot x_i \right) \Bigg] \cdot \chi_{[t_k,t_{k+1})}(t). \end{aligned} \tag{7}$$

Theorem 3.1. *For any $k \in \{1, \cdots, p\}$, if $F_k(\boldsymbol{x}, t_k)$ is a fuzzy interpolation function then the time-variant fuzzy system determined by (4) is a fuzzy interpolation function.*

Proof: It can be verified by Definitions 2.2 and 2.3 directly.

Corollary 3.1. *If the time-variant fuzzy systems are respectively determined by (5) – (7), then they are fuzzy interpolation functions.*

Remark 3.2. *When a fuzzy system is represented as an interpolation function, by numerical approximation theory, it possesses the universal approximation property. Theorem 3.1 and Definition 2.3 provide a theoretical foundation for the election of fuzzy system and variant weight in applying VWIM method.*

Further, we will investigate the approximation accuracy of time-variant fuzzy systems to continuously differentiable functions. Let $C^n(X)$ be the space of all n-times continuously differentiable functions on X. Particularly, $C(X) \triangleq C^0(X)$ denotes the space of continuous functions on X. If $g \in C(X)$, then ∞ norm of g is defined as: $\|g\|_\infty \triangleq \sup_{\boldsymbol{x}\in X} |g(\boldsymbol{x})|$. Furthermore, if $\boldsymbol{g} = (g_1, \cdots, g_n) \in C(X)$, then $\|\boldsymbol{g}\|_\infty = \bigvee_{i=1}^{n} \|g_i\|_\infty$.

In order to give a unified and compact expression of the universal approximation property of time-variant fuzzy systems, we need to introduce some notations.

Let $\alpha_i \triangleq \max_{1\le k\le p, 2\le j_i^{(k)}\le p_{i_k},} \left\{ \left| x_{kij_i^{(k)}} - x_{ki(j_i^{(k)}-1)} \right| \right\}$, $\alpha \triangleq \bigvee_{i=1}^{n} \alpha_i$ and $\beta \triangleq \max_{2\le k\le p} \{|t_k - t_{k-1}|\}$.

Theorem 3.2. *Under the above conditions, assume that $g \in C^1([a_1, b_1] \times \cdots \times [a_n, b_n] \times [0, T])$ and $g\left(\boldsymbol{x}_{kj_1^{(k)}\cdots j_n^{(k)}}, t_k\right) = y_{kj_1^{(k)}\cdots j_n^{(k)}} \left(j_i^{(k)} = 1, \cdots, p_{i_k};\ k = 1, \cdots, p;\ i = 1, \cdots, n\right)$.*

If the time -variant fuzzy system F is determined by (5), then

$$\|F-g\|_\infty \le \sum_{i=1}^{n} \|\tfrac{\partial g}{\partial x_i}\|_\infty \cdot \alpha_i + \|\tfrac{\partial g}{\partial t}\|_\infty \cdot \beta.$$

Proof: For any $(\boldsymbol{x}', t') = (x_1', \cdots, x_n', t') \in X_1 \times \cdots \times X_n \times U$, without loss of generality, we suppose that $t' \in [t_k, t_{k+1})$, $x_i' \in [x_{kil_i^{(k)}}, x_{ki(l_i^{(k)}+1)}] \bigcap [x_{(k+1)il_i^{(k+1)}}, x_{(k+1)i(l_i^{(k+1)}+1)}]$, where $i = 1, \cdots, n$. By differential mean valued theorem, we can prove that

$$|F(\boldsymbol{x}', t') - g(\boldsymbol{x}', t')| \le \sum_{i=1}^{n} \|\tfrac{\partial g}{\partial x_i}\|_\infty \cdot \alpha_i + \|\tfrac{\partial g}{\partial t}\|_\infty \cdot \beta.$$

This implies that $\|F-g\|_\infty \le \sum\limits_{i=1}^{n} \|\frac{\partial g}{\partial x_i}\|_\infty \cdot \alpha_i + \|\frac{\partial g}{\partial t}\|_\infty \cdot \beta$ holds.

