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A projection-filter method for solving nonlinear complementarity problems ☆

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ABSTRACT

The Josephy–Newton method attacks nonlinear complementarity problems which consists of solving, possibly inexactly, a sequence of linear complementarity problems. Under appropriate regularity assumptions, this method is known to be locally (superlinearly) convergent. Utilizing the filter method, we presented a new globalization strategy for this Newton method applied to nonlinear complementarity problem without any merit function. The strategy is based on the projection-proximal point and filter methodology. Our linesearch procedure uses the regularized Newton direction to force global convergence by means of a projection step which reduces the distance to the solution of the problem. The resulting algorithm is globally convergent to a solution. Under natural assumptions, locally superlinear rate of convergence was established.

1. Introduction

In this paper, the system of nonlinear complementarity problems (NCP(F)) [1,2], with the following form, is considered

$$x \geqslant 0, \quad F(x) \geqslant 0 \quad \text{and} \quad x^T F(x) = 0, \tag{1}$$

where $x \in R^n$, $F : R^n \to R^n$ is a given function which is assumed to be continuously differentiable in an open set containing R^n_+. While there exists a wide rang of approaches for solving the NCP(F), see [1–3,18], some of the most successful and widely used are Newton-type algorithms based on solving successive linearization of the problem. Given a point x^k, the (Josephy-) Newton method [3–7] generates a next iterate x^{k+1} by solving the linear complementarity problem ($LCP(\varphi_k)$)

$$x \geqslant 0, \quad \varphi_k(x) \geqslant 0, \quad x^T \varphi_k(x) = 0, \tag{2}$$

where $\varphi_k(\cdot)$ is the first-order approximation of $F(\cdot)$ at x^k:

$$\varphi_k(x) := F(x^k) + \nabla F(x - x^k). \tag{3}$$

This paper is organized as follows: In Section 2, we first review some preliminary results that will be used in the subsequence analysis, and then introduce the filter technique. In Section 3, a projection-filter method is put forward. The convergent analysis is given in Section 4. Some remarks and conclusion are listed in the last section.

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2. Preliminaries

2.1. Basic properties

We start with some equivalent formulations of NCP(F), each of which will be useful in the subsequent analysis. For convenience of writing, we define that [8]:

$$[x]_+ := \max\{0, x\}, \quad [x]_- := \max\{0, -x\}, \tag{4}$$

and for any $\alpha > 0$,

$$y_\alpha(x) := [x - \alpha F(x)]_+. \tag{5}$$

This function has the following properties.

Lemma 2.1 [9].

1. *For any $\alpha \geqslant 0$, $\|x - y_\alpha(x)\|$ is monotone nondescent.*
2. *For any $\alpha > 0$, $\frac{\|x - y_\alpha(x)\|}{\alpha}$ is monotone nonincrease.*

It is obviously that the following statements are equivalent [10]:

1. $\bar{x}$ solves NCP(F).
2. $\bar{x}$ is a solution of the variational inequality problem over the nonnegative orthant R^n_+:

$$\bar{x} \in R^n_+, \quad (x - \bar{x})^T F(\bar{x}) \geqslant 0, \quad \forall x \in R^n_+. \tag{6}$$

3. $\bar{x}$ is a zero of the natural (projection) residual:

$$0 = r(\bar{x}) := \min\{\bar{x}; F(\bar{x})\} = \bar{x} - [\bar{x} - F(\bar{x})]_+. \tag{7}$$

Given a current iterate x^k and a regularization parameter $\mu_k > 0$, consider the regularized linear complementarity problem ($LCP(\varphi_k)$) (2) in which (3) is replaced by

$$\varphi_k(x) := F(x^k) + G_k(x - x^k) + \mu_k(x - x^k), \tag{8}$$

where G_k is a positive semidefinite matrix (presumably, the Jacobian of F or its approximation, if F is differentiable at x^k). Suppose $z^k \geqslant 0$ is some approximate solution of this problem with e^k being the associated natural residual [6,11]:

$$\min\{z^k; \varphi_k(z^k)\} = e^k. \tag{9}$$

2.2. Filter technique

In succession, we introduce some definitions and properties for filter technique [12–17,19–27].

