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# NONTRIVIAL SOLUTIONS FOR SEMILINEAR SCHRÖDINGER EQUATIONS*

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**Abstract** The authors prove the existence of nontrivial solutions for the Schrödinger equation $-\Delta u + V(x)u = \lambda f(x,u)$ in $\mathbb{R}^N$, where $f$ is superlinear, subcritical and critical at infinity respectively, $V$ is periodic.

**Key words** Schrödinger equation, the relative Morse index, minimax method

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## 1 Introduction

In this article, we consider the existence of nontrivial solutions for nonlinear Schrödinger equation

$$-\Delta u + V(x)u = \lambda f(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $\lambda > 0$, $f$ is superlinear, subcritical and critical.

In the case that both $V(x)$ and $f(x,\cdot)$ are periodic, problem (1.1) has been studied by [8–11], [14], [15], [17] etc for subcritical case. It is well known, see for instance [19], that the spectrum $\sigma(-\Delta+V)$ of $-\Delta+V$ consists of essential spectrum. In general, one assumes that 0 belongs to the spectral gap of the operator $-\Delta+V$. Because the problem is setting on the whole space, the so-called Palais-Smale condition generally fails to be held. Using the concentration-compactness principle due to [12], [13], one may rule out the vanishing for the Palais-Smale sequence. The non-vanishing and the assumption of period of functions then allow one to obtain eventually a nontrivial solution of (1.1).

Critical problems

$$-\Delta u + V(x)u = \lambda[K(x)|u|^{2^*-2}u + f(x,u)] \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

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where $2^*=\frac{2N}{N-2}$ is a Sobolev exponent with $N\ge 4$, were considered in [4–6] and so on. It is assumed in these works either that the operator $-\Delta+V$ is compact, see [5], or that functions are periodic in $x$, see [4] and [6]. So in spirit of the work [2], one may obtain nontrivial solutions.

In this paper, we suppose that

$(V)$ $V(x)\in C(\mathbb{R}^N)$ is periodic and $\sigma(-\Delta+V)\cap(-\infty,0)\neq\emptyset$, $0\notin\sigma(-\Delta+V)$, where $\sigma$ denotes the spectrum in $L^2(\mathbb{R}^N)$;

$(f_1)$ $f\in C^1(\mathbb{R}^N\times\mathbb{R})$, $f'(x,t)\ge 0$ for $(x,t)\in\mathbb{R}^N\times\mathbb{R}$;

$(f_2)$ $f(x,0)=f'(x,0)=0$;

$(f_3)$ There exists $\theta>2$ such that $0\le\theta F(x,t)\le tf(x,t)$ for $t\in\mathbb{R}$ for $x\in\mathbb{R}^N$ and $\neq 0$, where $F(t)=\int_0^t f(s)\mathrm{d}s$;

$(f_4)$ $|f(x,t)|\le C(1+|t|^p)$, where $1<p<\frac{N+2}{N-2}$ if $N\ge 3; 1<p<\infty$ if $N=2$.

Our first result is concerning subcritical problems.

**Theorem 1.1** Suppose $(V)$ and $(f_1)-(f_4)$ hold. Then there exists $\lambda^*>0$ such that for $0<\lambda<\lambda^*$ problem (1.1) possesses at least a $H^1(\mathbb{R}^N)$ nontrivial solution.

Let $\bar\lambda=\sup\lambda^*$ so that problem (1.1) possesses at least a nontrivial solution if $0<\lambda<\bar\lambda$; it is not clear if $\bar\lambda$ is finite. By the elliptic regular theory, the solution obtained in Theorem 1.1 is a classical one.

We remark that we require neither $f(x,\cdot)$ is periodic nor the limiting behavior of $f(x,\cdot)$ as $x\to\infty$. Hence, the concentration-compactness principle and the arguments in previous works are not applicable. To prove Theorem 1.1, we consider problem

$$\begin{cases}-\Delta u+Vu=\lambda f(x,u) & \text{in}\quad B_n,\\ u=0 & \text{on}\quad \partial B_n,\end{cases}\tag{1.3}$$

where $B_n=B_n(0)$. Using linking type theorem, we obtain a sequence of solutions $\{u_n\}$ with the relative Morse index $\mathcal{M}(u_n)\le 1$. The fact $\mathcal{M}(u_n)\le 1$ allows us to show that $u_n$ weakly converges to a nontrivial solution of (1.1).

Next, we turn to critical problem (1.2). We assume further that

$(K)$ $K\in L^\infty(\mathbb{R}^N)$, $0<K(x_0)=\max\limits_{x\in B_1}K(x)$ and $K(x)=K(x_0)+O(|x-x_0|)$ for $x$ near $x_0$ and $K(x)$ is bounded from below on $B_1$ by a positive constant.

$(f_5)$ there exists a function $\bar f$ such that

$$f(x,u)\ge\bar f(u)\quad\text{a.e. for}\quad x\in\omega\quad\text{and}\quad u\ge 0,$$

where $\omega$ is some nonempty open set in $B_1$ and the function $\bar F(u)=\int_0^u\bar f(s)\mathrm{d}x$ satisfies

$$\lim_{\epsilon\to 0}\epsilon^{\min\{\frac{N+2}{2},\frac{p(N-2)}{2}\}}\int_0^{\epsilon^{-1}}\bar F\left[\left(\frac{\epsilon^{-\frac12}}{1+s^2}\right)^{\frac{N-2}{2}}\right]s^{N-1}\mathrm{d}x=\infty.\tag{1.4}$$

If $\bar F(s)=|s|^p$, then this condition is satisfied.

The fact that $K$ attains its maximum at the center of the ball $B_1$ is not essential.

**Theorem 1.2** Suppose $(V)$, $(K)$ and $(f_1)-(f_5)$ hold. Then there exists $\lambda^*>0$ such that for $0<\lambda<\lambda^*$ problem (1.2) possesses at least a $H^1(\mathbb{R}^N)$ nontrivial solution.

Theorem 1.2 is also proved by obtaining a sequence of solutions in balls $B_n$ with finite relative Morse index and by showing the weak limit function is a nontrivial solution of (1.2).

Our argument may be applied also to other potential $V$ such that the operator $-\Delta + V$ with $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$, for example, $V \in L^\infty(\mathbb{R}^N)$ and $V$ is Hölder continuous. Suppose further $\lim\limits_{|x|\to\infty} V(x) = V_\infty$. Then $\sigma(-\Delta+V)\cap(\infty, V_\infty) = \sigma_p(-\Delta+V)\cap(\infty, V_\infty)$, $\sigma_{\rm ess}(-\Delta+V) = [V\infty, \infty)$.

In Section 2, we state linking type theorem with the estimate of relative Morse index. Theorems 1.1 and 1.2 are proved in Sections 3 and 4, respectively.

## 2 Estimates of Relative Morse Index of Local Linking

Let $E$ be a Banach space with a direct sum decomposition $E = E^+ \oplus E^-$. Consider two sequences of subspaces:

$$E_0^+ \subset E_1^+ \subset \cdots \subset E^+, \quad E_0^- \subset E_1^- \subset \cdots \subset E^-,$$

such that

$$E^\pm = \overline{\bigcup_{n\in\mathbb{N}} E_n^\pm}.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we denote by $E_\alpha$ the space $E_{\alpha_1} \oplus E_{\alpha_2}$. We say $\alpha \le \beta$ if and only if $\alpha_1 \le \beta_1, \alpha_2 \le \beta_2$. A sequence $\{\alpha_n\} \in \mathbb{N}^2$ is admissible if, for every $\{\alpha\} \in \mathbb{N}^2$ there is $m \in \mathbb{N}$ such that $n \ge m$ implies $\alpha_n \ge \alpha$. For every $f : E \to \mathbb{R}$, we denote by $f_\alpha$ the restriction of the function $f$ on $E_\alpha$.