Remark 3.3. *By the numerical approximation theory, we know that any time-variant fuzzy system satisfying Theorem 3.2 is a first-order accurate approximator for the desired continuously differentiable function.*

Theorem 3.3. *Under the above conditions, suppose that 1) $g \in C^2([a_1, b_1] \times \cdots \times [a_n, b_n] \times [0, T])$ and $g(\boldsymbol{x}_{kj_1^{(k)}\cdots j_n^{(k)}}, t_k) = y_{kj_1^{(k)}\cdots j_n^{(k)}}$ ($j_i^{(k)} = 1, \cdots, p_{i_k}; k = 1, \cdots, p; i = 1, \cdots, n$); 2) the time-variant fuzzy system F is determined by (6); 3) the membership functions $A_{kij_i^{(k)}}(x_i)$ ($j_i^{(k)} = 1, \cdots, p_{i_k}; k = 1, \cdots, p; i = 1, \cdots, n$) are triangle-shaped membership functions; 4) the variable weight functions are chosen to be (i) of Example 2.1, then*

$$\|F-g\|_\infty \le \frac{1}{8} \cdot \sum_{i=1}^{n} \|\tfrac{\partial^2 g}{\partial x_i^2}\|_\infty \cdot \alpha_i^2 + \frac{1}{8}\|\tfrac{\partial^2 g}{\partial t^2}\|_\infty \cdot \beta^2.$$

Proof: For any $(\boldsymbol{x}', t') \in X_1 \times \cdots \times X_n \times U$, we also suppose that $t' \in [t_k, t_{k+1})$, $x_i' \in [x_{kil_i^{(k)}}, x_{ki(l_i^{(k)}+1)}]$ and $x_i' \in [x_{(k+1)il_i^{(k+1)}}, x_{(k+1)i(l_i^{(k+1)}+1)}]$, where $i = 1, \cdots, n$.

We define linear operators on $C^1\left(\left[x_{k1l_1}, x_{k1(l_1+1)}\right] \times \cdots \times \left[x_{knl_n}, x_{kn(l_n+1)}\right] \times [t_k, t_{k+1})\right)$ as

$$L_{(k,t')}g(\boldsymbol{x}', t') = \frac{t_{k+1}-t'}{t_{k+1}-t_k} \cdot g(\boldsymbol{x}', t_k) + \frac{t'-t_k}{t_{k+1}-t_k} \cdot g(\boldsymbol{x}', t_{k+1});$$

$$L_{kj}g(\boldsymbol{x}', t_k) = \sum_{r_j^{(k)}=l_j^{(k)}}^{l_j^{(k)}+1} A_{kjr_j^{(k)}}(x_j') \cdot g\left(x_1', \cdots, x_{kjr_j^{(k)}}, \cdots, x_n', t_k\right), \quad j = 1, \cdots, n;$$

$$L_k g(\boldsymbol{x}', t_k) = \sum_{r_1^{(k)}=l_1^{(k)}}^{l_1^{(k)}+1} \cdots \sum_{r_n^{(k)}=l_n^{(k)}}^{l_n^{(k)}+1} \left(\prod_{i=1}^{n} A_{kir_i^{(k)}}(x_i')\right) \cdot g\left(\boldsymbol{x}_{kr_1^{(k)}\cdots r_n^{(k)}}, t_k\right).$$

Similarly, we can also define linear operators $L_{(k+1)j}g$ ($j = 1, \cdots, n$) and $L_{(k+1)}g$ on $C^1\left(\left[x_{(k+1)1l_1'}, x_{(k+1)1(l_1'+1)}\right] \times \cdots \times \left[x_{(k+1)nl_n'}, x_{(k+1)n(l_n'+1)}\right] \times [t_k, t_{k+1})\right)$.

Similar to the proof of Theorem 3.2 in [18], we can prove that

$$|F(\boldsymbol{x}', t') - g(\boldsymbol{x}', t')| \le \frac{1}{8} \sum_{i=1}^{n} \parallel \tfrac{\partial^2 g}{\partial x_i^2} \parallel_\infty \cdot \alpha_i^2 + \frac{1}{8} \cdot \parallel \tfrac{\partial^2 g}{\partial t^2} \parallel_\infty \cdot \beta^2.$$

Hence, the assertion holds.