To handle (1), NCP(F) is usually described as follow optimization programme, too

$$\begin{aligned} \min \quad & m(x) = \|x^T F(x)\|^2 \\ \text{s.t.} \quad & F(x) \geqslant 0, \\ & x \geqslant 0. \end{aligned} \tag{10}$$

We hope to find a satisfying point, which relates not only to objective function but also to constraint conditions. Then we define two functions related close to constraint conditions and objective function as follow

$$\begin{aligned} h(x) &= \|[F(x)]_-\|, \\ p(x) &= \|x^T F(x)\|^2 + \sigma h(x), \end{aligned} \tag{11}$$

where σ is a constant. Moreover, $x \geqslant 0$ is always satisfied.

Definition 2.1. A pair $(h^{(k)}, p^{(k)})$ is said to dominate another pair $(h^{(i)}, p^{(i)})$ if and only if both $h^{(k)} \leqslant h^{(i)}$ and $p^{(k)} \leqslant p^{(i)}$. Note $x^{(k)} \prec x^{(i)}$.

With this concept it is now possible to define a *filter*, which will be used in our algorithm as a criterion for accepting or rejecting a trial step.

Definition 2.2. A filter is a list of pairs $(h^{(l)}, p^{(l)})$ such that no pair dominates any other. A point $(h^{(k)}, p^{(k)})$ is said to be acceptable for inclusion in the filter if it is not dominated by any point in the filter.

Filter method is that we put the *good* points into one set, and call it ***filter*** set, note $\mathscr{F}$. At the kth point, if a new point is accepted by the *filter*, then update $\mathscr{F}_k$. We denote

$$D_{k+1} = \{i | h^{(i)} \geqslant h^{(k)}, p^{(i)} \geqslant p^{(k)}, i \in \mathscr{F}_k\},$$

then

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{k+1\} \setminus D_{k+1}. \tag{12}$$

To obtain the convergent results, the strong condition which is our criterion is required:

$$\text{either } h(x^k) \leqslant \beta h(x^i) \quad \text{or} \quad p(x^k) \leqslant -\gamma h(x^i) + p(x^i). \tag{13}$$

For convenience, the following notation is applied throughout this paper.

Definition 2.3.

$$\begin{aligned} h_k^I &= \min\{h_i | h_i > 0,\ i = 1, 2, \ldots, n\}, \\ p_k^F &= \min\{p_i | h_i = 0,\ 1 \leqslant i \leqslant n\}, \end{aligned} \tag{14}$$

and p_k^I is the corresponding value to h_k^I.

3. Algorithm

Algorithm 3.1 (*Projection-Filter Algorithm*).

Step 0. Initialization: Select $0 < \gamma_1,\ \sigma, \eta, t < 1,\ \gamma \in \left(0, \frac{1}{2}\right),\ \beta \in \left(\frac{1}{2}, 1\right),\ \Delta_0 > 0$. Choose any $x^0 \in R^n$. Set $k := 0, \mathscr{F}_k := \{0\}$.
Step 1. Calculate $h_k^I,\ p_k^I,\ p_k^F$.
Step 2. Inexact Newton step: Stop if $\|r(x^k)\| = 0$. Otherwise, choose a positive semidefinite matrix G_k and set $\mu_k = \|r(x^k)\|^t$. Choose $\rho_k \in [0, 1)$ and compute $z^k \in R_+^n$, an inexact solution of $LCP(\varphi_k)$ given by (2) and (8), such that

$$\|e^k\| \leqslant \rho_k \mu_k \|x^k - z^k\| \tag{15}$$

and

$$(e^k)^T [\varphi_k(z^k) + z^k - x^k] \leqslant \rho_k \mu_k \|x^k - z^k\|^2. \tag{16}$$