Let $f \in C^1(E, \mathbb{R})$. We say that the function $f$ satisfies the (PS)$^*$ condition if every sequence $\{u_{\alpha_n}\}$ such that $\{\alpha_n\}$ is admissible and

$$u_{\alpha_n} \in E_{\alpha_n},\ \limsup_n f(u_{\alpha_n}) < \infty,\ f'_{\alpha_n}(u_{\alpha_n}) \to 0,$$

contains a convergent subsequence which converges to a critical point of $f$.

Let $E$ be a real Hilbert space. For a closed subspace $U \subset E$, we denote by $P_U$ the orthogonal projection onto $U$. Two closed subspaces $U$ and $W$ of $E$ are called commensurable if the operator $P_U - P_W$ is compact. The relative dimensions of $W$ with respect to $U$ is defined by the integer

$$\dim_U W = \dim(W \cap U^\perp) - \dim(W^\perp \cap U).$$

Consider a functional of the form

$$f(x) = \frac{1}{2}(Lx, x) + h(x), \tag{2.1}$$

where $L$ is an invertible self-adjoint operator, $h \in C^1(E)$ and its gradient $h' : E \to E$ is compact. Let $E^+$ and $E^-$ be the positive and negative eigenspaces of $L$ respectively. Then $E = E^+ \oplus E^-$ which is an orthogonal decomposition of $E$. Denote by $U^+(T), U^-(T)$ the positive and negative eigenspaces of the operator $T$.

Let $x$ be a critical point of the functional $f$. Assume that $h$ is twice differentiable at $x$. The relative Morse index of $f$ at $x$ with respect to the splitting $E^+ \oplus E^-$ is the integer

$$\mathcal{M}_{(E^+,E^-)}(x) := \mathcal{M}_{(E^+,E^-)}(x; f) = \dim_{E^-} U^-(f''(x)).$$

Let $E$ be a Hilbert space and $f \in C^2(E, \mathbb{R})$ have a form of (**??**). Let $\{P_n\}$ be an approximation scheme of $L$, i.e. $P_n \to Id$ strongly, while $P_nL - LP_n \to 0$ in the operator norm as $n \to \infty$. Denote, respectively, by $E_n^+$ and $E_n^-$ the positive and the negative eigenspaces of $P_nLP_n$ and by $P_n^+, P_n^-$ the orthogonal projections onto $E_n^+, E_n^-$. By Theorem 2.3 in [1], $P_nLP_n$ is invertible on $E_n := P_n(E)$ for large $n$ and so $E_n = E_n^+ \oplus E_n^-$.

The following result is Theorem 3.1 of [1].

**Theorem 2.1** Suppose $f$ and $E$ are described as above. Let $e \in \partial B_1 \cap E^+$ and set

$$S := \partial B_\rho \cap E^+, \quad Q := \{u + se \in E : \|u + se\| < r, s \geq 0, u \in E^-\},$$

where $r > \rho > 0$ and $s > 0$. Denote by $\partial Q$ the boundary of $Q$ in $E^- \oplus Re$. Assume that there exist numbers $\alpha < \beta$ such that

$$\sup_{\partial Q} f < \alpha < \inf_S f, \quad \sup_Q f < \beta,$$

and that $f$ satisfies the $(PS)^*$ condition with respect to $\{P_n\}$ for $c \in [\alpha, \beta]$. Then $f$ has a nontrivial critical point $x$ such that $\alpha \leq c = f(x) \leq \beta$, where

$$c = \inf_{\gamma \in \Gamma} \max_{u \in M} I(\gamma(u)),$$

$\Gamma = \{\gamma \in C(M, X);\ \gamma_{|\partial M} = \mathrm{id}\}$. Moreover, there holds

$$\mathcal{M}_{(E^+, E^-)}(x) \leq 1.$$

## 3 Subcritical Problems

Associated with problem (1.1), we consider the approximating problem in balls $B_n = B_n(0)$ in $\mathbb{R}^N$:

$$\begin{cases} -\Delta u + Vu = \lambda f(x, u) & \text{in} \quad B_n, \\ \qquad u = 0 & \text{on} \quad \partial B_n. \end{cases} \tag{3.1}$$

The operator $-\Delta + V$ on $H^2(B_n) \cap H_0^1(B_n)$ has discrete spectrum with eigenvalues $\lambda_{n,1} \leq \lambda_{n,2} \leq \cdots \to \infty$ and there exists a finite

$$j_n = \min\{i : \lambda_{n,i} > 0\}.$$

The eigenvalues $\lambda_{n,i}$ have the same variational characterization for each $i$. It is well known that the operator $-\Delta + V$ on the whole space has essential spectrum. Since 0 is in a gap of the spectrum $\sigma(-\Delta + V)$ in the whole space, namely, there exist $\alpha, \beta > 0$ such that $0 \in (-\alpha, \beta) \not\subset \sigma(-\Delta + V)$.

**Lemma 3.1** Suppose $\lambda_{n,i} \to \lambda_0$ as $n \to \infty$. Then $\lambda_0 \notin (-\alpha, \beta)$.

**Proof** Let $\varphi_n$ be the eigenfunction corresponding to $\lambda_{n,i}$, $\|\varphi_n\| = 1$. Let $\eta \in C_0^\infty(\mathbb{R}^N), 0 \leq \eta \leq 1, \eta = 1$ if $|x| \leq \frac{1}{2}$ and $\eta = 0$ if $|x| \geq 1$, $|\nabla \eta| \leq C$. Set $\psi_n(x) = \eta(\frac{x}{n})\varphi_n(x)$, and denote $\eta_n(x) = \eta(\frac{x}{n})$. There holds

$$\int_{\mathbb{R}^N} |\nabla \psi_n|^2 \mathrm{d}x = \int_{\mathbb{R}^N} \Big(\eta_n^2 |\nabla \varphi_n|^2 + \frac{2}{n} \eta_n \varphi_n \nabla \varphi_n \nabla \eta + \frac{1}{n^2} \varphi_n^2 |\nabla \eta|^2\Big) \mathrm{d}x.$$

It is apparently that
$$\int_{\mathbb{R}^N}\Big(\frac{2}{n}\eta_n\varphi_n\nabla\varphi_n\nabla\eta+\frac{1}{n^2}\varphi_n^2|\nabla\eta|^2\Big)\mathrm{d}x\to 0$$
as $n\to\infty$. Therefore, $\|\psi_n\|^2\to\|\varphi_n\|^2=1$ as $n\to\infty$. On the other hand,
$$\begin{aligned}&-\Delta\psi_n+V\psi_n-\lambda_0\psi_n\\ &=-\eta_n\Delta\varphi_n+\eta_nV\varphi_n-\lambda_{n,i}\eta_n\varphi_n-(\lambda_0-\lambda_{n,i})\psi_n+\frac{2}{n}\nabla\eta\cdot\nabla\varphi_n+\frac{1}{n^2}\varphi_n\Delta\eta\\ &=-(\lambda_0-\lambda_{n,i})\psi_n+\frac{2}{n}\nabla\eta\nabla\varphi_n+\frac{1}{n^2}\varphi_n\Delta\eta.\end{aligned}$$
It implies
$$-\Delta\psi_n+V\psi_n-\lambda_0\psi_n\to 0,$$
as $n\to\infty$ in weak sense and $L^2$ norm.

