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Minimizing total weighted flow time under uncertainty using dominance and a stability box

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ABSTRACT

We consider an uncertain single-machine scheduling problem, in which the processing time of a job can take any real value on a given closed interval. The criterion is to minimize the total weighted flow time of the n jobs, where there is a weight associated with a job. We calculate a number of minimal dominant sets of the job permutations (a minimal dominant set contains at least one optimal permutation for each possible scenario). We introduce a new stability measure of a job permutation (a stability box) and derive an exact formula for the stability box, which runs in $O(n \log n)$ time. We investigate properties of a stability box. These properties allow us to develop an $O(n^2)$-algorithm for constructing a permutation with the largest volume of a stability box. If several permutations have the largest volume of a stability box, the developed algorithm selects one of them due to a simple heuristic. The efficiency of the constructed permutation is demonstrated on a large set of randomly generated instances with $10 \le n \le 1000$.

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1. Introduction

Uncertainties are present in most real-life scheduling problems. Several approaches, which complement one another, are available for dealing with scheduling problems under uncertainty. In a *stochastic approach*, an uncertain scheduling parameter (e.g., the processing time of a job) is assumed to be a random variable with a specific probability distribution (see monograph [1], part II). However, in many real-life situations, one may have no sufficient information to characterize the probability distribution of each random parameter. In such situations, other approaches are needed. In the approach of seeking a *robust schedule* [2–9], a decision-maker prefers a schedule that hedges against the worst-case scenario. There is also available a *stability approach* [10–14], which combines a stability analysis, a multi-stage decision framework (the off-line planning stage and the online scheduling stages) and a solution concept of a minimal dominant set of the job permutations. A minimal dominant set optimally covers all the scenarios in the sense that for any possible scenario such a set contains at least one optimal permutation [12–14]. A minimal dominant set is useful for a scheduler to make an online decision whenever additional information on the processing times becomes available [12,13].

In this paper, we consider a single-machine scheduling problem with interval processing times of n jobs to be scheduled. In Section 2, we present different problem settings and the state-of-the-art. In Section 3, we calculate a number of minimal dominant sets and describe a modification of the problem establishing the uniqueness of a minimal dominant set. An illustrative example is given in Section 4. In Section 5, we introduce a stability box of a job permutation and derive an exact formula for characterizing the stability box, which runs in $O(n \log n)$ time. Properties of a stability box are investigated in Section 5. An $O(n^2)$-algorithm for finding a permutation with the largest volume of a stability box is developed in Section 6. Section 7 reports computational results. We conclude with Section 8.

2. Problem settings and state-of-the-art

There are $n \ge 2$ jobs $\mathcal{J} = \{J_1, J_2, \ldots, J_n\}$ to be processed on a single machine. Associated with job $J_i \in \mathcal{J}$, there is a weight $w_i > 0$ reflecting the importance of the job. The processing time p_i of job $J_i \in \mathcal{J}$ can take any real value between a lower bound $p_i^L > 0$ and an

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upper bound $p_i^U \geq p_i^L$. An exact value of the job processing time remains unknown until job completion. Let T denote the set of vectors $p=(p_1, p_2,\ldots,p_n)$ of the processing times in the space R_+^n of non-negative n-dimensional real vectors. Set T is the Cartesian product of the closed intervals:

$$T = \{p | p \in R_+^n, p_i^L \leq p_i \leq p_i^U,\ i \in \{1,2,\ldots,n\}\} = \times_{i=1}^{n} [p_i^L, p_i^U]. \tag{1}$$

A vector $p \in T$ is called a scenario. Let $S=\{\pi_1,\pi_2,\ldots,\pi_{n!}\}$ be the set of all permutations $\pi_k=(J_{k_1}, J_{k_2},\ldots J_{k_n})$ of n jobs $\mathcal{J}$. Given a permutation $\pi_k \in S$ and a scenario $p \in T$, let $C_i=C_i(\pi_k,p)$ denote the completion time of a job $J_i \in \mathcal{J}$ in the semi-active schedule [1,12] defined by the permutation π_k. The criterion $\sum w_i C_i$ denotes the minimization of the sum of the weighted completion times: $\sum_{J_i \in \mathcal{J}} w_i C_i(\pi_t,p)=\min_{\pi_k \in S}\{\sum_{J_i \in \mathcal{J}} w_i C_i(\pi_k,p)\}$, where permutation $\pi_t=(J_{t_1}, J_{t_2},\ldots J_{t_n}) \in S$ is optimal. By adopting the three-field notation $\alpha|\beta|\gamma$ from survey paper [15], this problem is denoted by $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$.

If a vector $p \in T$ of the processing times is fixed before scheduling ($p_i^L = p_i^U = p_i$, $i \in \{1,2,\ldots,n\}$, i.e., segment $[p_i^L, p_i^U]$ is degenerated into a point $p_i \in [p_i,p_i]$), then an uncertain problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ reduces to the deterministic problem $1\|\sum w_i C_i$, which can be solved in $O(n \log n)$ time [16]. Since a scenario $p \in T$ is not fixed for an uncertain problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$, the completion time C_i of a job $J_i \in \mathcal{J}$ cannot be calculated for a permutation $\pi_k \in S$. The value of the objective function for permutation π_k remains uncertain until the jobs have been completed. In the OR literature, several approaches for solving an optimization problem with uncertain objective function values have been developed. Next, we survey such settings and some results for scheduling jobs with uncertain processing times.

An uncertain problem with the objective function $\gamma=f(C_1,C_2,\ldots,C_n)$ is denoted by $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$, its deterministic counterpart by $\alpha\|\gamma$. For problem $1|p_i^L \leq p_i \leq p_i^U|\gamma$, there usually does not exist a permutation $\pi_t \in S$, which is optimal for each scenario from the set T. So, an additional criterion is often introduced for dealing with problem $1|p_i^L \leq p_i \leq p_i^U|\gamma$. In particular, a *robust schedule* minimizing the worst-case absolute (or the worst-case relative) deviation from optimality has been introduced in [3,4] to hedge against data uncertainty. In the robust approach, the scenario set T could contain a continuum of scenarios, i.e., T is the Cartesian product of the closed intervals as defined in (1), or just contains a finite number h of discrete scenarios [3,5,7,17,18]: $T=\{p^j=(p_1^j,p_2^j,\ldots,p_n^j) | p^j \in R_+^n, j \in \{1,2,\ldots,h\}\}$. Permutation $\pi_t \in S$ is optimal for problem $1\|\gamma$ with scenario p, if $f(C_1(\pi_t,p),C_2(\pi_t,p),\ldots,C_n(\pi_t,p))=\gamma_p^t = \min_{\pi_k \in S}\gamma_p^k = \min_{\pi_k \in S} f(C_1(\pi_k,p),C_2(\pi_k,p),\ldots,C_n(\pi_k,p))$. For permutation $\pi_k \in S$ and scenario $p \in T$, the difference $\gamma_p^k - \gamma_p^t = r(\pi_k,p)$ is called the *regret*. The value $Z(\pi_k)=\max\{r(\pi_k,p) | p \in T\}$ is called the worst-case absolute regret. The worst-case relative regret is defined as $Z'(\pi_k)=\max\{r(\pi_k,p)/\gamma_p^t | p \in T\}$, where $\gamma_p^t \neq 0$.

While a deterministic problem $1\|\sum C_i$ is polynomially solvable [16], finding a permutation $\pi_t \in S$ minimizing the worst-case absolute regret $Z(\pi_t)$ or the worst-case relative regret $Z'(\pi_t)$ are both binary NP-hard even for two scenarios [3,7], $h=2$. The latter problem becomes unary NP-hard for an unbounded number of discrete scenarios [7]. In [5], it was proven that minimizing the worst-case absolute regret $Z(\pi_k)$ for problem $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$ is binary NP-hard even if the intervals $[p_i^L, p_i^U]$ for all jobs $J_i \in \mathcal{J}$ have the same center $(p_i^U - p_i^L)/2$. In [19], the binary NP-hardness was proven for finding a permutation $\pi_t \in S$ that minimizes the worst-case absolute regret $Z(\pi_t)$ for an uncertain two-machine flow-shop problem with the makespan criterion $C_{\max}=\max\{C_i(\pi_t,p) | J_i \in \mathcal{J}\}$ even for two scenarios, $h=2$.

Only a few special cases of scheduling problems are known to be polynomially solvable for minimizing the worst-case regret. An $O(n^4)$-algorithm was developed [20] for minimizing the worst-case regret for problem $1|p_i^L \leq p_i \leq p_i^U, d_i^L \leq d_i \leq d_i^U|L_{max}$ with the criterion L_{max} of minimizing the maximum lateness $\max\{C_i(\pi_t,p)-d_i | J_i \in \mathcal{J}\}=\min_{\pi_k \in S}\{\max\{C_i(\pi_k,p)-d_i | J_i \in \mathcal{J}\}\}$, when the intervals of the job processing times and the intervals of the due dates d_i are given. In [5], it was proven that minimizing $Z(\pi_k)$ for problem $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$ can be realized in $O(n \log n)$ time, if the segments $[p_i^L, p_i^U]$, $J_i \in \mathcal{J}$, have the same center provided that the number n of the jobs is even. In [21], an $O(m)$-algorithm was proposed for minimizing the worst-case regret for the m-machine two-job flow-shop problem $Fm|p_i^L \leq p_i \leq p_i^U, n=2|C_{\max}$ provided that each of the m machines processes the jobs $\mathcal{J}=\{J_1,J_2\}$ in the same order.

In [8], a 2-approximation algorithm has been developed to minimize the worst-case regret for problem $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$. In [3,7,9], exact and heuristic algorithms were developed and tested to minimize the worst-case regret for the same problem.

In this paper, we adopt the *stability approach* [10–12] to problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$. The stability approach combines a stability analysis, a multi-stage decision framework, and the solution concept of a minimal dominant set of semi-active schedules.

Definition 1. The set of permutations (semi-active schedules) $S(T) \subseteq S$ is a *minimal dominant set* for an uncertain problem $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$, if

(a) for any fixed scenario $p \in T$, set $S(T)$ contains at least one permutation (a semi-active schedule), which is optimal for the deterministic problem $\alpha\|\gamma$ associated with scenario p,
(b) property (a) is lost for any proper subset of set $S(T)$.

The set $S(T)$ was investigated in [10–12] for the makespan criterion, and in [12,22,23] for the total flow time criterion. In [23], dominance relations were identified for a flow-shop problem $F2|p_i^L \leq p_i \leq p_i^U|\sum C_i$. In [22], for a job-shop problem $Jm|p_i^L \leq p_i \leq p_i^U|\sum C_i$, exact and heuristic algorithms were developed by using the disjunctive graph model.

Before presenting a new heuristic for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$, we mention some known results for an uncertain problem and for its deterministic counterpart. In [16], it was proven that problem $1\|\sum w_i C_i$ can be solved in $O(n \log n)$ time due to the following sufficient condition for the optimality of a permutation $\pi_k=(J_{k_1}, J_{k_2},\ldots J_{k_n}) \in S$:

$$\frac{w_{k_1}}{p_{k_1}} \geq \frac{w_{k_2}}{p_{k_2}} \geq \cdots \geq \frac{w_{k_n}}{p_{k_n}}, \tag{2}$$

where $p_{k_i} > 0$ for each job $J_{k_i} \in \mathcal{J}$. The deterministic problem $1\|\sum w_i C_i$ can be solved to optimality by the weighted shortest processing time rule: process the jobs in non-increasing order of their weight-to-process ratio w_{k_i}/p_{k_i}. Inequalities (2) provide also a necessary condition for the optimality of a permutation $\pi_k \in S$ as summarized in Theorem 1 [24].

Theorem 1 (*Smith [16] and Koffman [24]*). *Permutation* $\pi_k = (J_{k_1} J_{k_2}, \ldots J_{k_n}) \in S$ *is optimal for the deterministic problem* $1\|\sum w_i C_i$ *if and only if inequalities* (2) *hold.*

A minimal dominant set $S(T)$ for problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$ may be determined by using the precedence-dominance relation on the set of jobs $\mathcal{J}$.

Definition 2. Job J_u dominates job J_v with respect to T (it is denoted by $J_u \mapsto J_v$) if there exists a minimal dominant set $S(T)$ for problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$ such that job J_u precedes job J_v in every permutation from the set $S(T)$.

Theorem 2 (*Sotskov et al. [14]*). *For problem* $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$, *job* J_u *dominates job* J_v *with respect to* T *if and only if*

$$\frac{w_u}{p_u^U} \ge \frac{w_v}{p_v^L}. \tag{3}$$

The cardinality $|S(T)|$ of a minimal dominant set may be considered as a measure of uncertainty for problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$. In the least uncertain case of a cardinality being one, a minimal dominant set is a singleton, $\{\pi_k\} = S(T)$, which is also a solution to the deterministic problem $1\|\sum w_i C_i$ associated with any scenario $p \in T$.

Theorem 3 (*Sotskov et al. [14]*). *For the existence of a dominant singleton* $S(T) = \{\pi_k\} = \{(J_{k_1} J_{k_2}, \ldots J_{k_n})\}$ *for problem* $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$, *inequalities* (4) *are necessary and sufficient*:

$$\frac{w_{k_1}}{p_{k_1}^U} \ge \frac{w_{k_2}}{p_{k_2}^L}, \frac{w_{k_2}}{p_{k_2}^U} \ge \frac{w_{k_3}}{p_{k_3}^L}, \ldots, \frac{w_{k_{n-1}}}{p_{k_{n-1}}^U} \ge \frac{w_{k_n}}{p_{k_n}^L}. \tag{4}$$

The most uncertain case of problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$ is that with $|S(T)| = n!$.