Step 3. Set

$$y^k := z^k - e^k \quad \text{and} \quad v^k := F(y^k) - \varphi_k(z^k) + e^k.$$

Let

$$\varepsilon^k = -v^k + \mu_k (x^k - y^k). \tag{17}$$

If

$$\|\varepsilon^k\| \leqslant \sigma \mu_k \|x^k - y^k\|, \tag{18}$$

then goto Step5.
Step 4. Linesearch step: Find

$$y^k = x^k - \alpha_k (x^k - z^k),$$

where $\alpha_k = \beta^{m_k}$ with m_k being smallest nonnegative integer m such that

$$F(x^k - \beta^m (x^k - z^k))^T (x^k - z^k) \geqslant \lambda (1 - \rho_k) \mu_k \|x^k - z^k\|^2. \tag{19}$$

Set $v^k := F(y^k)$.
Step 5. Projection step: Calculate

$$\widetilde{x}^k := x^k - \frac{(x^k - y^k)^T v^k}{\|v^k\|^2} v^k, \tag{20}$$

$$\widehat{x}^k := [\widetilde{x}^k]_+. \tag{21}$$

Step 6. Compute $\widehat{h}_k = h(\widehat{x}^k),\ \widehat{p}_k = p(\widehat{x}^k)$. If $\widehat{x}^k$ is not acceptable to the filter, or $h(\widehat{x}^k) > \eta_1 \min\{h_k^I, \alpha_1 \Delta_k^2\}$, call Restoration Algorithm to produce a point $x^k = x_r^k$ accepted by the filter, update h_k^I, go to Step 1.
Step 7. $x^{k+1} = \widehat{x}^k$ is accepted by the filter, $h_{k+1} = \widehat{h}_k,\ p_{k+1} = \widehat{p}_k$ and remove the points dominated by (h_{k+1}, p_{k+1}) from the filter. Update the filter $\mathscr{F}_k$.
Step 8. If $h(x^{k+1}) \leqslant \eta_1 \min\{h_{k+1}^I, \alpha_1 \Delta_k^2\}$, then let $k := k+1$, go to Step 2. Otherwise, let $k := k+1$, and call Restoration Algorithm to produce a point $x^k = x_r^k$, and go to Step 1.

Remark 1. By Step 3, when $m_k > 1,\ m_k - 1$ is not the solution of (19), we must have

$$F(x^k - \beta^{m_k-1}(x^k - z^k))^T(x^k - z^k) < \lambda(1-\rho_k)\mu_k\|x^k - z^k\|^2.$$

Suppose that the sequences $\{x^k\},\ \{\mu_k\},\ \{\rho_k\}$ and $\{z^k\}$ are all bounded, as $k_{\subset K} \to \infty$, we have

$$F(x^*)^T(x^* - z^*) \leqslant \lambda(1-\rho^*)\mu^*\|x^* - z^*\|^2.$$

Algorithm 3.2 (*Restoration Algorithm*).

Step 0. Let $0 < c_2 < 1 < c_1,\ 0 < c_3 < 1 < c_4,\ x_0^k = x^k,\ \Delta_k^0 = \Delta_k,\ j := 0.$
Step 1. If $h(x_j^k) \leqslant \eta\min\{h_k^l, \gamma_1\Delta_k^2\}$, and x_j^k is accepted by the filter, let $x_r^k = x_j^k$, stop.
Step 2. Compute

$$\begin{aligned} \min \quad & \Psi_k^j(u) = h(x_j^k) - \|[\nabla F(x_j^k)u + F(x_j^k)]_-\| \\ \text{s.t.} \quad & \|u\| \leqslant \triangle_j^k, \end{aligned} \tag{22}$$

get u_j^k.
Step 3. Calculate:

$$r_k^j = \frac{h(x_j^k) - h(x_{j+1}^k)}{\Psi_k^j(u_j^k)}.$$

If $r_k^j \leqslant c_2$, then $x_{j+1}^k = x_j^k,\ \Delta_k^{j+1} = \min\{\Delta_0, c_3\Delta_k^j\},\ j := j+1$, go to Step 2. If $r_k^j \geqslant c_1$, then $x_{j+1}^k = x_j^k,\ \Delta_k^{j+1} = \max\{\Delta_0, c_4\Delta_k^j\}$, $j := j+1$, go to Step 2.
Step 4. Otherwise, let $x_{j+1}^k = x_j^k + u_j^k,\ \Delta_k^{j+1} = \max\{\Delta_0, c_3\Delta_k^j\},\ j := j+1$, go to Step 1.

4. The convergence properties

The analysis to the algorithm are based on the following standard assumptions. Further, to obtain the convergence, the sufficient reduction plays a crucial role throughout.

Assumption 4.1.

(1) The set $\{x^k\} \in X$ is nonempty and bounded.
(2) The function $F(x)$ is twice continuously differentiable on an open set containing X.
(3) When solving (22), we have

$$-\Psi_k^j(d) = h(x_j^k) - \|[F(x_j^k) + \nabla F(x_j^k)^T d]_-\| \geqslant \beta_2\min\{h(x_j^k), \Delta_k^j\}, \tag{23}$$

where $\beta_2 > 0$ is a constant.
(4) The matrix sequence $\{G_k\}$ is bounded.
(5) The Restoration Algorithm has a solution satisfying $\|d_r^k\| \leqslant \tau_0 h_k$.

(1) and (2) are the standard assumptions.
(3) is the sufficient reduction condition, which is a very weak condition because Cauchy step satisfies this condition. It is regarded as a condition in this paper. In a trust region method.
(3) guarantees the global convergence.
(4) plays an important role to obtain the convergent result. But it has minor effects to the local convergent rate. The following results are based on Assumption 4.1.
Analyzing the Restoration Algorithm, we obtain

Lemma 4.1. *The Restoration Algorithm terminates finitely under Assumption 4.1.*

Proof. If $h_k^j \to 0$, the conclusion is correct by Step 1 of Algorithm 3.2. We show it by contradiction. Assume the Restoration Algorithm does not terminate finitely. Now we consider when $h_k^j \nrightarrow 0$, namely $\forall j,\ \exists\varepsilon > 0$ and $h_k^j > \varepsilon$. Note

$$K = \left\{ j | r_k^j = \frac{h(x_j^k) - h(x_{j+1}^k)}{h(x_j^k) - \|[F(x_j^k) + \nabla F(x_j^k)^T d]_-\|} > c_2 > 0 \right\}.$$

From the above set K and (3) of Assumption 4.1, we have

$$+\infty > \sum_{j=1}^{\infty}(h_k^j - h_k^{j+1}) \geqslant \sum_{j\in K}(h_k^j - h_k^{j+1}) = \sum_{j\in K}\frac{[h(x_j^k) - h(x_{j+1}^k)]\cdot\{h(x_j^k) - \|[F(x_j^k) + \nabla F(x_j^k)^T d]_-\|\}}{h(x_j^k) - \|[F(x_j^k) + \nabla F(x_j^k)^T d]_-\|}$$
$$= \sum_{j\in K} r_k^j \cdot \{h(x_j^k) - \|[F(x_j^k) + \nabla F(x_j^k)^T d]_-\|\} \geqslant \sum_{j\in K} c_2\beta_2 \min\{h(x_j^k), \Delta_k^j\}. \tag{24}$$

So we have $\sum_{k\in K}\Delta_k^j < \infty$.