Theorem 3.4. *Under the above conditions, suppose that a) the conditions 1), 3) and 4) of Theorem 3.3 hold; b) the time-variant fuzzy system F is determined by (7), where*

$c_{kij_1^{(k)}\cdots j_n^{(k)}} = \left.\frac{\partial g}{\partial x_i}\right|_{(\boldsymbol{x},t_k)=\left(\boldsymbol{x}_{kj_1^{(k)}\cdots j_n^{(k)}},t_k\right)}, c_{k0j_1^{(k)}\cdots j_n^{(k)}} = g\left(\boldsymbol{x}_{kj_1^{(k)}\cdots j_n^{(k)}}, t_k\right) - \sum_{i=1}^{n} c_{kij_1^{(k)}\cdots j_n^{(k)}} \cdot x_{kij_i^{(k)}}$, $j_i^{(k)} = 1, \cdots, p_{i_k}$; $k = 1, \cdots, p$; $i = 1, \cdots, n$, *then*

$$\| F - g \|_\infty \leq \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \| \tfrac{\partial^2 g}{\partial x_i \partial x_j} \|_\infty \cdot\alpha^2 + \frac{1}{8}\cdot \| \tfrac{\partial^2 g}{\partial t^2} \|_\infty \cdot\beta^2.$$

Proof: Using Theorem 3.4 in [15] and similar to the proof of Theorem 3.3, we can prove that the assertion holds.

Remark 3.4. *Theorems 3.3 and 3.4 mean that choosing fuzzy systems to be Mamdani fuzzy systems with product engine or TS fuzzy systems with product engine, the membership functions to be triangle-shaped functions, and the variant weights to be (i) of Example 2.1, the time-variant fuzzy systems with second-order accurate approximators can be obtained.*

Remark 3.5. *Theorems 3.2 – 3.4 are different from the results of [10-16,18]. In the existing results on universal approximation properties of Mamdani fuzzy systems and TS fuzzy systems, fuzzy relation generalized by fuzzy rules are continuous functions. However, in time variant fuzzy system modeling, the fuzzy relation obtained by data are discrete valued functions. Thus, Mamdani fuzzy systems and TS fuzzy systems are not suitable to approximate such a time-variant model. From the mathematical point of view, the approximation mechanism of VWIM method is to combine local fuzzy system at each time interval to approximate the the desired time-variant model.*

4. **Approximation of Nonlinear Systems by VWIM Method.** In this section, we will answer the question: given a nonlinear system, how many fuzzy sets are needed for input and output variable in order to guarantee the desired approximation accuracy?

Consider a nonlinear system as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}, t), \tag{8}$$

where $\boldsymbol{x} = (x_1(t), \cdots, x_n(t))^T$ is the state vector, $x_i(t)$ $(i = 1, \cdots, n)$ are state variables. $X_1\times\cdots\times X_n$ and $\dot{X}_1\times\cdots\times\dot{X}_n$ are the universes of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ respectively, and $\boldsymbol{g} = (g_1 \cdots g_n)$ is a vector-valued function on $X_1\times\cdots\times X_n\times U$. Without loss of generality, we also assume that X_i, $\dot{X}_i$ $(i = 1, \cdots, n)$ and U are real number intervals respectively, i.e., $X_i = [a_i, b_i]$, $\dot{X}_i = [c_i, d_i]$ and $U = [0, T]$.

Similar to the Step 1, we can obtain a group of discrete-time fuzzy rules based on experiment data. The fuzzy rules at time t_k $(k = 1, \cdots, p)$ are given by:

$$\begin{cases} \text{If } x_1(t) \text{ is } A_{k1j_1^{(k)}} \text{ and } \cdots \text{ and } x_n(t) \text{ is } A_{knj_n^{(k)}} \text{ then } \dot{x}_1(t) \text{ is } B_{k1j_1^{(k)}\cdots j_n^{(k)}} \\ \text{If } x_1(t) \text{ is } A_{k1j_1^{(k)}} \text{ and } \cdots \text{ and } x_n(t) \text{ is } A_{knj_n^{(k)}} \text{ then } \dot{x}_2(t) \text{ is } B_{k2j_1^{(k)}\cdots j_n^{(k)}} \\ \qquad\qquad\qquad\qquad \cdots\cdots\cdots \\ \text{If } x_1(t) \text{ is } A_{k1j_1^{(k)}} \text{ and } \cdots \text{ and } x_n(t) \text{ is } A_{knj_n^{(k)}} \text{ then } \dot{x}_n(t) \text{ is } B_{knj_1^{(k)}\cdots j_n^{(k)}} \\ j_i^{(k)} = 1, \cdots, p_{i_k};\;\; i = 1, \cdots, n. \end{cases} \tag{9}$$