Theorem 4 (*Sotskov et al. [14]*). *Let* $p_i^L < p_i^U$, $J_i \in \mathcal{J}$. *For the existence of a minimal dominant set* $S(T)$ *for problem* $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$ *with a maximum cardinality* $|S(T)| = n!$, *inequality* (5) *is necessary and sufficient*:

$$\max\left\{\frac{w_i}{p_i^U}\middle| J_i \in \mathcal{J}\right\} < \min\left\{\frac{w_i}{p_i^L}\middle| J_i \in \mathcal{J}\right\}. \tag{5}$$

3. A minimal dominant set

We use the notation $1|p|\sum w_i C_i$ for indicating an individual problem (an instance) with the scenario p for the mass problem $1\|\sum w_i C_i$.

Lemma 1. *In each optimal permutation for the instance* $1|p|\sum w_i C_i$, *job* J_u *precedes job* J_v *if and only if*

$$\frac{w_u}{p_u} > \frac{w_v}{p_v}. \tag{6}$$

Proof. *Sufficiency*: By contradiction, we assume that there exists an optimal permutation $\pi_m \in S$ for the instance $1|p|\sum w_i C_i$ such that inequality (6) holds, however, job J_u follows job J_v in permutation π_m. Since the necessity of condition (2) given in Theorem 1 implies the inequalities $w_v/p_v \ge w_{v+1}/p_{v+1} \ge \cdots \ge w_u/p_u$, we obtain inequality $w_v/p_v \ge w_u/p_u$ contradicting (6).

Necessity: Let job J_u precede job J_v in each optimal permutation for the instance $1|p|\sum w_i C_i$.

We assume $w_u/p_u \le w_v/p_v$. Due to Theorem 1, job J_v must precede job J_u in any optimal permutation for the instance $1|p|\sum w_i C_i$. This contradiction completes the proof. □

Lemma 2. *For the instance* $1|p|\sum w_i C_i$, *there exist both an optimal permutation with job* J_u *preceding job* J_v *and an optimal permutation with job* J_v *preceding job* J_u *if and only if*

$$\frac{w_u}{p_u} = \frac{w_v}{p_v}. \tag{7}$$

Proof. *Sufficiency*: Since set S is finite, there exists a permutation π_l of the form $\pi_l = (\ldots J_u, \ldots J_v, \ldots) \in S$ or a permutation π_m of the form $\pi_m = (\ldots J_v, \ldots J_u, \ldots) \in S$ which is optimal for the instance $1|p|\sum w_i C_i$. Due to (7), a part of the necessary and sufficient condition (2) of the optimality of permutation π_l (Theorem 1) looks as follows:

$$\cdots \ge \frac{w_u}{p_u} = \cdots = \frac{w_v}{p_v} \ge \cdots, \tag{8}$$

and that of permutation π_m looks as

$$\cdots \ge \frac{w_v}{p_v} = \cdots = \frac{w_u}{p_u} \ge \cdots. \tag{9}$$

If equalities (8) hold, then equalities (9) hold, and vice versa. Due to Theorem 1, for the instance $1|p|\sum w_i C_i$, there exist both an optimal permutation of the form $\pi_l = (\ldots J_u, \ldots J_v, \ldots)$ and one of the form $\pi_m = (\ldots J_v, \ldots J_u, \ldots)$. Sufficiency is proven.

Necessity: Let there exist both an optimal permutation of the form π_l and one of the form π_m. Due to Theorem 1, this is possible only if equality (7) holds. □

The following claim directly follows from Lemma 2.

Lemma 3. *For the instance* $1|p|\sum w_i C_i$, *an optimal permutation is unique if and only if for any pair of jobs* $J_u \in \mathcal{J}$ *and* $J_v \in \mathcal{J}$ *equality* (7) *does not hold.*

Let $a=\min\{w_i/p_i^U|J_i\in\mathcal{J}\}$ and $b=\max\{w_i/p_i^L|J_i\in\mathcal{J}\}$. A subset $\mathcal{J}_r$ of the set $\mathcal{J}$ (where $r\in[a,b]$ is a real number) is crucial for calculating the number of minimal dominant sets:

$$\mathcal{J}_r=\left\{J_i\in\mathcal{J}\left|r=\frac{w_i}{p_i^U}=\frac{w_i}{p_i^L}\right.\right\}. \tag{10}$$

Theorem 5. *If inequality $|\mathcal{J}_{r_q}|\geq 2$ holds for each $r_q\in\{r_1,r_2,\ldots,r_m\}$, where integer $m\geq 1$ is maximal and $r_q\in[a,b]$, then the number of the minimal dominant sets existing for the instance $1|p_i^L\leq p_i\leq p_i^U|\sum w_iC_i$ is equal to $\prod_{q=1}^m|\mathcal{J}_{r_q}|!$.*

Proof. For any pair of jobs $J_t\in\mathcal{J}$ and $J_v\in\mathcal{J}$, we shall examine all the possible arrangements of the segments $[w_t/p_t^U,w_t/p_t^L]$ and $[w_v/p_v^U,w_v/p_v^L]$. W.l.o.g. assume $w_v/p_v^U\leq w_t/p_t^U$. Due to the symmetry of the jobs J_t and J_v, it is sufficient to examine the following nine cases (a)–(i).

Case (a): $w_t/p_t^U<w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^L<w_t/p_t^U$.

Inequality $w_v/p_v<w_t/p_t$ holds for each scenario $p\in T$. Due to Lemma 1, in any optimal permutation for the instance $1|p|\sum w_iC_i$, $p\in T$, job J_t precedes job J_v. Due to Definition 1, in any permutation from a minimal dominant set $S(T)$, job J_t precedes job J_v.

Case (b): $w_t/p_t^U<w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^L\leq w_t/p_t^U$.

If for the scenario $p\in T$ at least one of the conditions $w_v/p_v\neq w_v/p_v^L$ or $w_t/p_t\neq w_t/p_t^U$ holds, then arguing in the same way as in case (a), we obtain that in any permutation from a set $S(T)$, job J_t precedes job J_v. For the remaining vector $p'=(p'_1,p'_2,\ldots,p'_n)\in T$, for which both equalities $w_v/p'_v=w_v/p_v^L$ and $w_t/p'_t=w_t/p_t^U$ hold, we obtain $w_v/p'_v=w_t/p'_t$. Due to Lemma 2, for the instance $1|p'|\sum w_iC_i$, there exist both an optimal permutation of the form $\pi_l=(\ldots J_t,\ldots J_v,\ldots)\in S$ and one of the form $\pi_m=(\ldots J_v,\ldots J_t,\ldots)\in S$. However, no permutation of the form π_m may be contained in a set $S(T)$, since such a permutation is redundant. Indeed, a permutation of the form π_l is definitely contained in any minimal dominant set because of scenario $p\in T$, $p\neq p'$ (see condition (a) of Definition 1). The permutation π_l provides an optimal solution to the instance $1|p'|\sum w_iC_i$. If the set $S(T)$ contains a permutation of the form π_m, then $S(T)$ is not a minimal dominant set (condition (b) of Definition 1). We conclude that in any permutation from the set $S(T)$ job J_t precedes job J_v.

Case (c): $w_t/p_t^U<w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^L>w_t/p_t^U$.

Due to the strictness of the above inequalities, the length of the intersection of the segments $[w_t/p_t^U,w_t/p_t^L]$ and $[w_v/p_v^U,w_v/p_v^L]$ must be strictly positive. There exist both a scenario $p\in T$ and a scenario $p'=(p'_1,p'_2,\ldots,p'_n)\in T$ such that $w_t/p_t>w_v/p_v$ and $w_t/p'_t<w_v/p'_v$. Due to Lemma 1, in all optimal permutations for the instance $1|p|\sum w_iC_i$, job J_t precedes job J_v, and in all optimal permutations for the instance $1|p'|\sum w_iC_i$, job J_v precedes job J_t. Due to condition (a) of Definition 1, any minimal dominant set $S(T)$ constructed for problem $1|p_i^L\leq p_i\leq p_i^U|\sum w_iC_i$ contains both a permutation of the form $\pi_l=(\ldots J_t,\ldots J_v,\ldots)\in S$ and a permutation of the form $\pi_m=(\ldots J_v,\ldots J_t,\ldots)\in S$.

Case (d) with $w_t/p_t^U=w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^L<w_t/p_t^U$ is examined similarly as case (a).

Case (e) with $w_t/p_t^U=w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^L=w_t/p_t^U$ is examined similarly as case (b).

Case (f) with $w_t/p_t^U=w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^L<w_t/p_t^U<w_v/p_v^U$ is examined similarly as case (c).

Case (g) with $w_t/p_t^U=w_t/p_t^L$, $w_v/p_v^U<w_v/p_v^L$, $w_v/p_v^U=w_t/p_t^U$ is examined similarly as case (b).

Case (h): $w_t/p_t^U=w_t/p_t^L$, $w_v/p_v^U=w_v/p_v^L$, $w_v/p_v^L<w_t/p_t^U$.

The processing times of the jobs J_t and J_v are fixed and $w_v/p_v<w_t/p_t$. Due to Lemma 1, job J_v precedes job J_t in all optimal permutations. Due to condition (a) of Definition 1, job J_t precedes job J_v in any permutation from the set $S(T)$.

Case (i): $w_t/p_t^U=w_t/p_t^L$, $w_v/p_v^U=w_v/p_v^L$, $w_v/p_v^L=w_t/p_t^U$.

The processing times of the jobs J_t and J_v are fixed: $p_t^L=p_t^U=p_t$, $p_v^L=p_v^U=p_t$ and so $w_v/p_v=w_t/p_t$. Due to Lemma 3 (Lemma 2, respectively) for the instance $1|p|\sum w_iC_i$, the optimal permutation is not unique (there exist both an optimal permutation $\pi_l\in S$ with job J_t preceding job J_v and an optimal permutation $\pi_m\in S$ with job J_v preceding job J_t). Due to Definition 1, the jobs J_t and J_v generate two different minimal dominant sets $S(T)$ and $S'(T)$. The set $S(T)$ contains a permutation of the form π_l and does not contain a permutation of the form π_m, while the set $S'(T)$ is the other way around. Thus, at least two minimal dominant sets exist for such a problem $1|p_i^L\leq p_i\leq p_i^U|\sum w_iC_i$.

It is easy to convince that if case (i) occurs, then inequality $|\mathcal{J}_r|\geq 2$ must hold (where $r=w_t/p_t^U=w_t/p_t^L=w_v/p_v^U=w_v/p_v^L$), and vice versa.

From the above treated cases (a)–(i) for the jobs $J_t\in\mathcal{J}$ and $J_v\in\mathcal{J}$, we conclude that several minimal dominant sets $S(T)$ may occur only if there exist jobs J_t and J_v with the weight-to-process ratios satisfying case (i). Case (i) occurs if and only if $|\mathcal{J}_{r_q}|\geq 2$ for some real number $r_q\in[a,b]$. We enumerate minimal dominant sets generated by the sets $\mathcal{J}_{r_q}$ with $|\mathcal{J}_{r_q}|\geq 2$, $r_q\in\{r_1,r_2,\ldots,r_m\}$. Let $\mathcal{J}_{r_q}=\{J_{q(1)},J_{q(2)},\ldots J_{q(|\mathcal{J}_{r_q}|)}\}\subseteq\mathcal{J}$. Due to (10) of a set $\mathcal{J}_r$, we obtain the equalities:

$$r_q=\frac{w_{q(1)}}{p^U_{q(1)}}=\frac{w_{q(1)}}{p^L_{q(1)}}=\frac{w_{q(2)}}{p^U_{q(2)}}=\frac{w_{q(2)}}{p^L_{q(2)}}=\cdots=\frac{w_{q(|\mathcal{J}_{r_q}|)}}{p^L_{q(|\mathcal{J}_{r_q}|)}}=\frac{w_{q(|\mathcal{J}_{r_q}|)}}{p^U_{q(|\mathcal{J}_{r_q}|)}}, \tag{11}$$

which imply that the processing time of each job $J_{q(v)}\in\mathcal{J}_{r_q}$ is fixed and the weight-to-process ratios are the same for all jobs from the set $\mathcal{J}_{r_q}$. Due to Theorem 1, in any optimal permutation $\pi_l\in S$ for the instance $1|p|\sum w_iC_i$ with every scenario $p\in T$, all jobs from set $\mathcal{J}_{r_q}$ have to be located adjacently one by one: $\pi_l=(\ldots,\pi(\mathcal{J}_{r_q}),\ldots)$.

Hereafter, $\pi(\mathcal{J}_{r_q})$ denotes a permutation of the jobs $\mathcal{J}_{r_q}$. Since condition (7) holds for each pair of jobs from the set $\mathcal{J}_{r_q}$ (see (11)), we can implement Lemma 2 for them. It is clear that the set of jobs $\mathcal{J}_{r_q}$ generates $|\mathcal{J}_{r_q}|!$ optimal permutations for each instance $1|p|\sum w_iC_i$ with $p\in T$. Due to Definition 1, the set $\mathcal{J}_{r_q}$ generates $|\mathcal{J}_{r_q}|!$ minimal dominant sets. Each minimal dominant set $S(T)$ contains a permutation of the form $\pi_l=(\ldots,\pi(\mathcal{J}_{r_q}),\ldots)\in S$ provided that this set $S(T)$ does not contain other permutations of the form $\pi_m=(\ldots,\pi'(\mathcal{J}_{r_q}),\ldots)\in S$, where

$\pi(\mathcal{J}_{r_q}) \neq \pi'(\mathcal{J}_{r_q})$ (condition (b) of Definition 1). Since $\mathcal{J}_{r_q} \bigcap \mathcal{J}_{r_t} = \emptyset,\ q \neq t$, we conclude that the number of minimal dominant sets generated by the sets $\mathcal{J}_{r_q}$ with $|\mathcal{J}_{r_q}| \geq 2,\ r_q \in \{r_1, r_2, \ldots, r_m\}$, is equal to $\prod_{q=1}^{m} |\mathcal{J}_{r_q}|!$. $\square$

Since the cardinality $|S(T)|$ of a minimal dominant set could range from 1 (Theorem 3) to $n!$ (Theorem 4), it is impossible to generate in polynomial time all the elements of the set $S(T)$. However, one can construct a compact presentation of a minimal dominant set in the form of a digraph with the vertex set $\mathcal{J}$. To this end, one can check condition (3) of Theorem 2 for each pair of jobs $J_u \in \mathcal{J},\ J_v \in \mathcal{J}$ and construct a digraph $(\mathcal{J},\mathcal{A})$ of the precedence-dominance relation on the set $\mathcal{J}$: arc (J_u, J_v) belongs to set $\mathcal{A} \subseteq \mathcal{J} \times \mathcal{J}$ if and only if $J_u \mapsto J_v$. Such a construction of a digraph $(\mathcal{J},\mathcal{A})$ takes $O(n^2)$ time.