Because of terminating infinitely, $\forall j \in K$, we have $\Delta_k^j \to 0$. Thus, the radius Δ_k^j of trust region could not decrease, i.e. $\Delta_k^{j+1} \geqslant \Delta_k^j$. It is contrast to the assumption. Consequently, the result holds and the proof is complete. □

Lemma 4.2. *Every new iteration $x^{k+1} \neq x^k$ is acceptable to the filter set $\mathcal{F}$.*

Proof. From Algorithm 3.1, a new iteration x^{k+1} which is produced in Step 6 or Step 7 is accepted by the filter $\mathcal{F}_k$. The result therefore holds and the proof is complete. □

Theorem 4.1 [26]. *Under Assumption 4.1 , suppose there are infinitely many points added to the filter. Then*

$$\lim_{k\to\infty} h(x^k) = 0,$$

otherwise

$$h(x^k) = 0.$$

We now state a preliminary result.

Lemma 4.3 [10]. *Let $x,\ y,\ v,\ \bar{x}$ be any elements of R^n such that*

$$(x-y)^T v > 0, \quad \text{and} \quad (\bar{x}-y)^T v \leqslant 0.$$

Let

$$\widehat{x} = x - \frac{(x-y)^T v}{\|v\|^2} v.$$

Then

$$\|\widehat{x} - \bar{x}\|^2 \leqslant \|x - \bar{x}\|^2 - \|\widehat{x} - x\|^2.$$

Theorem 4.2. *Suppose that F is continuous and monotone. Then any sequence $\{x^k\}$ generated by Algorithm 3.1 is bounded.*

Suppose further that there exist constants $C_1, C_2, C_3 > 0$ such that $\|G_k\| \leqslant C_1$ for all k, and $C_3 \leqslant \mu_k \leqslant C_2$ starting with some index k_0.

Suppose that

$$\limsup_{k\to\infty} \rho_k < \min\{1; 1/C_2\}.$$

Then $\{x^k\}$ converges to some $\bar{x}$, which is a solution of NCP(F).

Proof. We discuss this theorem in two parts.

We prove that the sequence $\{x^k\}$ generated by Algorithm 3.1 is bounded. By properties of the projection [28, p. 121] it follows that

$$\{z^k - \varphi_k(z^k) - [z^k - \varphi_k(z^k)]_+\}^T\{x^k - [z^k - \varphi_k(z^k)]_+\} \leqslant 0.$$

Notice that

$$[z^k - \varphi_k(z^k)]_+ = z^k - e^k$$

and $x^k \in R_+^n$. Therefore,

$$[-\varphi_k(z^k) + e^k]^T(x^k - z^k + e^k) \leqslant 0.$$

Making use of the latter inequality, and by (8) and (16), the positive semidefinite matrix G_k, we further obtain

$$\begin{aligned}
F(x^k)^T(x^k - z^k) &= F(x^k)^T(x^k - z^k + e^k) - F(x^k)^T e^k \\
&= [F(x^k) - \varphi_k(z^k) + e^k]^T(x^k - z^k + e^k) - [-\varphi_k(z^k) + e^k]^T(x^k - z^k + e^k) - F(x^k)^T e^k \\
&\geqslant [F(x^k) - \varphi_k(z^k) + e^k]^T(x^k - z^k + e^k) - F(x^k)^T e^k \\
&= [F(x^k) - \varphi_k(z^k)]^T(x^k - z^k + e^k) + (e^k)^T(x^k - z^k + e^k) - F(x^k)^T e^k \\
&= [(G_k + \mu_k I)(x^k - z^k)]^T(x^k - z^k + e^k) + (e^k)^T[x^k - z^k - F(x^k)] + \|e^k\|^2 \\
&\geqslant [(G_k + \mu_k I)(x^k - z^k)]^T(x^k - z^k) - (e^k)^T\{[(G_k + \mu_k I)(z^k - x^k) + F(x^k)] + z^k - x^k\} \\
&\geqslant \mu_k\|x^k - z^k\|^2 - (e^k)^T[\varphi_k(z^k) + z^k - x^k] \geqslant \mu_k(1 - \rho_k)\|x^k - z^k\|^2.
\end{aligned} \tag{25}$$

