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A novel approach to topological defects in a vector order parameter system*

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Based on Duan's topological current theory, we propose a novel approach to study the topological properties of topological defects in a two-dimensional complex vector order parameter system. This method shows explicitly the fine topological structure of defects. The branch processes of defects in the vector order parameter system have also been investigated with this method.

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In recent years, a great deal of work on topological defects in a two-dimensional (2D) complex vector order parameter system has been done by physicists in various fields.[1−15] Topological defects in a 2D complex vector order parameter system appear in a variety of physical scenarios, including two-component Bose condensates,[1] counter-propagating waves in nonlinear media[2] and large aperture lasers.[3,4] (For other examples see Refs.[5]–[15].) In this communication, based on Duan's topological current theory,[16−21] we propose a novel approach to investigate the fine topological structure and branch processes of topological defects in the 2D complex vector order parameter system.

Usually, the dynamics of the 2D complex vector order parameter system is modeled by the complex vector Ginzburg–Landau equations,[5]

$$\begin{aligned}\partial_t A_\pm =&\ A_\pm + (1+\mathrm{i}\alpha)\nabla^2 A_\pm\\ &-(1+\mathrm{i}\beta)(|A_\pm|^2+\gamma\,|A_\pm|^2)A_\pm,\end{aligned}\qquad (1)$$

where $A_\pm$ are the two components of the complex vector field. In optics they are identified with the right and left circularly polarized components of the transverse field. The parameter α and β are two real parameters, and the parameter γ represents the coupling between A_+ and A_-. Topological defects in Eq.(1) can be classified in two groups:[5] vector and scalar defects. Vector defects are points where the two components, A_+ and A_-, vanish at the same time. Scalar defects are points at which only one of the two components, A_+ or A_-, vanishes. Usually, in order to investigate the properties of defects conveniently, $A_\pm$ are expressed as $|A_\pm|\,\mathrm{e}^{\mathrm{i}\theta_\pm}$. The phase singularities of $\theta_\pm$ correspond to the defects of $A_\pm$.

Charge density current of defects. In our novel theory of topological defects in the 2D complex vector order parameter system, we write

$$A_\pm(x,t)=\phi^1_{(\pm)}(x,t)+\mathrm{i}\phi^2_{(\pm)}(x,t),\qquad (2)$$

instead of adopting the traditional expression $A_\pm = |A_\pm|\,\mathrm{e}^{\mathrm{i}\theta_\pm}$; using the $\boldsymbol{\phi}_\pm$, we can define a unit vector $n^a_\pm = \dfrac{\phi^a_\pm}{\|\phi_\pm\|}(\|\phi_\pm\|^2=\phi^a_\pm\phi^a_\pm=\|A_\pm\|^2, a=1,2)$. Using this $\boldsymbol{n}$ field, we can define a density current as

$$j^\mu_\pm=\frac{1}{2\pi}\varepsilon^{\mu\nu\lambda}\varepsilon_{ab}\partial_\nu n^a_\pm\partial_\lambda n^b_\pm.\qquad (3)$$

Obviously, this is a topological current. The temporal component of Eq.(3) is defined as the density of the defect charge: $j^0_\pm=\rho_\pm$. It is clear that this topologically current is identically reserved, i.e. $\partial_\mu j^\mu_\pm=0$. According to Duan's topological current theory,[16−21] the topological current $j^\mu_\pm$ can be rewritten in a compact form

$$j^\mu_\pm=\delta^{(2)}(\boldsymbol{\phi}_\pm)D^\mu\left(\frac{\phi_\pm}{x}\right),\qquad (4)$$

where $D^\mu\left(\dfrac{\phi_\pm}{x}\right)=\dfrac{1}{2}\varepsilon^{\mu\nu\lambda}\varepsilon_{ab}\partial_\nu\phi^a_\pm\partial_\lambda\phi^b_\pm$ is the vector Jacobians of $\boldsymbol{\phi}_\pm$. The delta function expression in