Since $\|\psi_n\|\le C$, we may assume $\psi_n\rightharpoonup\psi$ as $n\to\infty$. It is well known that, see for instance Lemma 6.5.22 in [7], $\lambda_0\in\sigma_{\mathrm{ess}}(L)$, where $L$ is a self-adjoint operator, if and only if there exist $x_n\in D(L), \|x_n\|=1, n=1,2,\cdots$, satisfying
$$w-\lim_{n\to\infty}x_n=0,\quad \lim_{n\to\infty}(\lambda_0Id-L)x_n=0.$$
So if $\psi=0$, $\lambda_0\in\sigma_{\mathrm{ess}}(L)$, a contradiction to the fact $\lambda_0\in(-\alpha,\beta)$. If $\psi\not\equiv 0$, we see that $\psi$ solves the problem
$$-\Delta\psi+V\psi=\lambda_0\psi\quad\text{in }\mathbb{R}^N.$$
This means that $\lambda_0$ is an eigenvalue of $L$, again contradicting to $\lambda_0\in(-\alpha,\beta)$.

Fix a large $n$. By Lemma 3.1, $\lambda_{n,i}\notin(-\frac{1}{2}\alpha,\frac{1}{2}\beta)$, for every $i\in\mathbb{N}$. The eigenvalues of $(-\Delta+V,\ H_0^1(B_n))$ can be ordered as
$$\lambda_{n,1}<\cdots\le\lambda_{n,j_n-1}<0<\lambda_{n,j_n}\le\cdots\le\lambda_{n,k}\le\cdots.$$
Let $\varphi_i$, $i=1,2,\cdots$, be corresponding eigenfunctions. Set
$$E_n^+=\mathrm{span}\{\varphi_{j_n+i},i=0,1,\cdots\},\quad E_n^-=\mathrm{span}\{\varphi_i,i=1,\cdots,j_n-1.\}.\tag{3.2}$$
Critical points of the functional
$$I_n(u)=\frac{1}{2}\int_{B_n}(|\nabla u|^2+Vu^2)\mathrm{d}x-\int_{B_n}F(x,u)\mathrm{d}x$$
are solutions of problem (3.1). To solve problem (3.1), it is sufficient to look for critical points of $I_n$. We know that the functional $I_n$ is well defined on $E_n=H_0^1(B_n)=E_n^+\oplus E_n^-$. Denote by $\|\cdot\|$ the usual norm on $E_n$. Suppose $\lambda$ is finite, precisely, assume $0<\lambda\le\Lambda$ for some $\Lambda>0$, we have

**Proposition 3.1** Problem (3.1) possesses a nontrivial solution $u_n$ with $\mathcal{M}(u_n)\le 1$. Moreover, there exist positive constants $\sigma,C$ depending only on $\Lambda$ such that
$$\sigma\le I_n(u_n)\le C.$$

The proof of Proposition 3.1 relies on Benci-Rabinowitz linking theorem in the form of Theorem 2.1. We now verify the conditions in Theorem 2.1 by the following lemmas.

**Lemma 3.2** There exist real numbers $\rho, \delta > 0$ independent of $n$ such that

$$I_n(u) \geq \delta \quad \text{for all} \quad u \in S_{n,\rho} := \partial B_\rho \cap E_n^+,$$

where $B_\rho = \{u \in E_n : \|u\| \leq \rho\}$.

**Proof** Since $\lambda_{n,i} \in (\frac{1}{2}\alpha, \frac{1}{2}\beta)$, by $(f_2)$ and $(f_4)$, we have for $u \in E_n^+$ that

$$I_n(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{B_n} F(x,u)\mathrm{d}x \geq \frac{1}{2}\|u\|^2 - \epsilon\|u\|^2 - C_\epsilon\|u\|^{p+1} = \Big(\frac{1}{2} - \epsilon\Big)\rho^2 - C_\epsilon \rho^{p+1}.$$

The Lemma follows by choosing $\epsilon < \frac{1}{2}$ and $\rho > 0$ small.

For each $n$, we fix $v_n \in E_n^+$ with $\|v_n\| = 1$. $Q_n = \{u + sv_n : \|u + sv_n\| < r, s \geq 0, u \in E_n^-\}$.

**Lemma 3.3** There are constants $r = r(\Lambda) > \rho, C = C(\Lambda) > 0$ independent of $n$ such that

$$I_n|_{\partial Q_n} \leq 0, \quad I_n|_{Q_n} \leq C.$$

**Proof** We note that

$$\partial Q_n = \{u + sv_n : \|u + sv_n\| = r, s \geq 0, u \in E_n^-\},$$

or

$$\partial Q_n = \{u + sv_n : \|u + sv_n\| \leq r, s = 0, u \in E_n^-\}.$$

Since $I_n(0) = 0$, we show now that $\sup_{u \in \partial Q_n} I_n(u) \leq 0$. For $u + sv_n \in Q_n$ we have

$$I_n(u + sv_n) = \frac{1}{2}s^2 - \frac{1}{2}\|u\|^2 - \lambda \int_{Q_n} F(x, u + sv_n)\mathrm{d}x.$$

By the assumptions $(f_1) - (f_4)$, for every $\epsilon > 0$ there is a positive constant $C_\epsilon$ such that $F(x,t) \geq -\epsilon|t|^2 + C_\epsilon|t|^\theta$, where $2 < \theta < \frac{2N}{N-2}$. It follows that

$$\int_{Q_n} F(x, u + sv_n)\mathrm{d}x \geq -\epsilon\|u\|^2 - \epsilon s^2\|v_n\|^2 + C_\epsilon\|u + sv_n\|_{L^\theta}^\theta.$$

Therefore,

$$I_n(u + sv_n) \leq \frac{1}{2}s^2 - \frac{1}{2}\|u\|^2 + \lambda\epsilon\|u\|^2 + \lambda\epsilon s^2 - \lambda C_\epsilon\|u + sv_n\|_{L^\theta}^\theta.$$

We now observe that $X_n = E_n^- \oplus \mathbb{R}v_n$ is continuously embedded in $L^\theta(B_n)$, and there exists a continuous projection $\Pi_n : X_n \to \mathbb{R}v_n$ such that

$$\|sv_n\|_\theta \leq \|\Pi_n\|_\theta\|u + sv_n\|_\theta, \quad \|\Pi_n\|_\theta \geq 1.$$

Choosing $\epsilon\Lambda = \frac{1}{4}$, we obtain

$$I_n(u + sv_n) \leq -\frac{1}{4}\|u\|^2 + \frac{3}{4}s^2 - \lambda C_\epsilon\|u + sv_n\|_{L^\theta}^\theta.$$

Consequently, if $s = 0$, we have $I_n(u) \leq 0$, and for $\|u + sv_n\| \to \infty$, we have $I_n(u + sv_n) \to -\infty$. Our claim follows.

**Lemma 3.4** $I_n$ satisfies $(PS)^*$ condition.

**Proof** Let $\{u_m\} \subset E_n$ be a $(PS)^*$ sequence with respect to $\{Y_m\}$, where $Y_m := \mathrm{span}\{\varphi_{n,1}, \cdots, \varphi_{n,m}\}$ and $\varphi_{n,i}$ is an eigenfunction of $\lambda_{n,i}$, that is,

$$u_m \in Y_m,\ \limsup_{m\to\infty} I_n(u_m) < \infty,\ I'_n|_{Y_m}(u_m) \to 0 \text{ as } n \to \infty.$$

Suppose $\limsup\limits_{m\to\infty} I_n(u_m) \le C$. Then

$$C \ge I_n(u_m) - \frac{1}{2}\langle I'_n(u_m), u_m\rangle \ge \Big(\frac{1}{2} - \theta\Big)\lambda \int_{B_n} u_m f(x, u_m)\mathrm{d}x.$$

By $(f_1)$–$(f_4)$, we have that

$$|f(x,t)| \le Ctf(x,t) \quad \text{if} \quad |t| \le 1 \quad \text{and} \quad x \in \mathbb{R}^N \tag{3.3}$$

and

$$|f(x,t)|^{p'} \le C|t|^{p(p'-1)}|f(x,t)| = Ctf(x,t) \quad \text{if} \quad |t| > 1 \quad \text{and} \quad x \in \mathbb{R}^N, \tag{3.4}$$

where $p' = \frac{p+1}{p}$. Let $\Omega_m = \{x \in B_n : |u_m(x)| \le 1\}$, then

$$\frac{C}{\lambda} \ge \int_{\Omega_m} |f(x,u_m)|^2\mathrm{d}x + \int_{B_n\setminus\Omega_m} |f(x,u_m)|^{p'}\mathrm{d}x.$$