We now show that the linesearch procedure (19), if activated, always terminates with a positive stepsize α_k. Suppose that this is not the case for some iteration index k, i.e., for all integers m we have

$$F(x^k - \beta^m(x^k - z^k))^T(x^k - z^k) < \lambda(1-\rho_k)\mu_k\|x^k - z^k\|^2.$$

Since F is continuous, and $\lambda, \beta \in (0,1)$, passing onto the limit as $m \to \infty$, we obtain

$$F(x^k)^T(x^k - z^k) \leqslant \lambda(1-\rho_k)\mu_k\|x^k - z^k\|^2 < \mu_k(1-\rho_k)\|x^k - z^k\|^2. \tag{26}$$

Which contradicts (25). Therefore the linesearch step is well-defined.

By Step 4, Step 5 and Step 6 of Algorithm 3.1, when the point is accepted by the filter, $x^{k+1} = [\hat{x}^k]_+$. Let $\bar{x} \in R^n_+$ be any solution of NCP(F). It is easy to see that

$$\|x^{k+1} - \bar{x}\| \leqslant \|\hat{x}^k - \bar{x}\|. \tag{27}$$

Now, we discuss the two cases for (18) in Algorithm 3.1.

(1) If (18) is satisfied. By using (17) and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}(v^k)^T(x^k - y^k) &= [\mu_k(x^k - y^k) - \varepsilon^k]^T(x^k - y^k) = \mu_k\|x^k - y^k\|^2 - (\varepsilon^k)^T(x^k - y^k) \geqslant \mu_k\|x^k - y^k\|^2 - \|\varepsilon^k\|\|x^k - y^k\| \\ &\geqslant \mu_k(1-\sigma)\|x^k - y^k\|^2 > 0.\end{aligned} \tag{28}$$

(2) If (18) is not satisfied. By Step 4, (25) and $\limsup_{k\to\infty}\rho_k < \min\{1; 1/C_2\}$, we have

$$(v^k)^T(x^k - y^k) = F(y^k)^T(x^k - y^k) = F(y^k)^T\{x^k - [x^k + \alpha_k(z^k - x^k)]\} = \alpha_k F(y^k)^T(x^k - z^k) \geqslant \alpha_k\mu_k(1-\rho_k)\|x^k - z^k\|^2 > 0.$$

In this respect, the only difference is the choice of y^k and v^k.

By Lemma 4.3, in the either case $(v^k)^T(x^k - y^k) \leqslant 0$, it follows that

$$\|\hat{x} - \bar{x}\|^2 \leqslant \|x^k - \bar{x}\|^2 - \|\hat{x} - x^k\|^2.$$

Combining the latter relation with (27), we obtain

$$\|x^{k+1} - \bar{x}\|^2 \leqslant \|x^k - \bar{x}\|^2 - \|\hat{x} - x^k\|^2 \leqslant \|x^k - \bar{x}\|^2. \tag{29}$$

It immediately follows that the sequence $\{\|x^k - \bar{x}\|\}$ is monotone, so it converges. Therefore, $\{x^k\}$ is bounded.

Now, we prove that the sequence $\{x^k\}$ converges to the solution of NCP(F).

We consider the two possible cases:

$$\liminf_{k\to\infty}\|r(x^k)\| = 0 \tag{30}$$

and

$$\liminf_{k\to\infty}\|r(x^k)\| > 0. \tag{31}$$

(1) In the first case, by continuity of $r(\cdot)$ and boundedness of $\{x^k\}$, $\exists x^*$, s.t. $x^k \to x^*(k \to \infty)$ and $r(x^*) = 0$. Therefore x^* is a solution of NCP(F). We can choose $\bar{x} = x^*$ in (29). Because the sequence $\{\|x^k - x^*\|\}$ converges, it must be the case that $\{x^k\} \to x^*$ which is a solution of NCP(F).