Theorem 6. *The set $\mathcal{A} \subseteq \mathcal{J} \times \mathcal{J}$ constructed for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ defines a strict order relation on the set $\mathcal{J}$ if and only if there does not exist a $r \in [a,b]$ with $|\mathcal{J}_r| \geq 2$.*

Proof. If $w_u/p_u^U \geq w_v/p_v^L$ and $w_v/p_v^U \geq w_t/p_t^L$, then $w_u/p_u^U \geq w_t/p_t^L$. Due to Theorem 2, the set $\mathcal{A} \subseteq \mathcal{J} \times \mathcal{J}$ defines a transitive binary relation on $\mathcal{J}$. So, we need to test the anti-reflexivity of the relation $\mathcal{A}$.

Sufficiency: Assume that there does not exist a $r \in [a,b]$ with $|\mathcal{J}_r| \geq 2$. For any pair of jobs $J_t \in \mathcal{J}$ and $J_v \in \mathcal{J}$, one of the cases (a)–(h) is possible for their weight-to-process ratios, while case (i) is impossible (see the proof of Theorem 5). In each of the cases (a)–(e) and (h), inclusion $(J_v, J_t) \in \mathcal{A}$ holds. In case (g), $(J_t, J_v) \in \mathcal{A}$. In each of the cases (c) and (f), neither arc (J_v, J_t) nor arc (J_t, J_v) belongs to set $\mathcal{A}$ (Definition 2). For each pair of jobs $J_t \in \mathcal{J}$ and $J_v \in \mathcal{J}$, at most one arc which is incident to both vertices J_t and J_v, $t \neq v$, may belong to set $\mathcal{A}$. We conclude that relation $\mathcal{A}$ is anti-reflexive and so this binary relation is a strict order relation (the transitivity of $\mathcal{A}$ is already proven).

Necessity: Let there exist a $r \in [a,b]$ with $|\mathcal{J}_r| \geq 2$.

Due to definition (10) of a set $\mathcal{J}_r$, there exist jobs J_t and J_v such that their weight-to-process ratios satisfy case (i): $w_t/p_t^U = w_t/p_t^L$, $w_v/p_v^U = w_v/p_v^L$, $w_v/p_v^L = w_t/p_t^U$ and so intervals of the processing times degenerate into a point $r \in [r,r]$, where $r = w_u/p_u^L = w_u/p_u^U = w_v/p_v^L = w_v/p_v^U$. Due to Theorem 2, both inclusions $(J_t, J_v) \in \mathcal{A}$ and $(J_v, J_t) \in \mathcal{A}$ must hold. Since the contour (J_t, J_v, J_t) exists in the digraph $(\mathcal{J},\mathcal{A})$, a binary relation $\mathcal{A}$ is not anti-reflexive. $\square$

From Theorems 5 and 6, it follows that the existence of the sets $\mathcal{J}_{r_q}$ with $|\mathcal{J}_{r_q}| \geq 2,\ r_q \in \{r_1, r_2, \ldots, r_m\}$, implies that a minimal dominant set $S(T)$ loses useful properties. Indeed, if there exists at least one set $\mathcal{J}_{r_q}$ which is not a singleton, then $\mathcal{A} \subseteq \mathcal{J} \times \mathcal{J}$ is not a strict order relation (Theorem 6) and the number of the minimal dominant sets for the problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ may be rather large (Theorem 5).

Next, we show how to overcome these difficulties. Moreover, we show that such a bad set $\mathcal{J}_{r_q}$ is useful while solving a problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$: the problem size n can be reduced by $|\mathcal{J}_{r_q}| - 1$ for each non-singleton $\mathcal{J}_{r_q}$ via identifying the jobs of set $\mathcal{J}_{r_q}$ by one job.

As it was shown in the proof of Theorem 5, in any optimal permutation $\pi_l \in S$ for the instance $1|p|\sum w_i C_i$, all jobs from set $\mathcal{J}_{r_q} \subseteq \mathcal{J}$ must be adjacently located one by one: $\pi_l = (\ldots, \pi(\mathcal{J}_{r_q}), \ldots)$. Furthermore, the order of the jobs $\{J_{q(1)}, J_{q(2)}, \ldots J_{q(|\mathcal{J}_{r_q}|)}\} = \mathcal{J}_{r_q}$ in the permutation $\pi(\mathcal{J}_{r_q})$ does not influence the value of the objective function $\gamma = \sum_{i=1}^{n} w_i C_i$ calculated for permutation $\pi_k = (\ldots, \pi(\mathcal{J}_{r_q}), \ldots) \in S$. Indeed, the processing time of any job $J_{q(v)} \in \mathcal{J}_{r_q}$ is fixed and the weight-to-process ratios are the same for all jobs from the set $\mathcal{J}_{r_q}$. Thus, while looking for an optimal permutation for any instance $1|p|\sum w_i C_i$ generated by the problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ via fixing a scenario $p \in T$, one can treat all jobs $\{J_{q(1)}, J_{q(2)}, \ldots J_{q(|\mathcal{J}_{r_q}|)}\} = \mathcal{J}_{r_q}$ as one job with the weight and processing time equal to those of any job from set $\mathcal{J}_{r_q}$. By choosing only one job from each set $\mathcal{J}_{r_q},\ r_q \in \{r_1, r_2, \ldots, r_m\},\ |\mathcal{J}_{r_q}| \geq 2$, the original instance of an uncertain problem can be reduced to an equivalent instance (we denote this instance by $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$) with a smaller cardinality of the set of jobs to be scheduled (we denote this set by $\mathcal{J}^*$):

$$|\mathcal{J}^*| = |\mathcal{J}| - \sum_{q=1}^{m} (|\mathcal{J}_{r_q}| - 1) = n + m - \sum_{q=1}^{m} |\mathcal{J}_{r_q}|.$$

Summarizing, we derive Proposition 1, where $1^*|p|\sum w_i C_i$ denotes a deterministic instance generated by an uncertain instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ via fixing a scenario $p \in T$.

Proposition 1. *An instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ is equivalent to the original instance of problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ in the sense that for any fixed scenario $p \in T$, an optimal permutation π_k for the instance $1^*|p|\sum w_i C_i$ is obtained from an optimal permutation π_t for the instance $1|p|\sum w_i C_i$ via deleting the set of jobs $\mathcal{J} \backslash \mathcal{J}^*$ from permutation π_t.*

Along with a smaller size, the equivalent instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ has a unique minimal dominant set. Consequently, the set $S(T)$ is a minimal dominant set with respect to both inclusion and cardinality. Another useful property of the instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ is that the relation $\mathcal{A} \subseteq \mathcal{J} \times \mathcal{J}$ is a strict order relation (Theorem 6).

Instead of using digraph $(\mathcal{J},\mathcal{A})$, one can adopt a reduction $G = (\mathcal{J},\mathcal{A}^0)$ of the digraph $(\mathcal{J},\mathcal{A})$. The digraph G is obtained from $(\mathcal{J},\mathcal{A})$ via deleting the transitive arcs $\mathcal{A} \backslash \mathcal{A}^0$.

4. Example

The input data for Example 1 of problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ are given in columns 1–4 in Table 1. There exist two numbers $r_1 = 0.5$ and $r_2 = 4$ such that the sets $\mathcal{J}_{r_1}$ and $\mathcal{J}_{r_2}$ are not singletons: $\mathcal{J}_{r_1} = \{J_8, J_9, J_{10}\}$; $\mathcal{J}_{r_2} = \{J_4, J_{11}, J_{12}, J_{13}\}$. Due to Theorem 6, the binary relation $\mathcal{A} \subseteq \mathcal{J} \times \mathcal{J}$ is not a strict order relation and the digraph $(\mathcal{J},\mathcal{A})$ has contours. Due to Theorem 5, the number of the minimal dominant sets is equal to $|\mathcal{J}_{r_1}|! \cdot |\mathcal{J}_{r_2}|! = 3! \cdot 4! = 144$.

Due to Proposition 1, one can treat the jobs J_8, J_9, J_{10} (jobs $J_4, J_{11}, J_{12}, J_{13}$) as one job with the parameters equal to those of any job from set $\mathcal{J}_{r_1}$ (set $\mathcal{J}_{r_2}$). Let the set of jobs $\mathcal{J}_{r_1} = \{J_8, J_9, J_{10}\}$ be represented by job J_8, and the set of jobs $\mathcal{J}_{r_2} = \{J_4, J_{11}, J_{12}, J_{13}\}$ by job J_4. Example 1 can be

Table 1
Data for Example 1 (lines with $i \in \{1,2,\ldots,13\}$) and for Example 1* (lines with $i \in \{1,2,\ldots,8\}$).

1	2	3	4	5	6	7	8	9	10	11	12
i	p_i^L	p_i^U	w_i	$\frac{w_i}{p_i^L}$	$\frac{w_i}{p_i^U}$	d_i^-	d_i^+	$\frac{w_i}{d_i^+}$	$\frac{w_i}{d_i^-}$	$\frac{w_i}{d_i^-}-\frac{w_i}{d_i^+}$	$\left(\frac{w_i}{d_i^-}-\frac{w_i}{d_i^+}\right):(p_i^U-p_i^L)$
1	40	50	400	10	8	9	10	40	$44\frac{4}{9}$	$4\frac{4}{9}$	$\frac{4}{9}$
2	60	90	540	9	6	6	8	67.5	90	22.5	0.75
3	40	80	200	5	2.5	4	5	40	50	10	0.25
4	60	60	240	4	4	4	2.5	–	–	–	–
5	30	40	120	4	3	4	2.5	–	–	–	–
6	40	320	160	4	0.5	2	2.5	64	80	16	$\frac{2}{35}$
7	40	80	80	2	1	1	0.5	–	–	–	–
8	60	60	30	0.5	0.5	0.5	0.5	60	60	0	1
9	80	80	40	0.5	0.5						
10	100	100	50	0.5	0.5						
11	30	30	120	4	4						
12	40	40	160	4	4						
13	50	50	200	4	4						

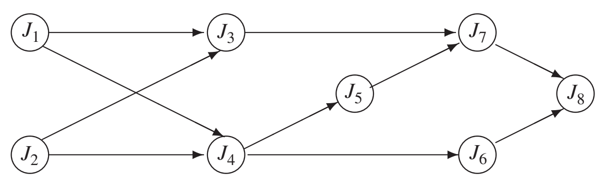

Fig. 1. Digraph $G=(\mathcal{J},\mathcal{A}^0)$ defining a unique minimal dominant set $S(T)$ for Example 1*.

reduced to an equivalent instance $1^*|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ with the set of jobs $\mathcal{J}^*$ of a smaller cardinality: $|\mathcal{J}^*| = n+m-\sum_{q=1}^{m}|\mathcal{J}_{r_q}| = 13+2-(4+3)=8$.

The input data for the instance $1^*|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ (we call this instance as Example 1*) are given in columns 1–4 of rows 1–8 in Table 1. We can consider Example 1* with the set of jobs $\mathcal{J}^* = \{J_1,J_2,\ldots,J_8\}$ instead of Example 1 with the set of jobs $\mathcal{J} = \{J_1,J_2,\ldots,J_{13}\} \supseteq \mathcal{J}^*$. For each pair of jobs $J_u \in \mathcal{J}^*$ and $J_v \in \mathcal{J}^*$, we check condition (3) of Theorem 2:

$$\frac{w_1}{p_1^U}=8\ge 5=\frac{w_3}{p_3^L},\quad \frac{w_1}{p_1^U}=8\ge 4=\frac{w_4}{p_4^L},\quad \frac{w_2}{p_2^U}=6\ge 5=\frac{w_3}{p_3^L},\quad \frac{w_2}{p_2^U}=6\ge 4=\frac{w_4}{p_4^L},\quad \frac{w_3}{p_3^U}=2.5\ge 2=\frac{w_7}{p_7^L},$$

$$\frac{w_4}{p_4^U}=4\ge 4=\frac{w_5}{p_5^L},\quad \frac{w_4}{p_4^U}=4\ge 4=\frac{w_6}{p_6^L},\quad \frac{w_5}{p_5^U}=3\ge 2=\frac{w_7}{p_7^L},\quad \frac{w_6}{p_6^U}=0.5\ge 0.5=\frac{w_8}{p_8^L},\quad \frac{w_7}{p_7^U}=1\ge 0.5=\frac{w_8}{p_8^L}.$$

Due to Theorem 2, the following relations hold: $J_1 \mapsto J_3$, $J_1 \mapsto J_4$, $J_2 \mapsto J_3$, $J_2 \mapsto J_4$, $J_3 \mapsto J_7$, $J_4 \mapsto J_5$, $J_4 \mapsto J_6$, $J_5 \mapsto J_7$, $J_6 \mapsto J_8$, $J_7 \mapsto J_8$. The unique minimal dominant set $S(T)$ for Example 1* is defined by the digraph $G=(\mathcal{J},\mathcal{A}^0)$ represented in Fig. 1.