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Eq.(4) of the topological current j^μ tells us that only when defects exist, i.e., $A_\pm = 0$, will j^μ not vanish. So it is necessary to study the zero points of $A_\pm$ to determine the nonzero solutions of $j^\mu_\pm$. The implicit function theorem[22] shows that under the regular condition

$$D^\mu\left(\frac{\phi_\pm}{x}\right) \neq 0, \tag{5}$$

the general solutions of $A_\pm = 0$, i.e., $\phi^a_\pm(t, x^1, x^2) = 0$ $(a = 1, 2)$ can be expressed as

$$x^a = x^a_{k\pm}(t), \quad (a = 1, 2; k_\pm = 1, 2, \ldots, N_\pm), \tag{6}$$

which represent the world lines of $N_\pm$ moving isolated singular points. These singular solutions are just the scalar defects located at the zero points of field $A_\pm$. Then a question arises naturally: what are the topological charges of the defects? Now, we will investigate them. Let $M_{k\pm}$ be the neighbourhood of $\boldsymbol{x}_{k\pm}$ with a boundary $\partial M_{k\pm}$ satisfying $\boldsymbol{x}_{k\pm} \notin \partial M_{k\pm}$, $M_{i\pm} \cap M_{j\pm} = \emptyset, (i \neq j)$. Then the generalized winding number $W_{k\pm}$ of $n^a_\pm$ at $\boldsymbol{x}_{k\pm}$ can be defined by the Gauss map $n : \partial M_{k\pm} \to S^1$

$$W_{k\pm} = \frac{1}{2\pi}\int_{\partial M_{k\pm}} n^*_\pm(\varepsilon_{ab} n^a_\pm \mathrm{d} n^b_\pm), \tag{7}$$

where $n^*_\pm$ is the pull back of map $n_\pm$. The generalized winding number is a topological invariant and is also called the degree of the Gauss map. Using Stokes's theorem in exterior differential form and the result in Eq.(4), we obtain

$$W_{k\pm} = \int_{M_{k\pm}} \delta^{(2)}(\boldsymbol{\phi}_\pm) D^0\left(\frac{\phi_\pm}{x}\right) \mathrm{d}^2 x, \tag{8}$$

which is just the integral of the temporal component of $j^\mu_\pm$ in $M_{k\pm}$. This explicitly shows that $j^0_\pm$ is just the charge density of the defect. Therefore, the winding number $W_{k\pm}$ represents the topological charge of the defect $\boldsymbol{x}_{k\pm}$.

Fine topological structure of defects. In order to explore the fine topological structure of $j^\mu_\pm$, one can expand the $\delta^{(2)}(\boldsymbol{\phi}_\pm)$ as is done in Refs.[16–21, 23]

$$\delta^{(2)}(\boldsymbol{\phi}_\pm) = \sum_{k_\pm}^{N_\pm} \beta_{k\pm}\eta_{k\pm}\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_{k\pm}), \tag{9}$$

where the positive integer $\eta_{k\pm}$ is called the Hopf index of map $x \to \phi_\pm$, and $\beta_{k\pm}$ is the Brouwer degree: $\beta_{k\pm} = \operatorname{sgn} D^0\left(\frac{\phi_\pm}{x}\right)\Big|_{\boldsymbol{x}_{k\pm}} = \pm 1$. It can be proved from Eq.(6) that the velocity of the $k_\pm$ defect is determined by

$$V^\mu_{k\pm} = \frac{\mathrm{d}x^\mu_{k\pm}}{\mathrm{d}t} = \left.\frac{D^\mu\left(\frac{\phi_\pm}{x}\right)}{D^0\left(\frac{\phi_\pm}{x}\right)}\right|_{\boldsymbol{x}_{k\pm}}. \tag{10}$$

Then substituting Eqs.(10) and (9) into Eq.(4), we obtain the dynamic form of the topological current $j^\mu_\pm$:

$$j^\mu_\pm = \sum_{k=1}^{N_\pm} \beta_{k\pm}\eta_{k\pm}\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_{k\pm})\frac{\mathrm{d}x^\mu_{k\pm}}{\mathrm{d}t} \tag{11}$$

and the topological charge of the $k_\pm$ defect

$$\begin{aligned} Q_{k\pm} &= \int_{M_{k\pm}} \rho_\pm \mathrm{d}^2 x \\ &= \int_{M_{k\pm}} \beta_{k\pm}\eta_{k\pm}\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_{k\pm})\mathrm{d}^2 x \\ &= \beta_{k\pm}\eta_{k\pm} \\ &= W_{k\pm}, \end{aligned} \tag{12}$$

where the integral is carried out over $M_{k\pm}$. When the integration is over the entire 2D space M, we obtain the total topological charge of the defect set in the 2D complex vector order parameter system,

$$\begin{aligned} Q_\pm &= \int_M \rho \mathrm{d}^2 x \\ &= \int_M \sum_{k=1}^{N_\pm} \beta_{k\pm}\eta_{k\pm}\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_{k\pm})\mathrm{d}^2 x \\ &= \sum_{k=1}^{N_\pm} W_{k\pm}. \end{aligned} \tag{13}$$