where $\{A_{kij_i^{(k)}}\}_{1\leq j_i^{(k)}\leq p_{i_k}}$ and $\{B_{kij_1^{(k)}\cdots j_n^{(k)}}\}_{1\leq j_1^{(k)}\leq p_{1_k},\cdots,1\leq j_n^{(k)}\leq p_{n_k}}$ are fuzzy partitions of X_i and $\dot{X}_i$ $(i = 1, \cdots, n)$ respectively; and the peak points of fuzzy sets $A_{kij_i^{(k)}}$ and $B_{kij_1^{(k)}\cdots j_n^{(k)}}$ are denoted by $x_{kij_i^{(k)}}$ and $\dot{x}_{kij_1^{(k)}\cdots j_n^{(k)}}$ respectively.

In the following, the time-variant fuzzy system determined by fuzzy rules (9) and VWIM method is denoted as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{10}$$

where $\boldsymbol{f} = (f_1, \cdots, f_n)$.

Further, in order to give compact proof of the conclusions, we need to give a lemma and an assumption.

Lemma 4.1. [21] *Consider two initial-value problems of first-order differential equations,*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0; \tag{11}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{12}$$

Suppose that $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *are continuous on the region* $D = \{(\boldsymbol{x}, t)\, ||t - t_0| \le a, \| \boldsymbol{x} - \boldsymbol{x}_0 \| \le b\}$ *and satisfy the Lipschitz condition about variable* $\boldsymbol{x}$ *respectively. The common Lipschitz constants of* $\boldsymbol{f}$ *and* $\boldsymbol{g}$ *are denoted by L. Let* $M \triangleq \max\{\| \boldsymbol{f} \|_\infty, \| \boldsymbol{g} \|_\infty\}$, $h \triangleq \min\{a, \frac{b}{M}\}$, $\boldsymbol{\varphi}$ *and* $\boldsymbol{\psi}$ *be the solutions of initial-value problems (11) and (12) on* $I \triangleq [t_0 - h, t_0 + h]$ *respectively. For any* $\delta > 0$, *as long as* $\| \boldsymbol{f} - \boldsymbol{g} \|_\infty < \delta$ *on* D , *then* $\| \boldsymbol{\varphi} - \boldsymbol{\psi} \|_\infty \le \frac{\delta}{L} e^{Lh}$ *holds on I.*

Assumption $(*)$: 1) the partition on $[0, T]$ is equidistant as t_k, $k = 1, \cdots, p$; 2) for any $k \in \{1, \cdots, p\}$, $i \in \{1, \cdots, n\}$ the partition on $[a_i, b_i]$ is equidistant; 3) $\boldsymbol{g}\Big(\boldsymbol{x}_{kj_1^{(k)}, \cdots, j_n^{(k)}}, t_k\Big) = \Big(\dot{x}_{k1j_1^{(k)} \cdots j_n^{(k)}}, \cdots, \dot{x}_{knj_1^{(k)} \cdots j_n^{(k)}}\Big)$, $j_i^{(k)} = 1, \cdots, p_{i_k}$; $k = 1, \cdots, p$; $i = 1, \cdots, n$.

Theorem 4.1. *Suppose that a) the assumption* $(*)$ *is satisfied and* $\boldsymbol{g} \in C^1([a_1, b_1] \times \cdots \times [a_n, b_n] \times [0, T])$; *b) the time-invariant fuzzy systems of VWIM method are chosen to be Mamdani fuzzy systems with min inference engine; c) nonlinear systems (8) and (10) have the same initial value. For any* $\varepsilon > 0$, *if* $p \ge \left(\left\|\frac{\partial \boldsymbol{g}}{\partial t}\right\|_\infty / \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right\|_\infty\right) \cdot T \cdot \frac{(n+1)}{\varepsilon} \cdot e^{\left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right\|_\infty \cdot T} + 1$ *and* $p_{i_k} \ge \left((b_i - a_i) / \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right\|_\infty\right) \cdot \frac{(n+1)}{\varepsilon} \cdot e^{\left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right\|_\infty \cdot T} + 1$, *then* $\| \boldsymbol{\varphi} - \boldsymbol{\psi} \|_\infty \le \varepsilon$.