5. A stability box

We define a *stability box* of permutation $\pi_k \in S$, which is a subset of the stability region [12,25]. Let $\mathcal{J}(k_i)=\{J_{k_1},\ldots,J_{k_{i-1}}\}$ and $\mathcal{J}[k_i]=\{J_{k_{i+1}},\ldots,J_{k_n}\}$. Let S_{k_i} denote a set of all permutations $(\overline{\mathcal{J}}(k_i),J_{k_i},\overline{\mathcal{J}}[k_i]) \in S$, where $\overline{\mathcal{J}}(k_i)(\overline{\mathcal{J}}[k_i])$ is a permutation of the jobs $\mathcal{J}(k_i)$ (jobs $\mathcal{J}[k_i]$, respectively). N_k denotes a subset of the set $N=\{1,2,\ldots,n\}$: $N_k \subseteq N$.

Definition 3. The maximal closed rectangular box $\mathcal{SB}(\pi_k,T)=\times_{k_i\in N_k}[l_{k_i},u_{k_i}]\subseteq T$ is called a stability box of permutation $\pi_k=(J_{k_1},J_{k_2},\ldots,J_{k_n})\in S$ with respect to T, if permutation $\pi_l=(J_{l_1},J_{l_2},\ldots,J_{l_n})\in S_{k_i}$ being optimal for the instance $1|p|\sum w_iC_i$ with a scenario $p=(p_1,p_2,\ldots,p_n)\in T$ remains optimal for the instance $1|p'|\sum w_iC_i$ with any scenario $p' \in \{\times_{j=1, j\ne i}^{n}[p_{k_j},p_{k_j}]\}\times[l_{k_i},u_{k_i}]$, $k_i \in N_k$. If there does not exist a scenario $p\in T$ such that permutation π_k is optimal for the instance $1|p|\sum w_iC_i$, then $\mathcal{SB}(\pi_k,T)=\emptyset$.

In [10–12,22,25], the stability ball and the stability region of an optimal semi-active schedule have been investigated for a job-shop scheduling problem. The definition of a stability region $\mathcal{K}(\pi_k,T)$ of a permutation $\pi_k \in S$ with respect to T is as follows:

$$\mathcal{K}(\pi_k,T)=\left\{p|p\in T, \sum_{J_i\in\mathcal{J}} w_iC_i(\pi_k,p)=\min_{\pi_l\in S}\left\{\sum_{J_i\in\mathcal{J}} w_iC_i(\pi_l,p)\right\}\right\}. \tag{12}$$

Since it is difficult to calculate a stability region, we adopt a stability box $\mathcal{SB}(\pi_k,T)\subseteq\mathcal{K}(\pi_k,T)$ as a substitute. Property 1 follows directly from the above definitions.

Property 1. *A stability box $\mathcal{SB}(\pi_k,T)$ and a stability region $\mathcal{K}(\pi_k,T)$ are empty, if and only if there is no a scenario $p\in T$ such that permutation π_k is optimal for the instance $1|p|\sum w_iC_i$.*

Non-empty sets $\mathcal{SB}(\pi_k,T)$ and $\mathcal{K}(\pi_k,T)$ may be characterized using the following claim.

Theorem 7. *There exists a scenario* $p \in T$ *such that permutation* $\pi_k = (J_{k_1} J_{k_2}, \ldots J_{k_n}) \in S$ *is optimal for the instance* $1|p|\sum w_i C_i$ *if and only if there is no a job* $J_{k_i}, i \in \{1,2,\ldots,n-1\}$, *that inequality*

$$\frac{w_{k_i}}{p^L_{k_i}} < \frac{w_{k_j}}{p^U_{k_j}} \tag{13}$$

holds for at least one job $J_{k_j}, j \in \{i+1, i+2, \ldots, n\}$.

Proof. *Sufficiency*: We assume that there is no a job J_{k_i}, $i \in \{1,2,\ldots,n-1\}$, such that inequality (13) holds. Thus, for each job J_{k_i}, $1 \le i < n$, the opposite inequality

$$\frac{w_{k_i}}{p^L_{k_i}} \ge \frac{w_{k_j}}{p^U_{k_j}} \tag{14}$$

must hold for each job $J_{k_j} j \in \{i+1, i+2, \ldots, n\}$. Next, we determine the components $p_{k_1}, p_{k_2}, \ldots, p_{k_n}$ of the desired scenario $p=(p_1, p_2, \ldots, p_n)$ via using an iterative procedure starting from determining p_{k_1}, then p_{k_2} and so on until p_{k_n} being determined.

At the first iteration, we set $p_{k_1} = p^L_{k_1}$. If the processing time $p^L_{k_1}$ turns out to be feasible for the job J_{k_j}, $1 < j \le u \le n$ (i.e., inclusion $p^L_{k_1} \in [p^L_{k_j}, p^U_{k_j}]$ holds), then we set $p_{k_j} = p^L_{k_1}$ for each $j \in \{2,3,\ldots,u-1\}$, provided that J_{k_u} is a job from the set $\mathcal{J}$ having the minimum index u with the processing time $p^L_{k_1}$ being infeasible for the job J_{k_u}. We set $p_{k_u} = p^L_{k_u}$.

If $u < n$, we repeat the above iteration using the index u instead of 1. Otherwise (if $u = n$), the desired scenario $p = (p_1, p_2, \ldots, p_n)$ is already determined.

Due to (14), inequalities (2) must hold. Due to Theorem 1, permutation $\pi_k = (J_{k_1} J_{k_2}, \ldots J_{k_n}) \in S$ is optimal for the instance $1|p|\sum w_i C_i$.

Necessity: Let there exist a job $J_{k_i}, 1 \le i < n$, such that inequality (13) holds for at least one job $J_{k_j} j \in \{i+1, i+2, \ldots, n\}$. From (13), it follows that for any processing times $p_{k_i} \in [p^L_{k_i}, p^U_{k_i}]$ and $p_{k_j} \in [p^L_{k_j}, p^U_{k_j}]$ inequality $w_{k_i}/p_{k_i} < w_{k_j}/p_{k_j}$ holds. Due to Theorem 1, there is no a scenario $p \in T$ such that a permutation of the form $(\ldots, p_{k_i}, \ldots, p_{k_j}, \ldots) \in S$ is optimal for the instance $1|p|\sum w_i C_i$. In particular, permutation π_k is not optimal for the instance $1|p|\sum w_i C_i$ with any scenario $p \in T$. □

Theorem 7 imply the following property of a stability box and region.

Property 2. *A stability box* $\mathcal{SB}(\pi_k, T)$ *and a stability region* $\mathcal{K}(\pi_k, T)$ *are empty if and only if there exists job* J_{k_i}, $i \in \{1,2,\ldots,n-1\}$, *such that inequality* (13) *holds for at least one job* $J_{k_j} j \in \{i+1, i+2, \ldots, n\}$.

Definitions 3 and (12) imply the following claim.

Property 3. *If there exists exactly one scenario* $p \in T$ *such that permutation* $\pi_k \in S$ *is optimal for the instance* $1|p|\sum w_i C_i$, *then* $\mathcal{SB}(\pi_k, T) = \{p\} = \mathcal{K}(\pi_k, T)$.

Another extreme case for a non-empty stability box (region) is characterized as follows.

Property 4. $\mathcal{SB}(\pi_k, T) = T = \mathcal{K}(\pi_k, T)$ *if and only if inequalities* (4) *hold.*

Proof. Inequalities (4) are necessary and sufficient for the existence of a dominant singleton $S(T) = \{\pi_k\} = \{(J_{k_1} J_{k_2}, \ldots J_{k_n})\}$ (Theorem 3). For any scenario $p \in T$, the permutation π_k is optimal for the instance $1|p|\sum w_i C_i$ if and only if inequalities (4) hold. From (12), it follows that inequalities (4) hold if and only if $\mathcal{K}(\pi_k, T) = T$. Due to inclusion $\mathcal{K}(\pi_k, T) \subseteq T$, equality $\mathcal{K}(\pi_k, T) = T$ implies equality $\mathcal{SB}(\pi_k, T) = T$ and vice versa. □

Next, we prove Theorem 8 which allows us to derive an $O(n \log n)$-algorithm for finding a stability box $\mathcal{SB}(\pi_k, T)$ for a permutation $\pi_k = (J_{k_1}, \ldots J_{k_{i-1}} J_{k_i} J_{k_{i+1}}, \ldots J_{k_n}) \in S$.

Proof of Theorem 8. Via testing inequalities (13) for each job $J_{k_i}, 1 \le i < n$, we can convince whether the stability box $\mathcal{SB}(\pi_k, T)$ is empty or not (Property 2). If there exists a job $J_{k_i}, i \in \{1,2,\ldots,n-1\}$, such that inequality (13) holds for a job $J_{k_j} j \in \{i+1, i+2, \ldots, n\}$, then $\mathcal{SB}(\pi_k, T) = \emptyset$ and the proof of Theorem 8 is done.

Let $\mathcal{SB}(\pi_k, T) \ne \emptyset$. Due to the additivity of the objective function $\gamma = \sum w_i C_i$, in order to find a rectangular box $\mathcal{SB}(\pi_k, T)$, it is sufficient to calculate the maximal range of a *possible variation* of the processing time p_{k_i}, $i \in \{1,2,\ldots,n\}$, which definitely preserves the optimality (if any) of the permutation π_k. Hereafter, a *possible variation* $[l_{k_i}, u_{k_i}]$ of the processing time p_{k_i} (a *possible variation* $[L_{k_i}, U_{k_i}]$ of the weight-to-process ratio) for the job J_{k_i} means the following. If π_k is an optimal permutation for the instance $1|p|\sum w_i C_i$ with $p = (p_1, p_2, \ldots, p_n) \in T$, then permutation π_k remains optimal for any instance $1|p'|\sum w_i C_i$ with $p' = (p'_1, p'_2, \ldots, p'_n) \in T$, where $p'_t = p_t$ for each $t \ne k_i$ and $p_{k_i} \in [l_{k_i}, u_{k_i}]$ (respectively, $w_{k_i}/p_{k_i} \in [L_{k_i}, U_{k_i}]$). We can compare the left bound $w_{k_i}/p^U_{k_i}$ of the segment $[w_{k_i}/p^U_{k_i}, w_{k_i}/p^L_{k_i}]$ (where $i \in \{1,2,\ldots,n-1\}$) with the right bounds $w_{k_j}/p^L_{k_j}$ of the segments $[w_{k_j}/p^U_{k_j}, w_{k_j}/p^L_{k_j}]$ for all the jobs $J_{k_j} j \in \{i+1, i+2, \ldots, n\}$. Due to Lemmas 1 and 2, we obtain the lower bound $d^-_{k_i}$ of a *possible variation* of the weight-to-process ratio as follows:

$$d^-_{k_i} = \max\left\{ \frac{w_{k_i}}{p^U_{k_i}}, \max_{i<j\le n} \left\{ \frac{w_{k_j}}{p^L_{k_j}} \right\} \right\}. \tag{15}$$

The lower bound $d^-_{k_n}$ is equal to $w_{k_n}/p^U_{k_n}$. Similarly, we compare the right bound $w_{k_i}/p^L_{k_i}$ of the segment $[w_{k_i}/p^U_{k_i}, w_{k_i}/p^L_{k_i}]$, where $i \in \{2,3,\ldots,n\}$, with the left bounds w_{kj}/p^U_{kj} of the segments $[w_{k_j}/p^U_{k_j}, w_{k_j}/p^L_{k_j}]$ for all the jobs $J_{k_j} j \in \{1,2,\ldots,i-1\}$. Due to Lemmas 1 and 2, we obtain the upper bound $d^+_{k_i}$ of a *possible variation* of the weight-to-process ratio as follows:

$$d^+_{k_i} = \min\left\{\frac{w_{k_i}}{p^L_{k_i}}, \min_{1 \le j < i}\left\{\frac{w_{k_j}}{p^U_{k_j}}\right\}\right\}. \tag{16}$$

The upper bound $d^+_{k_1}$ is equal to $w_{k_1}/p^L_{k_1}$. For $d^-_{k_i}$ and $d^+_{k_i}$, either case (j) or case (jj) occurs.

Case (j): $d^+_{k_i} \ge d^-_{k_i}$.

Due to Lemmas 1 and 2, the maximal range of a *possible variation* of the weight-to-process ratio of job J_{k_i} is equal to $[d^-_{k_i}, d^+_{k_i}]$. Therefore, the maximal range of a *possible variation* of the processing time p_{k_i} is equal to $[w_{k_i}/d^+_{k_i}, w_{k_i}/d^-_{k_i}]$.

Case (jj): $d^+_{k_i} < d^-_{k_i}$.

Due to Lemmas 1 and 2, the position *i* of the job J_{k_i} in permutation π_k may imply the non-optimality of permutation π_k: for each fixed processing time $p_{k_i} \in [p^L_{k_i}, p^U_{k_i}]$ there exists a scenario $p = (\dots, p_{k_i}, \dots) \in T$ such that a permutation of the form $(\pi(J_{k_1}, \dots J_{k_{i-1}}) J_{k_i}, \pi(J_{k_{i+1}}, \dots J_{k_n})) \in S$ is not optimal for the instance $1|p|\sum w_i C_i$. Thus, the optimality of a position *i* of the job J_{k_i} in a permutation $(\pi(J_{k_1}, \dots J_{k_{i-1}}) J_{k_i}, \pi(J_{k_{i+1}}, \dots J_{k_n}))$ cannot be guaranteed only by the processing time $p_{k_i} \in [p^L_{k_i}, p^U_{k_i}]$ (actually, the optimality of a position *i* of the job J_{k_i} depends on the processing times of the other jobs $J_l \in \mathcal{J}, l \ne k_i$). Consequently, the range for the *possible variation* of the processing time p_{k_i} is empty.