So far, we have only studied the charge density current and fine structure of the defect in the 2D complex vector order parameter system. Here we give some remarks arranged in order:

(i) The defects in the 2D complex vector order parameter system are generated at the zero points of the field $A_\pm$. For scalar defects, the two topological indices $W_{k\pm}$ are the topological charge of the defect $\boldsymbol{x}_{k\pm}$ in $A_\pm$. For vector defects, which are the zero points of both A_+ and A_-, the topological indices are characterized by both W_{k+} and W_{k-} i.e., (W_{k+}, W_{k-}). When $W_{k+} = W_{k-}$, the vector defect is of an argument type. If $W_{k+} = -W_{k-}$, the vector defect is of a director type. These results can be found in the Ref.[5], so they are not surprising.

(ii) The topological indices $W_{k\pm}$ have fine structures, i.e., $W_{k\pm} = \beta_{k\pm}\eta_{k\pm}$. Therefore, using $\beta_{k\pm}$ and

$\eta_{k\pm}$ to characterize the defect is more rigorous than only using its winding number $W_{k\pm}$. These results are more general than those usually given and will be helpful as a complement to previous researches.[1−15]

(iii) The density current $j^\mu_\pm$ is expressed as a delta function form. The fine topological structure of defects has been obtained directly from the density current $j^\mu_\pm$.

In the above discussions, we have used the regular condition (5). When this condition fails, a branch process will occur.[16−21,24] The solutions of $A_\pm = 0$ and $D^0(\phi/x) = 0$ are called branch points. There are two kinds of branch points, namely, limit points and bifurcation points. Each kind corresponds to a different case of branch processes. We denote one of the branch points as $(t^*_\pm, \boldsymbol{z}_{k\pm})$. In the following we will discuss them in detail.

Branch processes at limit points. The limit points are determined by $A_\pm = 0$ and

$$D^0\left(\frac{\phi_\pm}{x}\right)\bigg|_{(\boldsymbol{x}_{k\pm},t)} = 0,\quad D^1\left(\frac{\phi_\pm}{x}\right)\bigg|_{(\boldsymbol{x}_{k\pm},t)} \neq 0; \tag{14}$$

or

$$D^0\left(\frac{\phi_\pm}{x}\right)\bigg|_{(\boldsymbol{x}_{k\pm},t)} = 0,\quad D^2\left(\frac{\phi_\pm}{x}\right)\bigg|_{(\boldsymbol{x}_{k\pm},t)} \neq 0. \tag{15}$$

For simplicity, we only consider the case (14). Taking account of Eq.(14) and using the implicit function theorem,[22] we have a unique solution of $A_\pm$ in the neighbourhood of the limit point $(t^*_\pm, \boldsymbol{z}_{k\pm})$:

$$t = t(x^1), \quad x^2 = x^2(x^1), \tag{16}$$

with $t^*_\pm = t(z^1_{k\pm})$. In the present case, we know that

$$\frac{\mathrm{d}x^1}{\mathrm{d}t}\bigg|_{(t^*_\pm,\boldsymbol{z}_{k\pm})} = \frac{D^1\left(\dfrac{\phi_\pm}{x}\right)}{D^0\left(\dfrac{\phi_\pm}{x}\right)}\Bigg|_{(t^*_\pm,\boldsymbol{z}_{k\pm})} = \infty. \tag{17}$$