It implies

$$I_1 = \int_{\Omega_m} |f(x,u_m)|^2\mathrm{d}x \le \frac{C}{\lambda}, \quad I_2 = \int_{B_n\setminus\Omega_m} |f(x,u_m)|^{p'}\mathrm{d}x \le \frac{C}{\lambda}.$$

By Hödler's inequality,

$$\|u_m^+\|^2 = \lambda \int_{B_n} u_m^+ f(x,u_m)\mathrm{d}x \le \lambda(I_1^{\frac{1}{2}}\|u_m^+\|_{L^2} + I_2^{\frac{1}{p'}})\|u_m^+\|_{L^{p'}} \le \lambda C(I_1^{\frac{1}{2}} + I_2^{\frac{1}{p'}})\|u_m^+\|.$$

Thus

$$\|u_m^+\| \le C := C(\Lambda).$$

Similarly,

$$\|u_m^-\| \le C := C(\Lambda).$$

Consequently, $\|u_m\| \le C$. Note that for $m \ge k$,

$$\begin{aligned}\|u_m^+ - u_k^+\|^2 &= \int_{B_n} [|\nabla(u_m^+ - u_k^+)|^2 + V(u_m^+ - u_k^+)^2]\mathrm{d}x \\ &= \langle I'_n(u_m) - I'_n(u_k), u_m^+ - u_k^+\rangle + \lambda\int_{B_n} (f(x,u_m) - f(x,u_k)(u_m^+ - u_k^+)\mathrm{d}x \\ &\le (\epsilon_m + \epsilon_k)\|u_m^+ - u_k^+\| + C\|u_m^+ - u_k^+\|_{L^2} + \|u_m^+ - u_k^+\|_{L^{p+1}},\end{aligned}$$

where $\epsilon_m, \epsilon_k \to 0$ as $m, k \to \infty$. Since we may assume that $\{u_m\}$ strongly converges in $L^\gamma(B_n)$, $2 \le \gamma < \frac{2N}{N-2}$, we obtain that

$$\|u_m^+ - u_k^+\| \to 0,$$

as $m, k \to \infty$. Similarly,

$$\|u_m^- - u_k^-\| \to 0,$$

as $m, k \to \infty$. The $(PS)^*$ condition then follows.

**Proof of Proposition 3.1** By Theorem 2.1, Lemmas 3.2–3.4, there is a critical point $u_n$ of $I_n$ satisfying

$$\sigma \le I_n(u_n) \le C \quad \text{and} \quad \mathcal{M}(u_n) \le 1,$$

where $\sigma, C > 0$ are independent of $n$.

**Proof of Theorem 1.1** Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we may assume

$$u_n \rightharpoonup u \ \text{ in } \ H^1(\mathbb{R}^N), \quad u_n \to u \ \text{ a.e. } \mathbb{R}^N \ \text{ and } \ u_n \to u \ \text{ in } \ L^q_{\rm loc}(\mathbb{R}^N),$$

where $2 \le q < \frac{2N}{N-2}$ if $N \ge 3$ and $2 \le q < \infty$ if $N = 2$.

By the fact that $I_n'(u_n) = 0$, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ for large $n$, we have $supp\varphi \subset B_n$, $I_n'(u_n)\varphi = 0$. The weak convergence of $u_n$ implies $u$ is a weak solution of problem (1.1). To complete the proof, we only need to show $u \not\equiv 0$. We argue by contradiction. Suppose $u \equiv 0$. Denote by $I(u) := I_n(u)$ the corresponding functional of problem (3.1) in the ball $B_n$, and by $I''$ the second derivative of $I$. Since for any $\varphi_{n,i} \in E_n^-$, the subspace of $E_n = H_0^1(B_n)$, on which the operator $-\Delta + V$ is negative, for large $n$, we have

$$\langle I''(0)\varphi_{n,i}, \varphi_{n,i}\rangle = \int_{B_n} [|\nabla \varphi_{n,i}|^2 + V(\varphi_{n,i})^2]\mathrm{d}x \le -\frac{1}{2}\alpha\|\varphi_{n,i}\|^2,$$

where $(-\alpha, \beta)$, $\alpha, \beta > 0$, is the spectral gap of the operator $-\Delta + V$ in the whole space. Choosing $\psi_{n,1}, \psi_{n,2} \in E_n^+$, we may find $\epsilon_1^n, \epsilon_2^n > 0$ such that

$$\begin{aligned}\langle I''(0)(\varphi_{n,i} + \epsilon_i^n \psi_{n,i}), \varphi_{n,i} + \epsilon_i^n\psi_{n,i}\rangle &= \int_{B_n} [|\nabla \varphi_{n,i} + \epsilon_i^n \psi_{n,i}|^2 + V(\varphi_{n,i} + \epsilon_i^n \psi_{n,i})^2]\mathrm{d}x \\ &\le -\frac{1}{4}\alpha\|\varphi_{n,i}\|^2 \end{aligned} \tag{3.5}$$

for $i = 1, 2$. We may verify that $\{v_1^n, \cdots, v_{j_n-1}^n\} := \{\varphi_{n,1} + \epsilon_1^n\psi_{n,1}, \varphi_{n,2} + \epsilon_2^n \psi_{n,2}, \varphi_{n,3}, \cdots, \varphi_{n,j_n-1}\}$ are linearly independent.

We claim that there exists $\lambda^* > 0$ such that, for $0 < \lambda < \lambda^*$,

$$\langle I''(u_n)v_i^n, v_i^n\rangle = \int_{B_n} [|\nabla v_i^n|^2 + V(v_i^n)^2]\mathrm{d}x - \lambda \int_{B_n} f'(x, u_n)(v_i^n)^2 \mathrm{d}x < 0, \tag{3.6}$$

for $i = 1, \cdots, j_n - 1$. Indeed, for a fixed $R > 0$, by the convergence of $u_n \to 0$ in $L^\gamma(B_n)$, $2 \le \gamma < \frac{2N}{N-2}$, for any $\epsilon > 0$ there exists $n_0 > 0$ such that for, $n \ge n_0$,

$$\Big|\lambda \int_{B_R} f'(x, u_n)(v_i^n)^2\mathrm{d}x\Big| < \frac{1}{3}\epsilon\|v_i^n\|^2. \tag{3.7}$$

We note that $\{v_i^n\} \in L^\infty(B_n)$, and consequently

$$\Big|\lambda \int_{B_{R_n}\setminus B_R} f'(x, u_n)(v_i^n)^2\mathrm{d}x\Big| \le \lambda C \int_{B_{R_n}\setminus B_R} (1+|u_n|^{p-1})(v_i^n)^2\mathrm{d}x \le \lambda C(1+\|u_n\|^p)\|v_i^n\|^2. \tag{3.8}$$

Since $\|u_n\|$ is uniformly bounded in $0 < \lambda \le \Lambda$, we may choose $\lambda > 0$ such that

$$\Big|\lambda \int_{B_{R_n}\setminus B_R} f'(x, u_n)(v_i^n)^2\mathrm{d}x\Big| < \frac{1}{3}\epsilon\|v_i^n\|^2. \tag{3.9}$$

Thus (3.6) follows from (3.7) and (3.9) by a proper choice of $\epsilon > 0$.