(2) We consider now the second case. By (29), it follows that

$$\lim_{k\to\infty}\|\hat{x}^k - x^k\| = 0,$$

or by (20), equivalently,

$$\lim_{k\to\infty}\frac{(v^k)^T(x^k - y^k)}{\|v^k\|} = 0. \tag{32}$$

By (31) and Step 1 in Algorithm 3.1, it then follows that $\mu_k = \|r(x^k)\|^t > 0$ for all k. By (25) and the Cauchy–Schwarz inequality, we obtain

$$\|F(x^k)\|\|x^k - z^k\| \geqslant F(x^k)^T(x^k - z^k) \geqslant \mu_k(1-\rho_k)\|x^k - z^k\|^2 \geqslant C_3(1-\rho_k)\|x^k - z^k\|^2.$$

Hence,

$$\|F(x^k)\| \geqslant C_3(1-\rho_k)\|x^k - z^k\|.$$

Taking into account boundedness of $\{x^k\}$ and continuity of F, and $\limsup_{k\to\infty}\rho_k < \min\{1, 1/C_2\}$, we conclude that the sequence $\{x^k - z^k\}$ is bounded. It now easily follows that the sequences $\{z^k\}$, $\{e^k\}$ and $\{x^k\}$ are all bounded.

By (9), (7), (8), (15), the triangle and Cauchy–Schwarz inequalities, and the nonexpansiveness of the projection operator, we have

$$\begin{aligned}\|x^k - z^k\| &\geqslant \|x^k - (z^k - e^k)\| - \|e^k\| = \|x^k - [z^k - \varphi_k(z^k)]_+\| - \|e^k\| \\ &\geqslant \|x^k - [x^k - F(x^k)]_+\| - \|[x^k - F(x^k)]_+ - [z^k - \varphi_k(z^k)]_+\| - \|e^k\| \geqslant \|r(x^k)\| - \|x^k - F(x^k) - z^k + \varphi_k(z^k)\| - \|e^k\| \\ &\geqslant \|r(x^k)\| - \|x^k - z^k\| - \| - F(x^k) + \varphi_k(z^k)\| - \|e^k\| \geqslant \|r(x^k)\| - \|x^k - z^k\| - \|(G_k + \mu_k I)(x^k - z^k)\| - \|e^k\| \\ &\geqslant \|r(x^k)\| - \|x^k - z^k\| - \|G_k + \mu_k I\|\|x^k - z^k\| - \rho_k\mu_k\|x^k - z^k\| = \|r(x^k)\| - (1 + \|G_k + \mu_k I\| + \rho_k\mu_k)\|x^k - z^k\| \\ &\geqslant \|r(x^k)\| - (1 + C_1 + C_2)\|x^k - z^k\|.\end{aligned}$$

Therefore, $(2 + C_1 + C_2)\|x^k - z^k\| \geqslant \|r(x^k)\|$. Combining with (31), it is easy to see that

$$\liminf_{k\to\infty} \|x^k - z^k\| > 0. \tag{33}$$

Since $\|G_k\| \leqslant C_1,\ \mu_k = \|r(x^k)\|^t$ and $\{z^k\},\ \{e^k\}$ are bounded, it follows that $\{F(z^k - e^k)\}$ and $\varphi_k(z^k)$ are bounded. Therefore, by the triangle inequalities, for some $C_4 > 0$,

$$\|v^k\| = \|F(z^k - e^k) - \varphi_k(z^k) + e^k\| \leqslant \|F(z^k - e^k)\| + \|\varphi_k(z^k)\| + \|e^k\| \leqslant 1/C_4, \tag{34}$$

and by (18)

$$\|x^k - z^k + e^k\| \geqslant \|x^k - z^k\| - \|e^k\| \geqslant (1 - \rho_k\mu_k)\|x^k - z^k\| \geqslant (1 - \rho_k C_2)\|x^k - z^k\|. \tag{35}$$