Let the maximal range $[w_{k_i}/d^+_{k_i}, w_{k_i}/d^-_{k_i}]$ of a *possible variation* of the processing time p_{k_i} be calculated for each job $J_{k_i}, i \in \{1,2,\dots,n\}$, and let there exist at least one index $i \in \{1,2,\dots,n\}$ such that case (j) occurs. It is easy to convince that a stability box for a permutation π_k with respect to *T* is determined by the following Cartesian product:

$$\mathcal{SB}(\pi_k, T) = \times_{d^-_i \le d^+_i}\left[\frac{w_{k_i}}{d^+_{k_i}}, \frac{w_{k_i}}{d^-_{k_i}}\right]. \tag{17}$$

Indeed, equalities (15) and (16) imply inclusion $\times_{d^-_i \le d^+_i}[w_{k_i}/d^+_{k_i}, w_{k_i}/d^-_{k_i}] \times \{\times_{d^-_j > d^+_j}[p_{k_j}, p_{k_j}]\} \subseteq T$. Moreover, for any scenario $p' \in \{\times^n_{j=1, j \ne i}[p_{k_j}, p_{k_j}]\} \times [w_{k_i}/d^+_{k_i}, w_{k_i}/d^-_{k_i}]$ with any $k_i \in N_k = \{i \mid i \in N, d^-_i \le d^+_i\}$, inequalities (2) hold. Due to Theorem 1, Lemmas 1 and 2, permutation π_k is optimal for the instance $1|p'|\sum w_i C_i$. To convince that the Cartesian product $\times_{d^-_i \le d^+_i}[w_{k_i}/d^+_{k_i}, w_{k_i}/d^-_{k_i}]$ is a closed rectangular box, one needs to prove that for any small real $\varepsilon > 0$, there exists a scenario $p^\varepsilon = (p^\varepsilon_1, p^\varepsilon_2, \dots, p^\varepsilon_n) \in T$ with $p^\varepsilon_v \in [w_v/d^+_v - \varepsilon, w_v/d^-_v + \varepsilon]$ such that permutation π_k is not optimal for the instance $1|p^\varepsilon|\sum w_i C_i$. The latter claim follows from Theorem 1, Lemma 1 and definitions (15) and (16). Thus, the following theorem has been proven. □

Theorem 8. *If there is no a job* $J_{k_i}, i \in \{1,2,\dots,n-1\}$, *in permutation* $\pi_k = (J_{k_1}, J_{k_2}, \dots J_{k_n}) \in S$ *such that inequality* (13) *holds for at least one job* $J_{k_j}, j \in \{i+1, i+2, \dots, n\}$, *then a stability box* $\mathcal{SB}(\pi_k, T)$ *is calculated in* (17). *Otherwise,* $\mathcal{SB}(\pi_k, T) = \emptyset$.

The following claim follows from the above proof of Theorem 8.

Property 5. *If* $\mathcal{SB}(\pi_k, T) \ne \emptyset$, *then the singleton* $\{\pi_k\}$ *is the minimal dominant set for problem* $1|p^L_i \le p_i \le p^U_i|\sum w_i C_i$ *with the scenario set* $T^0 = \mathcal{SB}(\pi_k, T)$.

Property 6. *If* $\pi_k \in S(T)$, *then* $\mathcal{SB}(\pi_k, T) \ne \emptyset$ *and* $\mathcal{K}(\pi_k, T) \ne \emptyset$.

Proof. Due to condition (b) of Definition 1, the set $S(T) \setminus \{\pi_k\}$ cannot be a minimal dominant set for problem $1|p^L_i \le p_i \le p^U_i|\sum w_i C_i$. Therefore, there exists a scenario $p \in T$ such that permutation π_k is optimal for the instance $1|p|\sum w_i C_i$ (Definition 1). Thus, Lemma 1 implies $\mathcal{SB}(\pi_k, T) \ne \emptyset$ and $\mathcal{K}(\pi_k, T) \ne \emptyset$. □

Returning to Example 1*, we calculate a stability box for the permutation $\pi_1 = (J_1, J_2, \dots J_8)$. The intervals of the weight-to-process ratios are presented in a coordinate system in Fig. 2. The abscissa axis is used for indicating the weight-to-process ratios and the ordinate axis for the jobs from the set $\mathcal{J}^*$. The permutation π_1 does not violate the strong order relation on the set $\mathcal{J}^*$ defined by the digraph

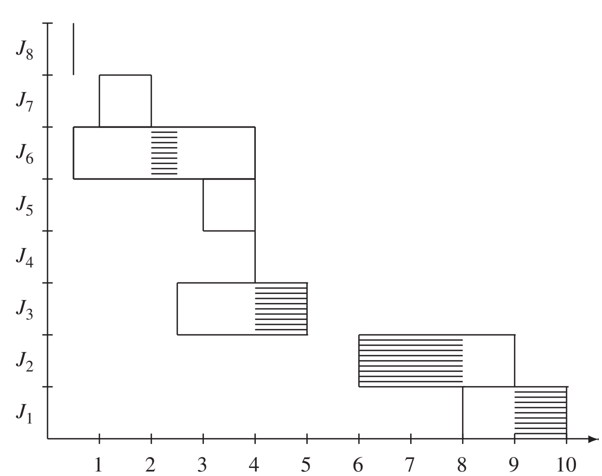

Fig. 2. *Possible variations* of the weight-to-process ratios in the permutation π_1 are dashed.

$(\mathcal{J}^*,\mathcal{A}^0)$ (Fig. 1). Hence, permutation π_1 belongs to the minimal dominant set $S(T)$, which is unique for Example 1*. The stability box $\mathcal{SB}(\pi_1,T)$ is not empty (Theorem 6). For each job $J_i \in \mathcal{J}^* \setminus \{J_8\}$, we calculate the value d_i^- using (15):

$$
\begin{aligned}
&d_1^- = \max\{w_1/p_1^U, \max_{1<j\le 8}\{w_j/p_j^L\}\} = 9, d_2^- = \max\{w_2/p_2^U, \max_{2<j\le 8}\{w_j/p_j^L\}\} = 6,\\
&d_3^- = \max\{w_3/p_3^U, \max_{3<j\le 8}\{w_j/p_j^L\}\} = 4, d_4^- = \max\{w_4/p_4^U, \max_{4<j\le 8}\{w_j/p_j^L\}\} = 4,\\
&d_5^- = \max\{w_5/p_5^U, \max_{5<j\le 8}\{w_j/p_j^L\}\}4, d_6^- = \max\{w_6/p_6^U, \max_{6<j\le 8}\{w_j/p_j^L\}\} = 2,\\
&d_7^- = \max\{w_7/p_7^U, \max_{7<j\le 8}\{w_j/p_j^L\}\} = 1.
\end{aligned}
$$

We calculate $d_8^- = w_8/p_8^U = 0.5$ and $d_1^+ = w_1/p_1^L = 10$. For each job $J_i \in \mathcal{J}^* \setminus \{J_1\}$, we calculate the value d_i^+ using (16):

$$
\begin{aligned}
&d_2^+ = \min\{w_2/p_2^L, \min_{1\le j<2}\{w_j/p_j^U\}\} = 8,\ \ d_3^+ = \min\{w_3/p_3^L, \min_{1\le j<3}\{w_j/p_j^U\}\} = 5,\\
&d_4^+ = \min\{w_4/p_4^L, \min_{1\le j<4}\{w_j/p_j^U\}\} = 2.5,\ \ d_5^+ = \min\{w_5/p_5^L, \min_{1\le j<5}\{w_j/p_j^U\}\} = 2.5,\\
&d_6^+ = \min\{w_6/p_6^L, \min_{1\le j<6}\{w_j/p_j^U\}\} = 2.5,\ \ d_7^+ = \min\{w_7/p_7^L, \min_{1\le j<7}\{w_j/p_j^U\}\} = 0.5,\\
&d_8^+ = \min\{w_8/p_8^L, \min_{1\le j<8}\{w_j/p_j^U\}\} = 0.5.
\end{aligned}
$$

For each index $i \in \{1,2,3,6,8\} = N_1$, inequality $d_{k_i}^+ \ge d_{k_i}^-$ holds (columns 7 and 8 in Table 1). We calculate the maximal ranges of the *possible variations* for the jobs $J_i \in \{J_1,J_2,J_3,J_6,J_8\}$: $[w_1/d_1^+, w_1/d_1^-] = [40,\frac{400}{9}]$; $[w_2/d_2^+, w_2/d_2^-] = [67.5,90]$; $[w_3/d_3^+, w_3/d_3^-] = [40,50]$; $[w_6/d_6^+, w_6/d_6^-] = [64,80]$; $[w_8/d_8^+, w_8/d_8^-] = [60,60]$ (columns 9 and 10). We obtain inequality $d_{k_j}^+ < d_{k_j}^-$ for each index $j \in \{4,5,7\}$ (columns 7 and 8). The range of the *possible variation* of each job $J_i \in \{J_4,J_5,J_7\}$ is empty. Due to Theorem 8, the stability box of permutation $\pi_1 = (J_1,J_2,\dots,J_8)$ with respect to T is defined as follows: $\mathcal{SB}(\pi_1,T) = [w_1/d_1^+, w_1/d_1^-] \times [w_2/d_2^+, w_2/d_2^-] \times [w_3/d_3^+, w_3/d_3^-] \times [w_6/d_6^+, w_6/d_6^-] \times [w_8/d_8^+, w_8/d_8^-] = [40,\frac{400}{9}] \times [67.5,90] \times [40,50] \times [64,80] \times [60,60]$. The ranges of the *possible variations* of the weight-to-process ratios for the jobs J_1,J_2,J_3,J_6 and J_8 are dashed in Fig. 2.

Hereafter, the relative volume of a stability box is defined as the product of the fractions

$$
\left(\frac{w_i}{d_i^-} - \frac{w_i}{d_i^+}\right) : (p_i^U - p_i^L) \tag{18}
$$

for the jobs $J_i \in \mathcal{J}^*$ having strictly positive ranges of their *possible variations* (for such a job $J_i \in \mathcal{J}^*$, inequality $d_i^- < d_i^+$ must hold). The calculation of the relative volume of a stability box for permutation $\pi_1 = (J_1,J_2,\dots J_8)$ is given in columns 7–12 in Table 1. For job J_8, column 12 contains 1 since $p_8^U = p_8^L$ and $d_8^- = 0.5 \le 0.5 = d_8^+$. Each job J_i, $i \in \{4,5,7\}$, has an empty range of their *possible variations* since $d_i^- > d_i^+$ (columns 7, 8 and 12).

Let L_{r_i} denote the element $w_{k_i}/p_{k_i}^L$ in the ordered set (list) $L = (w_{k_1}/p_{k_1}^L, w_{k_2}/p_{k_2}^L, \dots, w_{k_n}/p_{k_n}^L) = (L_1, L_2, \dots, L_n)$. Next, we describe the formal algorithm for calculating the stability box $\mathcal{SB}(\pi_k,T)$.

Algorithm STABOX.

Input: Segments $[p_i^L, p_i^U]$, weights w_i, $J_i \in \mathcal{J}$; permutation $\pi_k = (J_{k_1}, \dots, J_{k_n}) \in S$.
Output: Stability box $\mathcal{SB}(\pi_k,T)$, dimension $n_k = |N_k|$ of the stability box.
*Step*1: Construct the list $L = (L_1,\dots,L_n)$ of the fractions $w_{k_i}/p_{k_i}^L$ in non-increasing order.
*Step*2: Construct the list $U = (U_1,\dots,U_n)$ of the fractions $w_{k_i}/p_{k_i}^U$ in non-decreasing order.
*Step*3: Construct the list $U^0 = (U_1^0,\dots,U_n^0)$ of the fractions $w_{k_i}/p_{k_i}^U$ in non-increasing order.
*Step*4: **FOR** $i=1$ to $i=n$ **DO** set $U^0 := U^0 \setminus \{U_{t_i}^0 = w_{k_{t_i}}/p_{k_{t_i}}^U\}$;
  IF $w_{k_i}/p_{k_i}^L < U_1^0$ **THEN** $\mathcal{SB}(\pi_k,T) = \emptyset$, $n_k = 0$ **STOP**
  END FOR Set $n_k = 0$.
*Step*5: **FOR** $i=1$ to $i=n-1$ **DO** set $L := L \setminus \{L_{r_i} = w_{k_{r_i}}/p_{k_{r_i}}^L\}$;
  calculate $d_{k_i}^- = \max\{w_{k_i}/p_{k_i}^U, L_1\}$. **END FOR**
*Step*6: **FOR** $i=n$ to $i=2$ **DO** set $U := U \setminus \{U_{m_i} = w_{k_{m_i}}/p_{k_{m_i}}^U\}$;
  calculate $d_{k_i}^+ = \max\{w_{k_i}/p_{k_i}^U, U_1\}$. **END FOR**
*Step*7: Set $d_{k_n}^- = w_{k_n}/p_{k_n}^U$, $d_{k_1}^+ = w_{k_1}/p_{k_1}^L$.
*Step*8: **FOR** $J_i \in \mathcal{J}$ **DO**
  IF $d_{k_i}^+ < d_{k_i}^-$ **THEN** processing time p_{k_i} has to be fixed in $\mathcal{SB}(\pi_k,T)$; **ELSE**
  $[w_{k_i}/d_{k_i}^+, w_{k_i}/d_{k_i}^-]$ is the maximal range of a *possible variation* of p_{k_i}. Set $n_k := n_k + 1$.
  END FOR
*Step*9: $\mathcal{SB}(\pi_k,T) := \times_{d_i^- \le d_i^+}[w_{k_i}/d_{k_i}^+, w_{k_i}/d_{k_i}^-]$**STOP.**

The weight-to-process ratios are non-increasingly ordered in the list L (step 1). Therefore, after setting $L := L \setminus \{L_{r_v}\}$ for each $v = 1,2,\dots,i$ (step 5), we obtain $L_1 = \max_{i<j\le n}\{w_{k_j}/p_{k_j}^L\}$. Thus, the value $d_{k_i}^-$ calculated in step 5 satisfies (15). Arguing analogously, we can convince that the value $d_{k_i}^-$ calculated in step 6 satisfies (16). Inequality $w_{k_i}/p_{k_i}^L < U_1^0$ tested in step 4 is equivalent to inequality (13) used as a criterion for testing whether a stability box is empty (Property 2). Each of the steps 1–3 takes $O(n \log n)$ time and each of the steps 4–6, and 8 takes $O(n)$ time. Thus, the complexity of Algorithm STABOX is $O(n \log n)$.