Proof: It is easy to verify that $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfies the Lipschitz condition. By the uniqueness theorem for the initial-value problem, we know that nonlinear systems (8) and (10) have the sole solution respectively, denoted as $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$.

Since $\boldsymbol{g} \in C^1([a_1, b_1] \times \cdots \times [a_n, b_n] \times [0, T])$, we choose $\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \|_\infty$ as the Lipschitz constant of $\boldsymbol{g}$. By Theorem 3.2 and Lemma 4.1, we have

$$\|\boldsymbol{\varphi} - \boldsymbol{\psi}\|_\infty \le \frac{\sum\limits_{i=1}^{n} \left\|\frac{\partial \boldsymbol{g}}{\partial x_i}\right\|_\infty \cdot \frac{(b_i - a_i)}{p_{i_k} - 1} + \left\|\frac{\partial \boldsymbol{g}}{\partial t}\right\|_\infty \cdot \frac{T}{p-1}}{\left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right\|_\infty} \cdot e^{\left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right\|_\infty \cdot T} < \sum_{i=1}^{n} \frac{\varepsilon}{n+1} + \frac{\varepsilon}{n+1} = \varepsilon.$$

Remark 4.1. *Theorem 4.1 means that when fuzzy systems of VWIM method are chosen to be Mamdani fuzzy systems with min inference engine, then the corresponding time-variant fuzzy systems can approximate a class of nonlinear systems with first-order accuracy.*

Theorem 4.2. *Suppose that a') the condition a) and condition b) of Theorem 4.1 are satisfied and* $\boldsymbol{g} \in C^2([a_1, b_1] \times \cdots \times [a_n, b_n] \times [0, T])$; *b') the time-invariant fuzzy systems are chosen to be Mamdani fuzzy systems with product inference engine. For any* $\varepsilon > 0$, *if* $p \ge \frac{T}{2\sqrt{2}} \cdot \sqrt{\frac{n+1}{\varepsilon} \cdot \frac{\|\frac{\partial^2 \boldsymbol{g}}{\partial t^2}\|_\infty}{\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\|_\infty} \cdot e^{\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\|_\infty T}} + 1$ *and* $p_{i_k} \ge \frac{(b_i - a_i)}{2\sqrt{2}} \cdot \sqrt{\frac{n+1}{\varepsilon} \cdot \frac{\|\frac{\partial^2 \boldsymbol{g}}{\partial x_i^2}\|_\infty}{\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\|_\infty} \cdot e^{\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\|_\infty T}} + 1$ $(k = 1, \cdots, p;\ i = 1, \cdots, n)$, *then* $\| \boldsymbol{\varphi} - \boldsymbol{\psi} \|_\infty \le \varepsilon$.

Proof: By Theorem 3.3 and Lemma 4.1, we can prove that the assertion holds.

Theorem 4.3. *Suppose that the condition a') of Theorem 4.2 holds and the time-invariant fuzzy systems are chosen to be TS fuzzy systems with product inference engine. For any*

$\varepsilon > 0$, *if* $p_{i_k} \geq (b_i - a_i) \cdot \sqrt{\frac{n+1}{\varepsilon} \cdot \frac{\frac{1}{2}\sum_{k=1}^{n}\sum_{j=1}^{n} \| \frac{\partial^2 \boldsymbol{g}}{\partial x_k \partial x_j} \|_\infty}{\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \|_\infty} \cdot e^{\| \frac{\partial g}{\partial \boldsymbol{x}} \|_\infty T}} + 1$, $k = 1, \cdots, p$; $i = 1, \cdots, n$ *and* $p \geq \frac{T}{2\sqrt{2}} \cdot \sqrt{\frac{n+1}{\varepsilon} \cdot \frac{\| \frac{\partial^2 \boldsymbol{g}}{\partial t^2} \|_\infty}{\| \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} \|_\infty} \cdot e^{\| \frac{\partial g}{\partial \boldsymbol{x}} \|_\infty T}} + 1$, *then* $\| \boldsymbol{\varphi} - \boldsymbol{\psi} \|_\infty \leq \varepsilon$.