Next, we prove that (3.6) implies $\mathcal{M}(u_n) \geq 2$, which contradicts to the fact $\mathcal{M}(u_n) \leq 1$. Thus, $u \not\equiv 0$ and the proof will be completed. In fact, set $W^- = \mathrm{span}\{v_3^n, \cdots v_{j_n-1}^n\}$, $W^+$ the orthogonal complement of $W^-$ in $H_0^1(B_n)$. Then denoting by $U$ the negative eigenspace of $I''(u_n)$, we deduce

$$\dim_{W^-} U = \dim(U \cap W^+) - \dim(U^\perp \cap W^-) \geq 4,$$

and

$$\dim_{E^-} W^- = \dim(W^- \cap E^+) - \dim(W^+ \cap E^-) = -2.$$

Hence,

$$\mathcal{M}(u_n) = \dim_{W^-} U + \dim_{E^-} W^- \geq 2.$$

The conclusion then follows.

## 4 Critical Problems

We keep notations as in Section 3 and we prove Theorem 1.2 in this section. We consider the problem

$$-\Delta u + V(x)u = \lambda[K(x)|u|^{2^*-2}u + f(x,u)] \quad \text{in} \quad B_n, \quad u = 0 \quad \text{on} \quad \partial B_n. \tag{4.1}$$

Using the same argument as in the subcritical case, we see that it is sufficient to prove the following result.

**Proposition 4.1** Suppose $(V)$ and $(f_1)-(f_5)$ hold. Problem (4.1) possesses a nontrivial solution $u_n$ with $\mathcal{M}(u_n) \leq 1$. Moreover, there exist positive constants $\sigma, C$ independent of $n$ such that

$$\sigma \leq I_n(u_n) \leq C.$$

Let

$$J_n(u) = \frac{1}{2}\int_{B_n} (|\nabla u|^2 + V(x)|u|^2)\mathrm{d}x - \lambda \int_{B_n} \Big(\frac{1}{2^*}K(x)|u|^{2^*} + F(x,u)\Big)\mathrm{d}x$$

be the functional associated with problem (4.1), which bolongs to $C^2$. To find critical points of $J_n$, we start with following lemmas.

**Lemma 4.1** There exist constants $\rho > 0$ and $\alpha > 0$ independent of $n$ such that $\inf\limits_{u \in N_n} J_n(u) \geq \alpha$, where $N_n = \{u \in E_n^+; \|u\|_n = \rho\}$.

**Proof** Let $u \in E_n^+$, then

$$J_n(u) = \frac{1}{2}\|u\|_n^2 - \frac{\lambda}{2^*}\int_{B_n} K(x)|u|^{2^*}\mathrm{d}x - \lambda \int_{B_n} F(x,u)\mathrm{d}x.$$

It follows from $(f_2)$ and $(f_4)$, that, for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$|F(x,s)| \leq \epsilon s^2 + C_\epsilon |s|^{p+1}$$

for all $s \in \mathbb{R}$. Applying the Sobolev embedding theorem, we get that

$$\int_{B_n} F(x,u)\mathrm{d}x \leq C\big(\epsilon\|u\|_n^2 + C_\epsilon\|u\|_n^{p+1}\big)$$

for some constant $C>0$ independent of $n$. Consequently,

$$J_n(u) \geq \frac{1}{2}\|u\|_n^2 - \frac{\|K\|_\infty}{2^*}\|u\|_n^{2^*} - C\big(\epsilon\|u\|_n^2 + C_\epsilon\|u\|_n^p\big),$$

where $\|K\|_\infty = \sup\limits_{x\in\mathbb{R}^N}|K(x)|$. Choosing $\epsilon>0$ and $\rho>0$ sufficiently small, the result readily follows.

In the next lemma we find the energy range of the functional $J_n$ for which the Palais - Smale condition holds.

**Lemma 4.2** $I_n$ satisfies $(PS)_c^*$ condition for $c\in\big(0,\frac{1}{N}\|K\|_\infty^{-\frac{N-2}{2}}S^{\frac{N}{2}}\big)$.

**Proof** Let $\{u_m\}\subset E_n$ be a $(PS)_c^*$ sequence with respect to $\{Y_m\}$ with

$$c\in\Big(0,\frac{1}{N}\|K\|_\infty^{-\frac{N-2}{2}}S^{\frac{N}{2}}\Big),$$

where $Y_m:=\mathrm{span}\{\varphi_{n,1},\cdots,\varphi_{n,m}\}$ and $\varphi_{n,i}$ is an eigenfunction of $\lambda_{n,i}$, that is,

$$u_m\in Y_m,\ \limsup_{m\to\infty} I_n(u_m)=c,\ I_n'|_{Y_m}(u_m)\to 0 \text{ as } n\to\infty.$$

In the first step of the proof we show that the sequence $\{u_m\}$ is bounded in $E_n$. Indeed, we have

$$\begin{aligned}
c+o(1) &= J_n(u_m)-\frac{1}{2}\langle J_n'(u_m),u_m\rangle\\
&= \lambda\Big\{\frac{1}{N}\int_{B_n}K(x)|u_m|^{2^*}\mathrm{d}x+\frac{1}{2}\int_{B_n}u_mf(x,u_m)\mathrm{d}x-\int_{B_n}F(x,u_m)\mathrm{d}x\Big\}\\
&\geq \lambda\Big\{\frac{1}{N}\int_{B_n}K(x)|u_m|^{2^*}\mathrm{d}x+\big(\frac{1}{2}-\frac{1}{\theta}\big)\int_{B_n}u_mf(x,u_m)\mathrm{d}x\Big\}.
\end{aligned}\tag{4.2}$$

Letting $\Omega_m=\{x\in B_n;\,|u_m(x)|\leq 1\}$ we derive from (3.3), (3.4) and (4.2) that

$$c\geq\frac{\lambda}{N}\int_{B_n}K(x)|u_m(x)|^{2^*}\mathrm{d}x+\lambda C\Big(\int_{\Omega_m}|f(x,u_m)|^2\mathrm{d}x+\int_{B_n\setminus\Omega_m}|f(x,u_m)|^{p'}\mathrm{d}x\Big)+o(1),$$

for some constant $C>0$. Hence

$$\int_{B_n}|f(x,u_m)|^2\mathrm{d}x\leq\frac{c}{\lambda C}\quad\text{and}\quad\int_{B_n\setminus\Omega_m}|f(x,u_m)|^{p'}\mathrm{d}x\leq\frac{c}{\lambda C},$$

for all $m$. Since $\langle J_n'(u_m),u_m^+\rangle=\epsilon_m\|u_m^+\|_n$, with $\epsilon_m\to 0$, we deduce from the Hölder inequality that

$$\begin{aligned}
\|u_m^+\|_n^2 &= -\lambda\int_{B_n}K(x)|u_m|^{2^*-2}u_mu_m^+\mathrm{d}x-\lambda\int_{B_n}f(x,u_m)u_m^+\mathrm{d}x+\epsilon_m\|u_m^+\|_n\\
&\leq \lambda C_1\|K\|_\infty^{\frac{1}{2^*}}\Big(\int_{B_n}K(x)|u_m|^{2^*}\mathrm{d}x\Big)^{\frac{2^*-1}{2^*}}\|u_m^+\|_n+\epsilon_n\|u_m^+\|_n\\
&\quad+\lambda\Big(\frac{c}{\lambda C}\Big)^{\frac{1}{2}}\|u_m^+\|_{L^2}+\lambda\Big(\frac{c}{\lambda C}\Big)^{\frac{1}{p'}}\|u_m^+\|_{L^p},
\end{aligned}$$

for some constant $C_1>0$. This implies that $\{\|u_m^+\|_n\}$ is bounded. In a similar manner we show that $\{\|u_m^-\|_n\}$ is bounded. Consequently, $\{\|u_m\|_n\}$ is bounded and we may assume that $u_m\rightharpoonup u$ in $E_n$. Let $v_m=u_m-u$. According to Brézis–Lieb lemma [3] we have

$$J_n(u_m)=J_n(u)+J_n(v_m)+o(1),$$

and

$$\langle J_n'(u_m), u_m\rangle = \langle J_n'(u), u\rangle + \langle J_n'(v_m), v_m\rangle = o(1).$$