6. A job permutation with the largest volume of a stability box

Intuitively, a job permutation with a larger volume of the stability box seems better than one with a smaller volume. Next, we develop an $O(n^2)$-algorithm for finding a permutation $\pi_t \in S$ with the largest volume of a stability box $\mathcal{SB}(\pi_k,T) = \times_{k_i \in N_k}[l_{k_i},u_{k_i}]$. Due to Definition 3, for any job $J_i \in \mathcal{J}$, an open interval $(w_i/u_i,w_i/l_i)$ should not intersect with a closed interval $[w_v/p_v^U,w_v/p_v^L]$ for any job $J_v \in \mathcal{J}$, $v \neq i$:

$$\left(\frac{w_i}{u_i},\frac{w_i}{l_i}\right) \bigcap \left[\frac{w_v}{p_v^U},\frac{w_v}{p_v^L}\right] = \emptyset, \quad v \neq i. \tag{19}$$

Using (19), we can show how to define a position $x_i \in \{1,2,\ldots,n\}$ of a job $J_i \in \mathcal{J}$ in the permutation $\pi_t = (J_{t_1},\ldots J_{t_{x_i-1}} J_i J_{t_{x_i+1}},\ldots J_{t_n}) \in S$ having the largest volume of a stability box $\mathcal{SB}(\pi_t,T)$. To this end, it is sufficient to define the relative order of a job $J_i \in \mathcal{J}$ with respect to job $J_v \in \mathcal{J}$ for any $v \neq i$.

If the open interval $(w_i/p_i^U,w_i/p_i^L)$ does not intersect with the closed interval $[w_v/p_v^U,w_v/p_v^L]$, $J_v \in \mathcal{J}$:

$$(w_i/p_i^U,w_i/p_i^L) \bigcap \left[\frac{w_v}{p_v^U},\frac{w_v}{p_v^L}\right] = \emptyset, \quad v \neq i, \tag{20}$$

then the order of the jobs J_i and J_v in the desired permutation π_t is clearly defined by the minimal dominant set $S(T)$ (or similarly, by the digraph $(\mathcal{J},\mathcal{A})$). Namely: if $J_i \mapsto J_v$, then job J_i has to proceed job J_v in the permutation π_t, otherwise (if $J_v \mapsto J_i$), job J_v has to proceed job J_i. Due to Proposition 1 and Theorems 6, a minimal dominant set is assumed to be uniquely determined and the digraph $(\mathcal{J},\mathcal{A})$ is circuit-free for the problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ under consideration. Otherwise, an instance $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ has to be substituted by the instance $1^*|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ (see Proposition 1).

Let $(w_i/p_i^U,w_i/p_i^L)\bigcap[w_v/p_v^U,w_v/p_v^L] \neq \emptyset$ (i.e., equality (20) does not hold). There are four possible cases (I)–(IV) for an intersection of the intervals $(w_i/p_i^U,w_i/p_i^L)$ and $[w_v/p_v^U,w_v/p_v^L]$. An order of the jobs J_i and J_v in the desired permutation π_t may be defined in the cases (I)–(III) as follows.

Case (I): $w_v/p_v^U < w_i/p_i^U$ and $w_v/p_v^L \le w_i/p_i^L$.

In case (I), job J_v has to proceed job J_i in the desired permutation π_t. (Otherwise, if job J_i proceeds job J_v, then the *possible variation* $[l_i,u_i]$ of the processing time p_i is empty.)

Case (II): $w_v/p_v^U \ge w_i/p_i^U$ and $w_v/p_v^L > w_i/p_i^L$.

In case (II), job J_i has to proceed job J_v in the desired permutation π_t. (Otherwise, if job J_v proceeds job J_i, then the *possible variation* $[l_i,u_i]$ of the processing time p_i is empty.)

Case (III): $w_v/p_v^U \le w_i/p_i^U$ and $w_v/p_v^L \ge w_i/p_i^L$.

In case (III), the *possible variation* $[l_i,u_i]$ of the processing time p_i is empty for both orders of the jobs J_i and J_v: either job J_i is located before job J_v or job J_v before job J_i. Therefore, any of these two orders may be realized in the desired permutation π_t.

Case (IV): $w_v/p_v^U > w_i/p_i^U$ and $w_v/p_v^L < w_i/p_i^L$.

In case (IV), the *possible variation* $[l_i, u_i]$ of the processing time p_i may have a positive length both if job J_i is located before job J_v and if job J_v is located before job J_i. Moreover, these two lengths may be different.

Let $\mathcal{J}(J_i)$ denote the set of all jobs $J_v \in \mathcal{J}$ for which either equality (20) holds, or one of the cases (I), (II) or (III) occurs. As it is shown, the order of the job J_i with respect to job $J_v \in \mathcal{J}(J_i)$ is well defined in the desired permutation π_t. However, we cannot define the order of the jobs J_i and J_v in the permutation π_t if $J_v \in \mathcal{J}\setminus\mathcal{J}(J_i)$ (i.e., if case (IV) occurs). Let $\mathcal{J}_{J_i}^-$ denote the subset of the set $\mathcal{J}(J_i) \subset \mathcal{J}$ including all jobs $J_v \in \mathcal{J}(J_i)$ located before job J_i in the permutation π_t. The set $\mathcal{J}_{J_i}^+$ denotes the subset of the set $\mathcal{J}(J_i) = \mathcal{J}_{J_i}^- \cup \mathcal{J}_{J_i}^+ \subset \mathcal{J}$ including all jobs $J_v \in \mathcal{J}(J_i)$ located after job J_i in the permutation π_t.

If set $\mathcal{J}\setminus\{\mathcal{J}(J_i)\cup\{J_i\}\}$ is not empty (i.e., there exists a job J_v satisfying case (IV)), we order all the jobs $J_v \in \mathcal{J}\setminus\{\mathcal{J}(J_i)\cup\{J_i\}\} = \{J_v \in \mathcal{J} | w_v/p_v^U > w_i/p_i^U, w_v/p_v^L < w_i/p_i^L\}$ in non-decreasing of the fractions w_v/p_v^M (where $p_v^M = p_v^U - p_v^L/2$) as follows: $(J_{j_1} J_{j_2},\ldots J_{j_{c_i}})$, $c_i = |\{J_v \in \mathcal{J} | w_v/p_v^U > w_i/p_i^U, w_v/p_v^L < w_i/p_i^L\}|$. We treat c_i+1 permutations $(J_{j_1},\ldots J_{j_{t-1}} J_i J_{j_{t+1}},\ldots J_{j_{c_i}})$ for each index $t \in \{1,2,\ldots,c_i+1\}$ and choose from them a permutation $(J_{j_1},\ldots J_{j_{t^*-1}} J_i J_{j_{t^*+1}},\ldots J_{j_{c_i}})$ with the largest *possible variation* $[l_i,u_i]$ of the processing time p_i. A position x_i of the job J_i in the desired permutation $\pi_t = (J_{t_1},\ldots J_{t_{x_i-1}} J_i J_{t_{x_i+1}},\ldots J_{t_n}) \in S$ with the largest volume of a stability box $\mathcal{SB}(\pi_t,T)$ is calculated as follows:

$$x_i = t^* + |\mathcal{J}_{J_i}^-|. \tag{21}$$

Indeed, the *possible variation* $[l_v, u_v]$ for the job $J_v \in \mathcal{J}\setminus\{\mathcal{J}(J_i)\cup\{J_i\}\}$ is empty for any permutation from the set S since case (III) occurs (provided that job J_i is substituted by job J_v and vice versa). Thus, in the desired permutation $\pi_t \in S$, the order of the jobs $\mathcal{J}\setminus\mathcal{J}(J_i)$ has to maximize only the length of the *possible variation* $[l_i, u_i]$.

Furthermore, assigning a job J_i to a position m in a permutation $\pi_k = (J_{k_1} J_{k_2},\ldots J_{k_n}) \in S$ partitions the set of jobs $\mathcal{J}\setminus\{J_i\}$ into two subsets with respect to the permutation $\pi_k = (J_{k_1},\ldots J_{k_{m-1}} J_i J_{k_{m+1}} \ldots J_{k_n}) \in S$. One set is the set of jobs $\{J_{k_1},\ldots J_{k_{m-1}}\}$ located before job J_i, and the other set is the set of jobs $\{J_{k_{m+1}} \ldots J_{k_n}\}$ located after job J_i. The maximal *possible variation* of the weight-to-process ratio for job $J_{k_m} = J_i$ may be calculated using equalities (15) and (16). It is clear that the result of this calculation does not depend on the order of the jobs within the set $\{J_{k_1},\ldots J_{k_{m-1}}\}$ and within the set $\{J_{k_{m+1}} \ldots J_{k_n}\}$.

Next, we describe the formal algorithm for calculating the permutation $\pi_t \in S$ with the largest volume of a stability box $\mathcal{SB}(\pi_t,T)$.

Algorithm MAXSTABOX.

Input: Segments $[p_i^L, p_i^U]$, weights w_i, $J_i \in \mathcal{J}$.
Output: Permutation $\pi_t \in S$ with the largest volume of a stability box $\mathcal{SB}(\pi_t,T)$.
*Step*1: Construct the list $\mathcal{M} = (J_{k_1},\ldots J_{k_n})$ of the jobs $\mathcal{J}$ in non-increasing order of the fractions $w_{k_l}/p_{k_l}^M$ with $p_{k_l}^M = p_{k_l}^U - p_{k_l}^L/2$, $l \in \{1,\ldots,n\}$. Set $d=1$, $i=k_d$.

*Step*2: Construct the digraph $(\mathcal{J},\mathcal{A})$.

*Step*3: **FOR** $j=d+1$ to $j=n$ **DO** set $v=k_j$. **IF** the open interval $(w_i/p_i^U,w_i/p_i^L)$ does not intersect with the closed interval $[w_v/p_v^U,w_v/p_v^L]$ **THEN**

IF $J_i \mapsto J_v$ **THEN** job J_i proceeds job J_v in the permutation π_t;
ELSE job J_v proceeds job J_i in the permutation π_t;

*Step*4: **ELSE IF** there is no job $J_v \in \mathcal{J}$ with $w_v/p_v^U > w_i/p_i^U$, $w_v/p_v^L < w_i/p_i^L$

THEN define the order of the jobs J_i and J_v in the permutation π_t according to the above rule for the corresponding case (I), (II) or (III) occurring for the job J_i. Namely:

IF $w_v/p_v^U < w_i/p_i^U$, $w_v/p_v^L \le w_i/p_i^L$ **THEN** job J_v proceeds job J_i in $\pi_t \in S$;
IF $w_v/p_v^U \ge w_i/p_i^U$, $w_v/p_v^L > w_i/p_i^L$ **THEN** job J_i proceeds job J_v in $\pi_t \in S$;
IF $w_v/p_v^U \le w_i/p_i^U$, $w_v/p_v^L \ge w_i/p_i^L$ **THEN** job J_i proceeds job J_v in $\pi_t \in S$.

GOTO step 6.

*Step*5: **ELSE** Let $(J_{h_1},\dots J_{h_{c_i}})$ be the list of jobs $\mathcal{J}(J_i)$ ordered as in the list $\mathcal{M}$.

Via treating c_i+1 permutations $(J_{h_1},\dots J_{h_{t-1}} J_i J_{h_{t+1}},\dots J_{h_{c_i}})$ for each $t \in \{1,2,\dots,c_i+1\}$, choose permutation $(J_{h_1},\dots J_{h_{t^*1}} J_i J_{h_{t^*+1}},\dots J_{h_{c_i}})$ having the largest *possible variation* $[l_i,u_i]$ of the processing time p_i.

Calculate the position $x_i = t^* + |\mathcal{J}_{J_i}^-|$ of job J_i in the desired permutation $\pi_t = (J_{t_1},\dots J_{t_{x_i-1}} J_i J_{t_{x_i+1}},\dots J_{t_n}) \in S$ **GOTO** step 7.

*Step*6: Calculate the position $x_i = |\mathcal{J}_{J_i}^-|+1$ of job J_i in the desired permutation $\pi_t = (J_{t_1},\dots J_{t_{x_i-1}} J_i J_{t_{x_i+1}},\dots J_{t_n}) \in S$.

END FOR

*Step*7: Set $d := d+1$, $i=k_d$. **IF** $d<n$ **GOTO** step 3.

ELSE Construct the permutation $\pi_t \in S$ via setting every job $J_i \in \mathcal{J}$ on the position x_i calculated either in step 5 or in step 6 **STOP.**

Step 1 takes $O(n \log n)$ time. Step 2 takes $O(n^2)$ time. In step 5, the order of the jobs in the list $(J_{h_1},\dots J_{h_{c_i}})$ is the same as in the list $\mathcal{M}$ already constructed in step 1.

Treating c_i+1 permutations $(J_{h_1},\dots J_{h_{t-1}} J_i J_{h_{t+1}},\dots J_{h_{c_i}})$ for each $t \in \{1,2,\dots,c_i+1\}$ takes $O(c_i)$ time in step 5. If $J_v \in \mathcal{J}(J_i)$, then job J_v does not belong to the set $\mathcal{J}(J_k)$ for any job $J_k \in \mathcal{J}\setminus\{J_i\}$. Therefore, inequality $\sum_{i=1}^n c_i \le n-1$ holds (where equality $c_k=0$ may hold for some jobs $J_k \in \mathcal{J}$). Thus, the complexity $O(n(n+\sum_{i=1}^n c_i))$ of the steps 3–6 is $O(n^2)$ and so the whole Algorithm MAXSTABOX runs in $O(n^2)$ time as well.