Proof: By the Theorem 3.4 and Lemma 4.1, we can prove that Theorem 4.3 holds.

Remark 4.2. *The results of Theorems 4.2 and 4.3 show that when fuzzy systems of VWIM are chosen to be Mamdani fuzzy systems or TS fuzzy systems with product inference engine, then the time-variant fuzzy systems constructed by VWIM method are universal approximations to a class of nonlinear systems with second order accuracy.*

Remark 4.3. *The majority of existing results on sufficient conditions for fuzzy systems as universal approximators concentrated on answering the question: given a continuous function, how many rules and fuzzy sets are needed to achieve predefined approximation accuracy. Theorems 4.1 – 4.3 give some sufficient conditions for time-variant fuzzy systems as universal approxiamtors to a class of nonlinear systems. These results are meaningful to the optimum design and stability analysis of fuzzy systems.*

5. **Simulation Experiment.**

Example 5.1. *Consider the following nonlinear system:*

$$\begin{cases} \dot{x}_1 = 0.1 \cdot x_2^2 - 0.1 \cdot x_1 + \cos t, \\ \dot{x}_2 = -0.3 \cdot x_2 \cdot \sin t + 0.1 \cdot x_1^2 \cdot \sin(2\pi t/5), \end{cases} \tag{13}$$

where the initial state $x_1(0) = 0.8$, $x_2(0) = -0.5$ *and the time universe is* $[0, 5]$.

The simulation steps are shown as follows.

Step 1. Make equidistant partition on $[0, 5]$ as $t_k = \frac{5(k-1)}{p-1}$, $k = 1, \cdots, p$.

Step 2. Determine the input universe. For any $k \in \{1, \cdots, p\}$, we substitute t_k into Equation (13). By Matlab, we can determine the maximum values and minimum values with respect to x_1 and x_2 respectively, i.e., $x^{(1)}_{\max} = b^{(1)}_k$, $x^{(1)}_{\min} = a^{(1)}_k$, $x^{(2)}_{\max} = b^{(2)}_k$, $x^{(2)}_{\min} = a^{(2)}_k$.

Step 3. Determine the fuzzy sets on the input universe. For each index k, we also make equidistant partition on $[a^{(1)}_k, b^{(1)}_k]$ and $[a^{(2)}_k, b^{(2)}_k]$ as $x^{(1)}_{ki} = a^{(1)}_k + \frac{(i-1)(b^{(1)}_k - a^{(1)}_k)}{n-1}$ $(i = 1, \cdots, n)$ and $x^{(2)}_{kj} = a^{(2)}_k + \frac{(j-1)(b^{(2)}_k - a^{(2)}_k)}{m-1}$ $(j = 1, \cdots, m)$. The fuzzy sets A_{ki} and B_{kj} are defined as $A_{ki}(x_1) = trimf(x_1, [x^{(1)}_{k(i-1)}, x^{(1)}_{ki}, x^{(1)}_{k(i+1)}])$ and $B_{kj}(x_2) = trimf(x_2, [x^{(2)}_{k(j-1)}, x^{(2)}_{kj}, x^{(2)}_{k(j+1)}])$.

Step 4. Substituting $(x^{(1)}_{ki}, x^{(2)}_{kj}, t_k)$ into nonlinear Equation (13), the corresponding elements $\dot{x}^{(1)}_{kij}$ and $\dot{x}^{(2)}_{kij}$ can be computed respectively.

Step 5. Time-invariant fuzzy system is chosen to be Mamdani fuzzy system with product inference engine. The variable weights are chosen to be (i) of Example 2.1.

Based on VWIM method, the time-variant fuzzy systems can be expressed as:

$$\begin{cases} \dot{x}_1 = \sum_{k=1}^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{m} \left(\frac{t_{k+1}-t}{t_{k+1}-t_k} \cdot C_{kij} \cdot \dot{x}^{(1)}_{kij} + \frac{t-t_k}{t_{k+1}-t_k} \cdot C_{(k+1)ij} \cdot \dot{x}^{(1)}_{(k+1)ij} \right) \cdot \chi_k(t) \\ \dot{x}_2 = \sum_{k=1}^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{m} \left(\frac{t_{k+1}-t}{t_{k+1}-t_k} \cdot C_{kij} \cdot \dot{x}^{(2)}_{kij} + \frac{t-t_k}{t_{k+1}-t_k} \cdot C_{(k+1)ij} \cdot \dot{x}^{(2)}_{(k+1)ij} \right) \cdot \chi_k(t) \\ C_{kij} \triangleq A_{ki}(x_1) \cdot B_{kj}(x_2), \chi_k(t) \triangleq \chi_{[t_k, t_{k+1})}(t) \end{cases} \tag{14}$$