Suppose that condition (18) in Algorithm 3.1 holds an infinite number of times. For such iterations k, by (28), (34) and (35), we have (recall also that $v^k \neq 0$)

$$\begin{aligned}\frac{(v^k)^T(x^k - y^k)}{\|v^k\|} &\geqslant \frac{\mu_k(1-\sigma)\|x^k - y^k\|^2}{\|v^k\|} \geqslant C_4C_3(1-\sigma)\|x^k - y^k\|^2 = C_4C_3(1-\sigma)\|x^k - z^k + e^k\|^2 \\ &\geqslant C_4C_3(1-\sigma)(1-\rho_k C_2)\|x^k - z^k\|^2.\end{aligned} \tag{36}$$

Passing onto the limit in (36) and taking into account (32), we obtain

$$\liminf_{k\to\infty} \|x^k - z^k\|^2 = 0,$$

which contradicts (33). We conclude that if $\liminf_{k\to\infty}\|r(x^k)\| > 0$, then condition (18) in Algorithm 3.1 may not more than a finite number of times.

Hence, we can assume that for all k sufficiently large, y^k and v^k are obtain through the linesearch step (19), in which case

$$\frac{(v^k)^T(x^k - y^k)}{\|v^k\|} = \frac{\alpha_k F(y^k)^T(x^k - z^k)}{\|F(y^k)\|} \geqslant \frac{\alpha_k(1-\rho_k)\mu_k\|x^k - z^k\|^2}{\|F(y^k)\|}.$$

Using (32), taking into account boundedness of $\{F(y^k)\}$, and the fact $\mu_k \geqslant C_4$ and $\limsup_{k\to\infty}\rho_k < 1$, we have

$$\lim_{k\to\infty} \alpha_k\|x^k - z^k\| = 0.$$

Because of (33), we conclude that it must be the case that

$$\lim_{k\to\infty} \alpha_k = 0.$$

Because of $\alpha_k = \beta^{m_k}$ and $\beta \in (0,1)$, it is equivalent to saying that $m_k \to \infty$. By Remark 1 (behind Algorithm 3.1), taking into account boundedness of the sequences $\{x^k\},\ \{\mu_k\},\ \{\rho_k\}$ and $\{z^k\}$, and passing onto a subsequence if necessary, as $k \to \infty$, and taking into account that $\mu^* > 0$, $\|x^* - z^*\| > 0$ (by (34)) and $\rho^* \leqslant \limsup_{k\to\infty}\rho_k < 1,\ \lambda \in (0,1)$, it is easy to see that

$$F(x^*)^T(x^* - z^*) \leqslant \lambda(1-\rho^*)\mu^*\|x^* - z^*\|^2 < (1-\rho^*)\mu^*\|x^* - z^*\|^2. \tag{37}$$

On the other hand, passing onto the limit in (25), we have that

$$F(x^*)^T(x^* - z^*) \geqslant (1-\rho^*)\mu^*\|x^* - z^*\|^2,$$

which contradicts to (37). Hence the case $\liminf_{k\to\infty}\|r(x^k)\| > 0$ is not possible. This completes the proof. □

Using the similar proof to [10], we can get the superlinear convergence.

Theorem 4.3. *Let F be monotone and continuous on R^n. Let x^* be the (unique) solution of NCP(F) at which F is differentiable with $\nabla F(\bar{x})$ positive definite. Let $\nabla F(\bar{x})$ be locally Hölder continuous around $\bar{x}$ with degree $p \in (0,1]$. Suppose that*

$$\lim_{k\to\infty} \rho_k = 0,$$

and starting with some $k_0, G_k = \nabla F(x^k)$. Then the sequence $\{x^k\}$ converges to x^* Q-supperlinearly.

5. Concluding remarks

We presented a new globalization strategy for the Newton method applied to nonlinear complementarity problem. Our strategy is based on the projection-proximal point and filter methodology. The resulting algorithm is globally convergent to a solution. Under natural assumptions, locally superlinear rate of convergence was also established.

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