Hence

$$c + o(1) \geq \frac{1}{2}\int_{B_n} |\nabla v_m|^2 \mathrm{d}x - \frac{1}{2^*}\int_{B_n} K(x)|v_m|^{2^*}\mathrm{d}x. \tag{4.3}$$

Since $v_m \to 0$ in $L^2(B_n)$, applying the Sobolev inequality we get

$$\int_{B_n} |\nabla v_m|^2 \mathrm{d}x = \int_{B_n} K(x)|v_m|^{2^*}\mathrm{d}x + o(1) \leq \|K\|_\infty \Big(S^{-1}\int_{B_n} |\nabla v_m|^2 \mathrm{d}x\Big)^{\frac{2^*}{2}}. \tag{4.4}$$

If $\int_{B_n} |\nabla v_m|^2 \mathrm{d}x \to l > 0$, then we deduce from (4.4) that

$$l \geq \|K\|_\infty^{-\frac{N-2}{2}} S^{\frac{N}{2}},$$

which combined with (4.3) and (4.4) gives

$$c \geq \frac{l}{N} \geq \frac{\|K\|_\infty^{-\frac{N-2}{2}}}{N} S^{\frac{N}{2}},$$

which is impossible. Therefore $l = 0$ and the result follows.

Let

$$\psi_\epsilon = \left(\frac{\sqrt{N(N-2)}\epsilon}{\epsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}, \ \epsilon > 0.$$

Let $B(x_0, 2r) \subset B_{n_0}$ for large $n_0$, where $x_0$ is the center of the ball $B_{n_0}$. By $\zeta \in C_0^1(\mathbb{R}^N)$ we denote the function that satisfies $\zeta(x) = 1$ in $B(x_0, r)$ and $\zeta(x) = 0$ on $\mathbb{R}^N \setminus B(x_0, 2r)$ and $0 \leq \zeta(z) \leq 1$ on $\mathbb{R}^N$. Set $\varphi_\epsilon(x) = \zeta(x)\psi_\epsilon(x)$, $\varphi_\epsilon \in H_0^1(B_n)$. We define

$$Q_n(\epsilon) = \{y + tP_n^+\varphi_\epsilon;\ y \in E_n^-,\ t \geq 0\},$$

where $P_n^+$ is the projection from $E_n$ to $E_n^+$.

**Lemma 4.3** There exists $\epsilon_0 > 0$ such that $P_n^+\varphi_\epsilon \not\equiv 0$ for $0 < \epsilon \leq \epsilon_0$.

The proof is identical to that of Lemma 5 in [5].

In the sequel we need the following asymptotic estimates of norms of $\varphi_\epsilon$:

$$\|\nabla\varphi_\epsilon\|_2^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \tag{4.5}$$

$$\|\varphi_\epsilon\|_{2^*}^{2^*} = S^{\frac{N}{2}} + O(\epsilon^{N}), \tag{4.6}$$

$$\|\varphi_\epsilon\|_2^2 = \begin{cases} K_1\epsilon^2 + O(\epsilon^{N-2}) & \text{if} \quad N \geq 5, \\ K_1\epsilon^2|\log\epsilon^2| + O(\epsilon^2) & \text{if} \quad N = 4, \end{cases} \tag{4.7}$$

$$\|\varphi_\epsilon\|_1 \leq K_2\epsilon^{\frac{N-2}{2}}, \tag{4.8}$$

and

$$\|\varphi_\epsilon\|_{2^*-1}^{2^*-1} \leq K_3\epsilon^{\frac{N-2}{2}}, \tag{4.9}$$

for some constants $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ (see [2]).

**Lemma 4.4** We have

$$\sup_{u\in Q_n(\epsilon)} J_n(u) < \frac{1}{N}\|K\|_\infty^{-\frac{N-2}{2}} S^{\frac{N}{2}}.$$

**Proof** We follow some ideas from the article [5] (see Lemma 6). First, we observe that if $u \in E_n$ with $u \not\equiv 0$, then

$$\begin{aligned} J_n(su) &= \frac{s^2}{2}\int_{B_n}\big(|\nabla u|^2+V(x)u^2\big)\mathrm{d}x - \frac{\lambda s^{2^*}}{2^*}\int_{B_n}K(x)|u|^{2^*}\mathrm{d}x - \lambda\int_{B_n}F(x,su)\mathrm{d}x \\ &\le \frac{s^2}{2}\int_{B_n}\big(|\nabla u|^2+V(x)u^2\big)\mathrm{d}x - \frac{\lambda s^{2^*}}{2^*}\int_{B_n}K(x)|u|^{2^*}\mathrm{d}x. \end{aligned}$$

From this estimate we deduce that $\lim\limits_{s\to\infty} J_n(su) = -\infty$. Hence there exists $s_\epsilon \ge 0$ such that

$$J_n(s_\epsilon u) = \sup_{t\ge 0} J_n(tu).$$

We may assume that $s_\epsilon > 0$ and it satisfies

$$s_\epsilon\int_{B_n}\big(|\nabla u|^2+Vu^2\big)\mathrm{d}x - \lambda s_\epsilon^{2^*-1}\int_{B_n}K|u|^{2^*}\mathrm{d}x - \lambda\int_{B_n}uf(x,s_\epsilon u)\mathrm{d}x = 0.$$

This equation implies that

$$s_\epsilon \le \left(\frac{\int_{B_n}\big(|\nabla u|^2+Vu^2\big)\mathrm{d}x}{\lambda\int_{B_n}K|u|^{2^*}\mathrm{d}x}\right)^{\frac{N-2}{4}} = A.$$

Since the function

$$s \to \frac{s^2}{2}\int_{B_n}\big(|\nabla u|^2+Vu^2\big)\mathrm{d}x - \frac{\lambda s^{2^*}}{2^*}\int_{B_n}K|u|^{2^*}\mathrm{d}x$$

is increasing on the interval $[0,A]$ we see that

$$J_n(su) \le \frac{1}{N}\left[\frac{\int_{B_n}\big(|\nabla u|^2+Vu^2\big)\mathrm{d}x}{\Big(\lambda\int_{B_n}K|u|^{2^*}\mathrm{d}x\Big)^{\frac{2}{2^*}}}\right]^{\frac{N}{2}} - \int_{B_n}F(x,s_\epsilon u)\mathrm{d}x. \tag{4.10}$$

For simplicity we may assume that $K(0) = \max\limits_{x\in B_1} K(x)$, as $J_n$ is a translation invariant. For $u = u^- + tP_n^+\varphi_\epsilon \in B_n(\epsilon)$, with $\|u\|_{2^*,K} = 1$, we write

$$\int_{B_n}\big(|\nabla u|^2+Vu^2\big)\mathrm{d}x = -\|u^-\|_k^2 + \frac{\|\nabla\big(tP_n^+\varphi_\epsilon\big)\|_2^2}{\|tP_n^+\varphi_\epsilon\|_{2^*,K}^2}\|tP_n^+\varphi_\epsilon\|_{2^*,K}^2 + t^2\int_{B_n}V\big(P_n^+\varphi_\epsilon\big)^2\mathrm{d}x. \tag{4.11}$$