The predefined error accuracy is chosen to be 0.1. Let $p = 22$, $n = 5$ and $m = 3$. The error curves of variables x_1 and x_2 are shown in Figures 2 and 3. The solution curves and the phase plane curve of system (14) and the comparison with corresponding curves on real system (13) are shown in Figures 4 – 6, where "$\cdots$" denotes the curve of system (13) and "–" denotes the curve of (14).

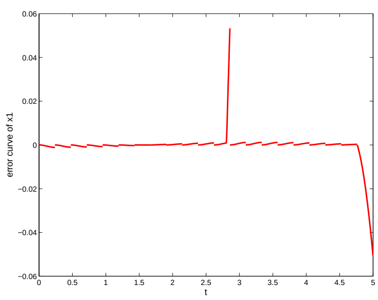

FIGURE 2. Error curve of variable x_1

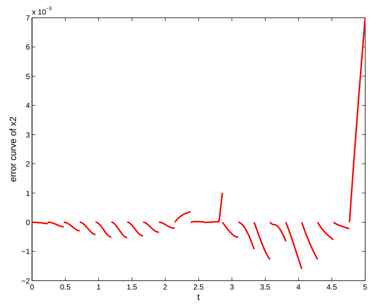

FIGURE 3. Error curve of variable x_2

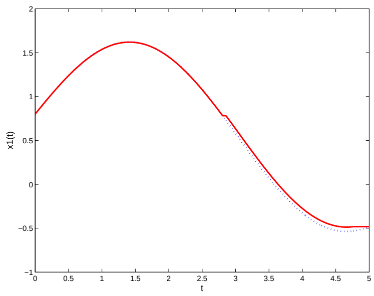

FIGURE 4. Comparison curve of variable x_1

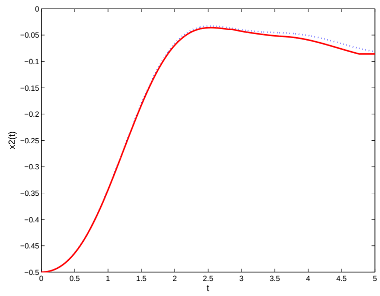

FIGURE 5. Comparison curve of variable x_2

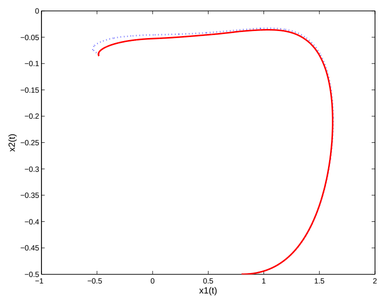

FIGURE 6. Comparison curve of phase plant (x_1, x_2)

From the simulation results, we can see that time-variant fuzzy system (14) based on VWIM method can approximate system (13) with the desired accuracy.

6. **Conclusions.** In this paper, we investigate the design and approximation problem of time-variant fuzzy systems. The main contributions include:

(1) A novel modeling method for time-variant fuzzy systems, called VWIM method has been proposed. From the mathematical point of view, the time-variant fuzzy systems based on VWIM method can be regarded as some interpolation functions. And, the approximation accuracy of various classes of time-variant fuzzy systems has been analyzed.

(2) The time-variant fuzzy systems based on VWIM method have been introduced to approximate a class of nonlinear systems. Some sufficient conditions for time-variant

fuzzy systems as universal approximators to a class of nonlinear systems have been given. By them, we can determine the numbers of fuzzy sets and fuzzy rules which are needed for approximating any nonlinear systems with given precision.

From the theoretical analysis and simulation result, we can learn that only if experts master adequate input-output data information of the systems, the time-variant fuzzy systems determined by VWIM method could reflect the dynamic behavior of systems with high precision. This conclusion shows that VWIM method can be used as a new tool to deal with the design and analysis of fuzzy controller.

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