Then the Taylor expansion of $t = t(x^1)$ at the limit point $(t^*_\pm, \boldsymbol{z}_{k\pm})$ is

$$t - t^*_\pm = \frac{1}{2}\,\frac{\mathrm{d}^2 t}{(\mathrm{d}x^1)^2}\bigg|_{(t^*_\pm,\boldsymbol{z}_{k\pm})} (x^1 - z^1_{k\pm})^2, \tag{18}$$

which is a parabola in $x^1 - t$ plane. If $\mathrm{d}^2t/(\mathrm{d}x^1)^2\big|_{(t^*_\pm,\boldsymbol{z}_{k\pm})} > 0$, we have the branch solutions for $t > t^*_\pm$; otherwise, we have the branch solutions for $t < t^*_\pm$. These two cases are related to the origin and the annihilation of the defects. One of the results of Eq.(17) is that the velocity of defects is infinite when they are being annihilated or generated, which is gained only from the topology of $A_\pm$. This agrees with the result obtained by Bray.[24] Since topological current is identically conserved, the topological charges of the generated or annihilated defect pair must be opposite to each other at the limit point, i.e., $\beta_{l_1\pm}\eta_{l_1\pm} = -\beta_{l_2\pm}\eta_{l_2\pm}$, indicating that $\beta_{l_1\pm} = \beta_{l_2\pm}$ and $\eta_{l_1\pm} = -\eta_{l_2\pm}$. One can see that the Brouwer degree $\eta_{k\pm}$ being indefinite at the limit points implies a discontinuous change at limit points along the world lines of the defects (from ±1 to ∓1).

Branch processes at bifurcation points. For a limit point it is required that $D^1\left(\frac{\phi_\pm}{x}\right)\Big|_{(\boldsymbol{x}_{k\pm},t)} \neq 0$. As to a bifurcation point $(t^*, \boldsymbol{z}_l)$,[25] it must satisfy a more complex condition,

$$D^\mu\left(\frac{\phi_\pm}{x}\right)\bigg|_{\boldsymbol{z}_{k\pm}} = 0 \quad (\mu = 0,1,2), \tag{19}$$

which leads to the important fact that the function relationship between t and x^1 is not unique in the neighbourhood of the bifurcation point $(t^*_\pm, \boldsymbol{z}_{k\pm})$. It is easy to see that

$$V^1_{k\pm} = \frac{\mathrm{d}x^1}{\mathrm{d}t}\bigg|_{\boldsymbol{z}_{k\pm}} = \frac{D^1\left(\dfrac{\phi_\pm}{x}\right)}{D^0\left(\dfrac{\phi_\pm}{x}\right)}\Bigg|_{\boldsymbol{z}_{k\pm}},\quad V^2_{k\pm} = \frac{\mathrm{d}x^2}{\mathrm{d}t}\bigg|_{\boldsymbol{z}_{k\pm}} = \frac{D^2\left(\dfrac{\phi_\pm}{x}\right)}{D^0\left(\dfrac{\phi_\pm}{x}\right)}\Bigg|_{\boldsymbol{z}_{k\pm}}. \tag{20}$$

This directly shows that the direction of the integral curve of Eq.(20) is indefinite at $(t^*_\pm, \boldsymbol{z}_{k\pm})$, i.e., the velocity field of the defect is indefinite at $(t^*_\pm, \boldsymbol{z}_{k\pm})$. In addition, according to Duan's topological current theory,[16−21] the Taylor expansion of the solution of $A_\pm = 0$ in the neighbourhood of the bifurcation point can be expressed generally as

$$A(x^1 - z^1_{k\pm})^2 + 2B(x^2 - z^2_{k\pm})(t - t^*_\pm) + (t - t^*_\pm)^2 = 0, \tag{21}$$

which leads to

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}t} + C = 0 (A \neq 0), \tag{22}$$

where A, B, and C are three constants. The solution of Eq.(22) gives different motion directions of defects at the bifurcation point. There are two important cases.

Case 1 $\Delta = 4(B^2 - AC) > 0$.

From Eq.(22) we can obtain two different motion directions of the defects: $(\mathrm{d}x^1/\mathrm{d}t)\big|_{1,2} = (-B \pm \sqrt{B^2 - AC})/A$. This is the intersection of the moving directions of the two defects, i.e. the two defects meet and then depart into different directions at the bifurcation point.

Case 2 $\Delta = 4(B^2 - AC) = 0$).

From Eq.(22), we obtain only one motion direction of the defects at the bifurcation point: $(\mathrm{d}x^1/\mathrm{d}t)\big|_{1,2} = -B/A$, which includes three sub-cases: (a) one defect splits into two defects; (b) two defects merge into one; (c) two defects tangentially intersect at the bifurcation point.

In both cases 1 and 2, we know that the sum of the topological charges of the final defects must be equal to that of the initial defects at the bifurcation point.

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