As in [5] (see formula (20) there) we have the following estimate

$$\Big|\int_{B_n}K\big(|P_n^+\varphi_\epsilon|^{2^*} - |\varphi_\epsilon|^{2^*}\big)\mathrm{d}x\Big| \le C_2\epsilon^{N-2}$$

for some constant $C_2 > 0$. Using this, $(K)$ and (4.6), we get

$$\begin{aligned} \|P_n^+\varphi_\epsilon\|_{2^*,K}^2 &= \Big(\|P_n^+\varphi_\epsilon\|_{2^*,K}^{2^*}\Big)^{\frac{2}{2^*}} = \Big(\|\varphi_\epsilon\|_{2^*,K}^{2^*} + O(\epsilon^{N-2})\Big)^{\frac{N-2}{N}} \\ &= \big(K(0)S^{\frac{N}{2}} + O(\epsilon) + O(\epsilon^{N-2})\big)^{\frac{N-2}{N}} \\ &= K(0)^{\frac{N-2}{N}}S^{\frac{N-2}{2}} + O\big(\epsilon^{\frac{N-2}{N}}\big). \end{aligned} \tag{4.12}$$

As in [5] (see p288) we can derive the following estimate

$$\Big|\int_{B_n}|\nabla\varphi_\epsilon|^2\mathrm{d}x-\int_{B_N}|\nabla\big(P_n^+\varphi_\epsilon\big)|^2\mathrm{d}x\Big|=O(\epsilon^{N-2}).\tag{4.13}$$

Inserting (4.12) and (4.13) into (4.11) and using (4.5), we get

$$\begin{aligned}\int_{B_n}\big(|\nabla u|^2+Vu^2\big)\mathrm{d}x=-\|u\|_k^2+\Big(K(0)^{-\frac{N-2}{N}}S+O\big(\epsilon^{N-2}\big)\Big)\|tP_n^+\varphi_\epsilon\|_{2^*,K}^2&\\+t^2\int_{B_n}V\big(P_n^+\varphi_\epsilon\big)^2\mathrm{d}x,&\end{aligned}\tag{4.14}$$

As in [5] (see (25), (26) and (27) there) we derive the estimate

$$\begin{aligned}1=\|u\|_{2^*,K}^{2^*}&\geq\|tP_n^+\varphi_\epsilon\|_{2^*,K}^{2^*}+\frac{1}{2}\|u^-\|_{2^*,K}^{2^*}-C_4t^{2^*}\epsilon^{\frac{(N-2)N}{N+2}}\\&\geq t^{2^*}\|\varphi_\epsilon\|_{2^*,K}^{2^*}-C_3t^{2^*}\epsilon^{N-2}-C_4t^{2^*}\epsilon^{\frac{(N-2)N}{N+2}}+\frac{1}{2}\|u^-\|_{2^*,K}^{2^*},\end{aligned}\tag{4.15}$$

for some constants $C_3>0$ and $C_4>0$. This estimate implies that $t$ is bounded. We now distinguish two cases:

(i) $\|u^-\|_{2^*,K}^{2^*}\leq 2C_4t^{2^*}\epsilon^{\frac{(N-2)N}{N+2}}$ or

(ii) $\|u^-\|_{2^*,K}^{2^*}>2C_4t^{2^*}\epsilon^{\frac{(N-2)N}{N+2}}$.

In the first case we have (see [5] p289 formula (26))

$$\|tP_n^+\varphi_\epsilon\|_{2^*,K}^2\leq 1+C_5\epsilon^{N-2},\tag{4.16}$$

for some constant $C_5>0$. If the case (ii) prevails, then by the first part of the inequality (4.15) we have

$$\|tP_n^+\varphi_\epsilon\|_{2^*,K}^{2^*}\leq 1.\tag{4.17}$$

Since $s_\epsilon$ satisfies

$$\begin{aligned}&\int_{B_n}\big(|\nabla(u^-+tP_n^+\varphi_\epsilon)|^2+V(x)(u^-+tP_n^+\varphi_\epsilon)^2\big)\mathrm{d}x\\&-\lambda s_\epsilon^{2^*-2}\int_{B_n}K(x)|u^-+tP_n^+\varphi_\epsilon|^{2^*}\mathrm{d}x\\&-\lambda\int_{B_n}\frac{(u^-+tP_n^+\varphi_\epsilon)f(x,s_\epsilon(u^-+tP_n^+\varphi_\epsilon))}{s_\epsilon}\mathrm{d}x=0,\end{aligned}$$

we get that

$$\lim_{\epsilon\to0}\int_{B_n}|\nabla(u^-+tP_n^+\varphi_\epsilon)|^2+V(x)|u^-+tP_n^+\varphi_\epsilon|^2\mathrm{d}x\geq\lim_{\epsilon\to0}s_\epsilon^{2^*-2}.$$

In both cases (4.16) and (4.17) we deduce from (4.14) that

$$\lim_{\epsilon\to0}s_\epsilon^{2^*-2}\leq K(0)^{-\frac{N-2}{N}}S,$$

and $s_\epsilon$ is bounded for small $\epsilon>0$. We now estimate the integral involving $F$:

$$\Big|\int_{B_n}F(x,u^-+tP_n^+\varphi_\epsilon)\mathrm{d}x-\int_{B_n}F(x,u^-)\mathrm{d}x-\int_{B_n}F(x,tP_n^+\varphi_\epsilon)\mathrm{d}x\Big|$$

$$
\begin{aligned}
&= \Big|\int_{B_n}\Big[\int_0^{tP_n^+\varphi_\epsilon} f(x,u^-+s)\mathrm{d}s - \int_0^{tP_n^+\varphi_\epsilon} f(x,s)\mathrm{d}s\Big]\mathrm{d}x\Big| \\
&\le C_6\Big[\int_{B_n}|(tP_n^+\varphi_\epsilon)|\big(1+|u^-+tP_n^+\varphi_\epsilon|^{p-1}\big)\mathrm{d}x + \int_{B_n}|(tP_n^+\varphi_\epsilon)|\big(1+|tP_n^+\varphi_\epsilon|^{p-1}\big)\mathrm{d}x\Big] \\
&\le C_6\Big[\int_{B_n}\big(|u^-|^{p-1}|tP_n^+\varphi_\epsilon|+|tP_n^+\varphi_\epsilon|+|tP_n^+\varphi_\epsilon|^p\big)\mathrm{d}x\Big]. \qquad (4.18)
\end{aligned}
$$

We deduce from the condition $\|(u^-+tP_n^+\varphi_\epsilon)\|_{2^*,K}=1$ that $\|u^-\|_\infty$ is uniformly bounded. As in [5] (see formula (20) there) we have

$$
\begin{aligned}
\Big|\int_{B_n}\big(|P_n^+\varphi_\epsilon|^p-|\varphi_\epsilon|^p\big)\mathrm{d}x\Big| &\le C_7\big(\|\varphi_\epsilon\|_{p-1}^{p-1}\|P_n^-\varphi_\epsilon\|_\infty+\|P_n^-\varphi_\epsilon\|_p^p\big) \\
&\le \big(\epsilon^{N-\frac{(N-2)(p-1)}{2}}\epsilon^{\frac{N-2}{2}}+\epsilon^{\frac{p(N-2)}{2}}\big)=O\big(\epsilon^{\frac{N-2}{2}}\big).
\end{aligned}
$$

Therefore it follows from (4.18) that

$$
\Big|\int_{B_n}\big[F(x,u)-F(x,u^-)-F(x,tP_n^+\varphi_\epsilon)\big]\mathrm{d}x\Big| \le C_8\big(\epsilon^{\frac{N-2}{2}}+\epsilon^{N-\frac{p(N-2)}{2}}\big).
$$

Consequently,

$$
\begin{aligned}
&\int_{B_n}F(x,s_\epsilon(u^-+tP_n^+\varphi_\epsilon))\mathrm{d}x \\
&\ge \int_{B_n}F(x,s_\epsilon u^-)\mathrm{d}x+\int_{B_n}F(x,s_\epsilon tP_n^+\varphi_\epsilon)\mathrm{d}x+O\big(\epsilon^{\frac{N-2}{2}}\big). \qquad (4.19)
\end{aligned}
$$