7. Computational results

There might be several permutations with the largest relative volume of a stability box in the set $S(T)$, e.g., if several consecutive jobs in a permutation π_k has zero *possible variations* of their weight-to-process ratios. We break ties in ordering such jobs by adopting one of the following heuristics. The lower-point heuristic generates an optimal permutation $\pi_l \in S$ for the instance $1|p^L|\sum w_i C_i$, where $p^L=(p_1^L,p_2^L,\dots,p_n^L)$. The upper-point heuristic generates an optimal permutation $\pi_u \in S$ for the instance $1|p^U|\sum w_i C_i$ with $p^U=(p_1^U,p_2^U,\dots,p_n^U)$. The mid-point heuristic generates an optimal permutation $\pi_m \in S$ for the instance $1|p^M|\sum w_i C_i$ where $p^M=((p_1^U-p_1^L)/2, (p_2^U-p_2^L)/2,\dots,(p_n^U-p_n^L)/2)$. Algorithm MAXSTABOX combined with the lower-point heuristic, upper-point heuristic and mid-point heuristic is called Algorithm SL, Algorithm SU and Algorithm SM, respectively. These algorithms were coded in C++ and tested on a Laptop with AMD Turien (tm) 64×2 Mobile Technology TL-52 1.61 GHz 1,00 GB RAM. Tables 2 and 3 represent the computational results for randomly generated instances of the problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$ with $n \in \{10,20,\dots,100\}$ and those with $n \in \{200,300,\dots,1000\}$. We solved (either exactly or approximately) more than 300 series of randomly generated instances. Each series contains 100 instances with the same combination of number n of jobs and the maximal possible error δ of the random processing times. Numbers n is given in column 1 in Tables 2 and 3.

An integer lower bound p_i^L and an integer upper bound p_i^U of the values $p_i \in R_+^1$ of the job processing times, $p_i \in [p_i^L,p_i^U]$, have been generated in the following way. An integer center C of a segment $[p_i^L, p_i^U]$ was generated using the uniform distribution in the range $[L,U]: L \le C \le U$. The lower bound p_i^L was defined as $p_i^L = C\cdot(1-\delta/100)$, the upper bound p_i^U as $p_i^U = C\cdot(1+\delta/100)$. A maximum possible error of the random processing time (in percentages) is equal to $\delta\%$ given in column 2. We tested instances of problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$ with $\delta\% \in \{0.1\%,0.2\%,\dots,1\%,2.5\%,5\%,10\%,15\%,20\%,25\%\}$. The same range $[L, U]$ for the varying center C of the segment $[p_i^L, p_i^U]$ was used for all jobs $J_i \in \mathcal{J}$, namely: $L=1$ and $U=100$. For each job $J_i \in \mathcal{J}$, the weight $w_i \in R_+^1$ was uniformly distributed in the range [1, 50]. Note that the weight w_i was assumed to be known before scheduling.

In the experiments, we answered the question of how large the relative error Δ of the value $\gamma_{p^*}^t$ of the objective function $\gamma = \sum_{i=1}^n w_i C_i$ was obtained for the permutation π_t with the largest relative volume of a stability box $\mathcal{SB}(\pi_t,T)$ with respect to the actually optimal objective function value $\gamma_{p^*}^*$ calculated for the actual processing times $p^*=(p_1^*,p_2^*,\dots,p_n^*) \in T: \Delta = \gamma_{p^*}^* - \gamma_{p^*}^t/\gamma_{p^*}^*$. The relative volume of a stability box $\mathcal{SB}(\pi_t,T)$ is defined as the product of the fractions (18). In contrast to the weights w_i, the actual processing times p_i^* of the jobs $J_i \in \mathcal{J}$ have been assumed to be unknown before scheduling. Columns 9–11 (columns 12–14) present the average (maximal) error Δ for the corresponding series of instances obtained by Algorithm SL, Algorithm SU and Algorithm SM, respectively. As a measure of uncertainty for problem $1|p_i^L \le p_i \le p_i^U|\sum w_i C_i$, we use the relation $|\mathcal{A}|:((n(n-1))/2)$, where $|\mathcal{A}|$ is the number of arcs in the digraph $(\mathcal{J},\mathcal{A})$ and $(n(n-1))/2$ is the number of arcs in the complete circuit-free digraph. The average relation $|\mathcal{A}|:((n(n-1))/2)\cdot 100\%$ is given in column 3 in Tables 2 and 3. Columns 4 represents the average dimension n_t of the stability box with the largest relative volume. Columns 5 represents the average largest relative volume of the stability box.

Table 2
Randomly generated instances with $[L,U]=[1,100]$, $w_i \in [1,50]$ and $n \in \{10,20,\ldots,100\}$.

n	δ (%)	$\|\mathcal{A}\|$ (%)	n_t	Volume of $\mathcal{SB}(\pi_t,T)$	Exact solutions			Average error			Maximal error			CPU-time
					SL	SU	SM	SL	SU	SM	SL	SU	SM	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
10	0.1	100	10	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
10	0.2	100	10	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
10	0.3	100	10	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
10	0.4	100	10	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
10	0.5	100	10	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
10	0.6	99.933333	9.97	0.9917838	99	97	**100**	0.000001	0.000017	**0**	0.000069	0.001328	**0**	0
10	0.7	99.577778	9.92	0.8987959	94	**96**	**96**	**0.000015**	0.000021	0.000021	**0.000437**	0.001265	0.001265	0
10	0.8	99.6	9.91	0.9174445	95	95	**98**	0.000014	0.000015	**0.000003**	0.001029	0.001105	**0.000214**	0
10	0.9	99.644444	9.96	0.9015215	**96**	**96**	**96**	**0.000015**	0.000019	**0.000015**	**0.001159**	**0.001159**	**0.001159**	0
10	1	99.533333	9.87	0.9092345	92	90	**95**	0.000017	0.000037	**0.000013**	0.000789	0.000743	**0.000589**	0
10	2.5	97.311111	9.66	0.513894	79	**87**	**87**	0.000152	0.000097	**0.000079**	0.002142	0.002611	**0.001819**	0
10	5	95.555556	9.63	0.2295068	69	71	**72**	0.000313	0.000241	**0.000236**	**0.003975**	**0.003975**	**0.003975**	0
10	10	90.333333	9	0.0661674	42	**45**	**45**	0.001534	0.00154	**0.001447**	**0.009765**	0.010319	0.010319	0
10	15	85.622222	8.44	0.0242103	34	**36**	35	**0.002910**	0.002968	0.002938	**0.020817**	**0.020817**	**0.020817**	0
10	20	81.844444	7.8	0.0160475	24	26	**30**	0.004946	0.00452	**0.004466**	**0.03258**	0.033525	0.032974	0
10	25	77	6.94	0.0212792	**18**	17	17	**0.007555**	0.007569	0.007721	0.046069	**0.042479**	**0.042479**	0
20	0.1	100	20	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
20	0.2	100	20	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
20	0.3	100	20	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
20	0.4	100	20	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
20	0.5	100	20	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
20	0.6	99.878947	19.85	0.9171307	90	91	**92**	0.000009	0.00001	**0.000005**	0.000296	0.000296	**0.00022**	0
20	0.7	99.710526	19.67	0.7754711	86	81	**92**	0.000024	0.000021	**0.000012**	0.000479	**0.000356**	0.000395	0
20	0.8	99.5	19.57	0.6563203	67	73	**78**	0.000036	0.00003	**0.00002**	**0.000289**	0.000397	0.000334	0
20	0.9	99.3	19.38	0.5173639	64	70	**76**	0.000045	0.000035	**0.000029**	**0.000719**	**0.000719**	**0.000719**	0
20	1	99.189474	19.44	0.4158016	69	74	**77**	0.000045	0.000037	**0.000035**	0.000649	0.000477	**0.000411**	0
20	2.5	97.668421	18.93	0.0838408	39	37	**46**	0.000139	0.000163	**0.000133**	0.001064	**0.00087**	0.001475	0
20	5	94.926316	17.37	0.0121054	18	16	**21**	0.000448	0.000463	**0.000428**	**0.002496**	0.002906	**0.002496**	0
20	10	90.189474	14.68	0.0006664	4	6	**7**	**0.001633**	0.001693	0.001647	0.006251	**0.006032**	**0.006032**	0
20	15	85.063158	11.65	0.001938	0	0	0	0.003874	**0.003784**	0.003809	0.017293	0.017494	**0.017106**	0
20	20	80.415789	9.31	0.0050749	**1**	0	**1**	0.006213	0.006369	**0.006179**	**0.021536**	0.022109	**0.021536**	0
20	25	74.352632	7.33	0.011858	0	0	0	**0.011219**	0.01129	0.011408	**0.033143**	0.033884	0.033884	0
30	0.1	100	30	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
30	0.2	100	30	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
30	0.3	100	30	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
30	0.4	100	30	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
30	0.5	100	30	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
30	0.6	99.873563	29.55	0.8463245	83	78	**86**	0.000014	0.000021	**0.000005**	0.000466	0.000251	**0.0001**	0
30	0.7	99.698851	29.3	0.5438828	64	64	**79**	0.000022	0.000027	**0.000008**	**0.000195**	0.000292	**0.000195**	0
30	0.8	99.531034	29.02	0.390187	59	50	**65**	0.000031	0.000039	**0.000023**	**0.000264**	0.000283	**0.000264**	0
30	0.9	99.386207	28.77	0.2859986	54	45	**61**	0.00004	0.000039	**0.000024**	0.000374	0.000254	**0.000176**	0
30	1	99.266667	28.68	0.1899473	43	**49**	**49**	0.000043	0.000037	**0.000031**	0.00026	0.000365	**0.000236**	0
30	2.5	97.714943	27.03	0.0077736	16	13	**19**	**0.000146**	0.000161	0.00015	**0.000678**	0.000725	**0.000678**	0
30	5	95.032184	23.59	0.0000551	2	3	**5**	0.00049	0.000485	**0.000482**	0.00239	**0.00187**	0.002177	0.01
30	10	90.234483	17.04	0.0000339	0	0	0	0.001737	0.001668	**0.001643**	**0.007396**	0.007661	0.007661	0
30	15	84.751724	11.86	0.0013088	0	0	0	0.003789	0.003757	**0.003746**	0.010754	0.012262	**0.010137**	0
30	20	79.857471	9.26	0.0036536	0	0	0	0.006993	0.006991	**0.00691**	0.019905	0.019421	**0.019391**	0
30	25	74.825287	7.29	0.0151472	0	0	0	0.010592	**0.010509**	0.010513	0.026652	**0.024**	0.024102	0
40	0.1	100	40	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
40	0.2	100	40	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
40	0.3	100	40	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
40	0.4	100	40	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0