It then follows from (4.14) and (4.19) (taking into account both cases (4.16) and (4.17)) that

$$
\begin{aligned}
&J_n\big(s_\epsilon(u^-+tP_n^+\varphi_\epsilon)\big) \\
&\le \frac{1}{N}K(0)^{-\frac{N-2}{2}}S^{\frac{N}{2}}+O\big(\epsilon^{\frac{N-2}{2}}\big)+O\big(\epsilon^{N-\frac{p(N-2)}{2}}\big)-\int_{B_n}F(x,s_\epsilon u)\mathrm{d}x \\
&\le \frac{1}{N}K(0)^{-\frac{N-2}{2}}S^{\frac{N}{2}}+O\big(\epsilon^{\frac{N-2}{2}}\big)+O\big(\epsilon^{N-\frac{p(N-2)}{2}}\big)-\int_{B_n}F(x,s_\epsilon u^-)\mathrm{d}x \\
&\quad -\int_{B_n}F(x,s_\epsilon tP_n^+\varphi_\epsilon)\mathrm{d}x \\
&\le \frac{1}{N}K(0)^{-\frac{N-2}{2}}S^{\frac{N}{2}}+O\big(\epsilon^{\frac{N-2}{2}}\big)+O\big(\epsilon^{N-\frac{p(N-2)}{2}}\big)-\int_{B_n}F(x,s_\epsilon tP_n^+\varphi_\epsilon)\mathrm{d}x. \qquad (4.20)
\end{aligned}
$$

We now observe that

$$
\begin{aligned}
&\Big|\int_{B_n}\big(F(x,s_\epsilon tP_n^+\varphi_\epsilon)-F(x,s_\epsilon t\varphi_\epsilon)\big)\mathrm{d}x\Big| \\
&\le \int_{B_n}\Big|\int_{s_\epsilon t\varphi_\epsilon}^{s_\epsilon tP_n^+\varphi_\epsilon}f(x,s)\mathrm{d}s\Big|\mathrm{d}x \le C\big(\|P_n^+\varphi_\epsilon\|_2^2+\|P_n^+\varphi_\epsilon\|_p^p\big)=o\big(\epsilon^{\frac{N-2}{2}}\big). \qquad (4.21)
\end{aligned}
$$

Therefore, by (4.20), (4.21) and with the aid of assumption $(f_5)$, we get

$$
\begin{aligned}
&J_n\big(s(u^-+tP_n^+\varphi_\epsilon)\big) \\
&\le \frac{1}{N}K(0)^{-\frac{N-2}{2}}S^{\frac{N}{2}}+O\big(\epsilon^{\frac{N-2}{2}}\big)+O\big(\epsilon^{N-\frac{p(N-2}{2}}\big)-\int_{B_n}\bar{F}(s_\epsilon t\varphi_\epsilon)\mathrm{d}x \\
&\le K(0)^{-\frac{N-2}{2}}S^{\frac{N}{2}}+O\big(\epsilon^{\frac{N-2}{2}}\big)+O\big(\epsilon^{N-\frac{p(N-2}{2}}\big)-\int_{B(0,R)}\bar{F}\Big(\frac{A\epsilon^{\frac{N-2}{2}}}{(\epsilon^2+|x|^2)^{\frac{N-2}{2}}}\Big)\mathrm{d}x.
\end{aligned}
$$

We now observe that assumption $(f_5)$ implies that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{\frac{N-2}{2}}}\int_{B(0,R)}\bar{F}\Big(\frac{A\epsilon^{\frac{N-2}{2}}}{(\epsilon^2+|x|^2)^{\frac{N-2}{2}}}\Big)\mathrm{d}x=\infty$$

and

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{N-\frac{p(N-2)}{2}}}\int_{B(0,R)}\bar{F}\Big(\frac{A\epsilon^{\frac{N-2}{2}}}{(\epsilon^2+|x|^2)^{\frac{N-2}{2}}}\Big)\mathrm{d}x=\infty,$$

from which we deduce that

$$J_n\big(s(u^-+tP_n^+\varphi_\epsilon)\big)<\frac{S^{\frac{N}{2}}}{N}K(0)^{-\frac{N-2}{2}}.$$

**Lemma 4.5** Let

$$M_n(\epsilon)=\{u+tP_n^+\varphi_\epsilon;\ \|u+tP_n^+\varphi_\epsilon\|\le R,\ t\ge 0,\ u\in E_n^-\},$$

then, for $R>0$ sufficiently large,

$$c_n=\inf_{h\in\Gamma_n}\sup_{u\in M_n(\epsilon)}J_n(h(u))$$

are critical values of $J_n$.

**Proof** Let $\rho$ be a constant from Lemma 4.1. We claim that for sufficiently large $R>\rho$ $\sup\limits_{u\in\partial M_n(\epsilon)}J_n(u)=0$. If $u\in\partial M_n(\epsilon)$ and $t=0$, then $J_n(u)\le 0$. So let $R=\|u+tP_n^+\varphi_\epsilon\|$, with $t>0$. It follows from assumptions $(f_2)-(f_4)$ that for every $\eta>0$ there exists $C_\eta>0$ such that

$$F(x,u)\ge -\eta u^2+C_\eta|u|^\theta,$$

with $2<\theta<2^*$. This implies that

$$\int_{B_n}F(x,u+tP_n^+\varphi_\epsilon)\mathrm{d}x\ge -\eta\|u\|_2^2-\eta t^2\|P_n^+\varphi_\epsilon\|_2^2+C_\eta\|u+tP_n^+\varphi_\epsilon\|_\theta^\theta.$$

By the Sobolev inequality we have

$$\begin{aligned}J_n(u+tP_n^+\varphi_\epsilon)\le{}&-\frac{1}{2}\|u\|^2+\eta C\|u\|^2+\frac{1}{2}t^2\|P_n^+\varphi_\epsilon\|_k^2+C\eta t^2\|P_n^+\varphi\|^2\\&-C_\eta\|u+tP_n^+\varphi_\epsilon\|_\theta^\theta-\frac{m}{2^*}\|u+tP_n^+\varphi_\epsilon\|_{2^*}^{2^*},\end{aligned}$$

for some constant $C>0$ and $m=\inf\limits_{x\in\mathbb{R}^N}K(x)$. We now observe that $X_n=E_n^-\oplus\mathbb{R}P_n^+\varphi_\epsilon$ is continuously embedded in $L^q(B_n)$ for $2\le q\le 2^*$ and there exists a continuous projection $\Pi_k:\ X_n\to\mathbb{R}P_n^+\varphi_\epsilon$ such that

$$\|tP_n^+\varphi_\epsilon\|_q\le\|\Pi_n\|_q\|u+tP_n^+\varphi_\epsilon\|_q\ \text{ and }\ \|\Pi_n\|_q\ge 1.$$

Choosing $\eta$ such that $\eta C=\frac{1}{4}$ we get

$$J_n(u+tP_n^+\varphi_\epsilon)\le-\frac{1}{4}\|u\|^2+\frac{3}{4}\|tP_n^+\varphi_\epsilon\|^2-C_1\big(t^\theta\|P_n^+\varphi_\epsilon\|_\theta^\theta+t^{2^*}\|P_n^+\varphi_\epsilon\|_{2^*}^{2^*}\big),$$

where $C_1>0$ is a constant depending on $\|\Pi_k\|_q$, $\|\Pi_n\|_{2^*}$, $m$, $N$ and $C_\eta$. Consequently, we see that $J_n(u+tP_n^+\varphi_\epsilon)\to-\infty$ as $\|u+tP_n^+\varphi_\epsilon\|\to\infty$ and our claim follows.

**Proof of Proposition 4.1** We now observe that, by Lemma 4.4 $c_n<\frac{S^{\frac{N}{2}}}{N}\|K\|_\infty^{-\frac{N-2}{2}}$, and by virtue of Lemma 4.2 the $(PS)_c^*$ condition holds at the level $c_n$. Therefore the results follows from Theorem 2.1.

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