40	0.5	99.997436	39.98	0.9920779	**100**	98	**100**	**0**	**0**	**0**	**0**	0.00002	**0**	0
40	0.6	99.826923	39.02	0.6075943	63	56	**71**	0.000012	0.000017	**0.000008**	0.000215	0.000189	**0.000153**	0
40	0.7	99.64359	38.49	0.3081518	36	51	**57**	0.000032	0.000021	**0.000013**	0.000191	0.000153	**0.000115**	0
40	0.8	99.471795	37.93	0.1781846	32	34	**53**	0.000034	0.000035	**0.000018**	0.000189	0.000193	**0.000158**	0.01
40	0.9	99.323077	37.63	0.1107597	24	27	**36**	0.00005	0.000049	**0.000039**	**0.00024**	0.000369	0.00025	0
40	1	99.252564	37.43	0.0716628	18	20	**24**	0.000052	0.00005	**0.000042**	0.000271	**0.000234**	0.000283	0
40	2.5	97.642308	34.09	0.0004278	4	4	**6**	0.000144	0.000155	**0.000134**	0.000487	0.000524	**0.000447**	0
40	5	94.915385	27.55	0.0000036	0	0	0	0.000513	0.000515	**0.000483**	0.001526	**0.00141**	0.001677	0
40	10	89.40641	17.61	0.0000011	0	0	0	0.001968	**0.001911**	0.001949	**0.004101**	0.004221	0.004364	0
40	15	84.441026	11.59	0.0045304	0	0	0	0.003827	0.003903	**0.003826**	**0.008565**	0.010147	0.00874	0
40	20	79.284615	8.27	0.0075389	0	0	0	0.00692	**0.006801**	0.006808	0.013952	**0.013522**	0.014075	0
40	25	74.234615	6.84	0.0123886	0	0	0	**0.010471**	0.010515	0.01051	0.022627	0.020185	**0.020071**	0
50	0.1	100	50	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
50	0.2	100	50	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
50	0.3	100	50	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
50	0.4	100	50	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
50	0.5	99.998367	49.97	0.9954546	**100**	99	**100**	**0**	**0**	**0**	**0**	0.000009	**0**	0
50	0.6	99.821224	48.56	0.4573814	48	37	**62**	0.000015	0.000016	**0.000008**	0.000146	**0.000083**	0.000102	0
50	0.7	99.639184	47.41	0.1820659	26	22	**48**	0.000028	0.000029	**0.000014**	0.000182	0.000165	**0.000111**	0.01
50	0.8	99.471837	46.56	0.0681998	11	10	**23**	0.000041	0.000035	**0.000025**	0.000184	0.000144	**0.000115**	0
50	0.9	99.326531	46.22	0.0275715	8	10	**20**	0.000047	0.000043	**0.000029**	0.000204	0.000136	**0.000117**	0
50	1	99.245714	46.11	0.0225534	6	11	**15**	0.000055	0.000052	**0.000039**	0.000195	0.000239	**0.000171**	0
50	2.5	97.670204	40.62	0.0000008	**1**	**1**	**1**	0.000172	0.000176	**0.000155**	0.000628	0.000549	**0.000487**	0
50	5	95.005714	30.75	0.01	**1**	**1**	**1**	0.000516	0.000515	**0.000492**	0.001407	0.001473	**0.001373**	0.01
50	10	89.920816	17.99	0.0005546	0	0	0	0.001806	0.001823	**0.001798**	**0.003586**	0.003817	0.003677	0
50	15	84.440816	11.37	0.0016523	0	0	0	0.003862	0.003819	**0.003803**	0.006838	0.006965	**0.006621**	0
50	20	79.208163	8.34	0.0082483	0	0	0	0.007084	**0.006924**	0.007009	0.013978	**0.013416**	**0.013416**	0
50	25	74.041633	6.69	0.0128007	0	0	0	0.010999	**0.010926**	0.010972	0.02207	**0.02073**	0.02109	0
60	0.1	100	60	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
60	0.2	100	60	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
60	0.3	100	60	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
60	0.4	100	60	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
60	0.5	99.99887	59.98	0.9905815	**99**	98	**99**	**0**	**0**	**0**	**0.000008**	0.000019	**0.000008**	0.01
60	0.6	99.814689	57.51	0.3163	24	29	**49**	0.000017	0.000018	**0.000008**	0.000109	0.000136	**0.000046**	0
60	0.7	99.649718	56.48	0.1061775	14	22	**32**	0.000033	0.000029	**0.000017**	0.000135	0.000179	**0.000112**	0
60	0.8	99.468362	55.16	0.0289076	7	5	**13**	0.000044	0.000046	**0.000029**	0.000163	**0.000156**	**0.000156**	0.01
60	0.9	99.316384	54.31	0.0095535	3	3	**12**	0.000051	0.000043	**0.000027**	0.000181	0.000208	**0.00015**	0
60	1	99.188701	53.59	0.0019831	3	4	**8**	0.00006	0.000054	**0.000038**	0.000244	0.000187	**0.000105**	0
60	2.5	97.629379	45.66	≈ 0	0	0	0	0.000177	0.000177	**0.000158**	0.000448	0.000471	**0.000378**	0.01
60	5	95.076271	32.99	0.0000001	0	0	0	0.000496	0.00049	**0.000472**	**0.001034**	0.001091	0.001068	0
60	10	89.482486	17.32	0.0000072	0	0	0	0.001931	**0.001911**	0.001922	0.003342	**0.003211**	0.003244	0
60	15	84.623729	11.49	0.0040151	0	0	0	0.004135	0.004025	**0.004003**	0.007468	0.007117	**0.007078**	0
60	20	79.323729	8.2	0.0046016	0	0	0	0.007482	**0.007444**	0.007465	0.012847	**0.01214**	0.012333	0.01
60	25	74.327684	6.67	0.02173	0	0	0	0.011204	**0.01116**	0.011177	**0.022117**	0.022474	0.022726	0
70	0.1	100	70	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
70	0.2	100	70	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
70	0.3	100	70	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
70	0.4	100	70	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
70	0.5	99.999586	69.99	0.9984616	**100**	99	**100**	**0**	**0**	**0**	**0**	0.000001	**0**	0
70	0.6	99.827743	67	0.233306	22	23	**44**	0.000015	0.00002	**0.000008**	0.000075	0.000118	**0.000053**	0.01
70	0.7	99.638509	64.84	0.0488726	4	5	**13**	0.000032	0.00003	**0.000017**	0.000098	0.000084	**0.000074**	0
70	0.8	99.481988	63.76	0.0611658	2	**5**	**5**	0.00004	0.000035	**0.000022**	0.000122	0.000101	**0.000074**	0.01
70	0.9	99.337474	62.72	0.0171889	1	2	**10**	0.000046	0.000047	**0.000031**	0.00013	0.000117	**0.000109**	0
70	1	99.21118	61.37	0.0005221	0	2	**5**	0.000056	0.000052	**0.000037**	0.000184	0.000158	**0.000140**	0
70	2.5	97.660455	50.51	≈ 0	0	0	0	0.000163	0.00017	**0.00015**	**0.000303**	0.000369	0.000319	0.01
70	5	94.888613	33.57	≈ 0	0	0	0	0.000523	0.000529	**0.000508**	0.001091	0.001073	**0.001064**	0
70	10	89.796273	17.56	0.0000175	0	0	0	0.001796	0.001793	**0.001788**	**0.003046**	0.003056	0.003078	0.01
70	15	84.413251	10.86	0.0005931	0	0	0	**0.004112**	0.004122	0.004119	0.008153	0.008121	**0.007798**	0
70	20	79.687785	7.78	0.0052374	0	0	0	0.007117	**0.007101**	0.007122	0.01754	0.016522	**0.016504**	0.01
70	25	74.69648	6.37	0.0100287	0	0	0	0.010946	**0.010945**	0.010964	0.019794	0.019873	**0.019234**	0

Table 2 (*continued*)

n	δ (%)	$\lvert\mathcal{A}\rvert$ (%)	n_t	Volume of $\mathcal{SB}(\pi_t,T)$	Exact solutions			Average error			Maximal error			CPU-time
					SL	SU	SM	SL	SU	SM	SL	SU	SM	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
80	0.1	100	80	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
80	0.2	100	80	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0
80	0.3	100	80	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
80	0.4	100	80	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
80	0.5	99.998418	79.94	0.9878789	96	**99**	96	**0**	**0**	**0**	0.00002	**0.000003**	0.00002	0
80	0.6	99.813924	75.86	0.1725169	14	17	**33**	0.000019	0.000013	**0.000007**	0.000088	0.000072	**0.000065**	0.01
80	0.7	99.601899	72.84	0.0126145	5	0	**9**	0.000035	0.000034	**0.00002**	0.000124	0.000104	**0.000099**	0.01
80	0.8	99.518987	72.28	0.0014432	1	0	**3**	0.00004	0.000038	**0.000023**	0.000154	0.00011	**0.000109**	0
80	0.9	99.327848	70.72	0.0000176	0	0	**3**	0.000049	0.000048	**0.000032**	0.000137	0.000154	**0.000121**	0.01
80	1	99.19019	69.06	0.0000396	0	0	0	0.000058	0.000058	**0.000043**	0.000175	**0.000164**	0.000175	0
80	2.5	97.613924	54.06	≈ 0	0	0	0	0.000179	0.000174	**0.000154**	0.000483	0.000401	**0.000342**	0.01
80	5	94.923101	34.36	≈ 0	0	0	0	0.000531	0.000529	**0.000511**	0.001184	**0.001064**	0.001188	0.01
80	10	89.441456	16.53	0.0000619	0	0	0	0.001847	0.001819	**0.001811**	0.00283	0.002968	**0.002615**	0
80	15	84.247785	11.04	0.0026216	0	0	0	0.004147	0.00412	**0.004104**	0.006911	0.007253	**0.006867**	0.01
80	20	79.317089	8	0.0034028	0	0	0	0.007321	0.007296	**0.007282**	**0.011022**	0.011262	0.01117	0.01
80	25	74.320253	6.11	0.0155625	0	0	0	0.011628	**0.011564**	0.011598	0.017525	**0.017237**	0.017386	0
90	0.1	100	90	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
90	0.2	100	90	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
90	0.3	100	90	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
90	0.4	100	90	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
90	0.5	99.997753	89.91	0.9820893	**96**	95	**96**	**0**	**0**	**0**	0.000013	**0.000008**	0.000013	0.01
90	0.6	99.820974	84.81	0.1248793	9	13	**31**	0.000018	0.000016	**0.000008**	0.000085	0.000055	**0.00005**	0
90	0.7	99.630961	81.41	0.0061758	2	0	**6**	0.000031	0.00003	**0.000015**	0.000089	0.000112	**0.000064**	0.01
90	0.8	99.476404	79.52	0.0002124	0	0	**5**	0.000041	0.000039	**0.000022**	0.000114	0.000126	**0.000078**	0.01
90	0.9	99.342322	78.38	0.000009	0	0	**1**	0.000051	0.000046	**0.000033**	0.000173	0.000153	**0.000097**	0.01
90	1	99.185268	75.78	0.0000009	0	0	0	0.00006	0.000059	**0.000042**	0.000131	0.000127	**0.000104**	0.01
90	2.5	97.60799	57.83	≈ 0	0	0	0	0.00017	0.000164	**0.000144**	0.000365	0.000332	**0.000293**	0.01
90	5	94.853683	34.73	≈ 0	0	0	0	0.000534	0.000532	**0.000511**	0.001055	**0.000924**	0.001029	0.01
90	10	89.605243	16.65	0.000029	0	0	0	0.001892	0.00187	**0.001865**	0.003357	**0.003177**	0.003229	0.01
90	15	84.416479	10.59	0.0027047	0	0	0	0.004241	0.004227	**0.004197**	0.008959	0.008594	**0.008589**	0.01
90	20	79.296879	7.62	0.0077067	0	0	0	0.007256	0.007299	**0.007234**	0.012012	0.011768	**0.011491**	0
90	25	74.480649	5.37	0.0172727	0	0	0	0.011469	0.011472	**0.011459**	**0.017259**	0.017678	0.017662	0.01
100	0.1	100	100	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.02
100	0.2	100	100	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
100	0.3	100	100	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
100	0.4	100	100	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.01
100	0.5	99.99899	99.94	0.9922799	95	**99**	95	**0**	**0**	**0**	0.000016	**0.000008**	0.000016	0.01
100	0.6	99.804444	93.48	0.0648904	5	1	**18**	0.000017	0.000019	**0.000008**	0.000051	0.000064	**0.000048**	0.01
100	0.7	99.630101	89.87	0.0018207	1	0	**5**	0.000031	0.000032	**0.000017**	0.000087	0.000096	**0.000052**	0.01
100	0.8	99.470505	86.99	0.0000507	0	1	**3**	0.000044	0.000041	**0.000026**	0.000125	0.000112	**0.000087**	0.02
100	0.9	99.317576	46.22	0.0000034	0	0	0	0.000052	0.000049	**0.000035**	0.000114	0.000101	**0.000093**	0.01
100	1	99.216768	83.17	≈ 0	0	0	0	0.000055	0.000054	**0.000037**	0.000136	0.000126	**0.000082**	0.01
100	2.5	97.595152	59.87	≈ 0	0	0	0	0.000176	0.000173	**0.000154**	0.000344	0.00039	**0.000308**	0.01
100	5	94.97697	34.64	≈ 0	0	0	0	0.000511	0.000503	**0.000486**	0.000841	0.000805	**0.000766**	0.01
100	10	89.650707	16.2	0.0002331	0	0	0	0.001894	**0.001846**	0.00185	0.002927	0.00304	**0.002912**	0.02
100	15	84.311717	10.47	0.001412	0	0	0	0.004305	**0.004244**	0.004255	0.006093	0.006235	**0.006015**	0.01
100	20	79.057172	7.32	0.0246701	0	0	0	0.007373	**0.007356**	0.007377	**0.011932**	0.012269	0.012514	0.01
100	25	74.157172	5.92	0.0142538	0	0	0	0.011441	0.011428	**0.011396**	0.018428	0.018026	**0.017491**	0.01
The worst results					39	32	3	67	39	6	55	49	17	
The best results					53	54	106	58	65	137	73	70	124	
The best results when $(\mathcal{J},\mathcal{A})$ is not complete					10	11	63	15	22	94	30	27	81	

Table 3
Randomly generated instances with $[L,U]=[1,100]$, $w_i \in [1,50]$ and $n \in \{200,300,\ldots,1000\}$.

n	δ (%)	$\lvert\mathcal{A}\rvert$ (%)	n_t	Volume of $\mathcal{SB}(\pi_t,T)$	Exact solutions SL	SU	SM	Average error SL	SU	SM	Maximal error SL	SU	SM	CPU-time
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
200	0.1	100	200	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.08
200	0.2	100	200	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.08
200	0.3	100	200	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.08
200	0.4	100	200	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.09
200	0.5	99.997688	199.54	0.9008472	78	**90**	78	**0**	**0**	**0**	0.000007	**0.000003**	0.000007	0.08
200	0.6	99.804472	174.45	0.0004153	0	0	0	0.00002	0.000021	**0.00001**	0.000044	0.000048	**0.000026**	0.08
200	0.7	99.631005	161.56	≈0	0	0	0	0.000033	0.000031	**0.000017**	0.000059	0.000061	**0.000038**	0.09
200	0.8	99.460101	150.7	≈0	0	0	0	0.000043	0.000043	**0.000026**	0.000072	0.000071	**0.000051**	0.08
200	0.9	99.318040	142	≈0	0	0	0	0.000051	0.00005	**0.000034**	0.000078	0.000088	**0.00006**	0.09
200	1	99.199749	136.14	≈0	0	0	0	0.000058	0.000054	**0.000041**	0.000103	0.000083	**0.000077**	0.08
200	2.5	97.57608	71.48	≈0	0	0	0	0.000176	0.000178	**0.000155**	0.000303	0.000308	**0.000264**	0.09
200	5	94.829497	33.45	≈0	0	0	0	0.000536	0.000529	**0.000513**	0.000813	**0.000685**	0.000716	0.08
200	10	89.62	14.93	0.0011442	0	0	0	0.001979	0.00197	**0.001955**	**0.002694**	0.002764	0.002701	0.09
200	15	84.361859	9.53	0.0136524	0	0	0	0.004197	0.004193	**0.004173**	**0.005407**	0.005471	0.005485	0.09
200	20	79.290653	6.53	0.0130892	0	0	0	0.00766	0.007616	**0.007603**	**0.009749**	0.009891	0.009822	0.09
200	25	74.281206	4.76	0.0392243	0	0	0	**0.011833**	0.011838	0.011845	**0.01741**	0.01746	0.017492	0.09
300	0.1	100	300	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.27
300	0.2	100	300	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.27
300	0.3	100	300	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.27
300	0.4	100	300	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.26
300	0.5	99.997748	298.97	0.7989643	67	**73**	71	**0**	**0**	**0**	0.000002	0.000004	**0.000001**	0.27
300	0.6	99.811839	247.58	0.0000001	0	0	0	0.000019	0.00002	**0.000009**	0.000035	0.000041	**0.000021**	0.28
300	0.7	99.643501	221.84	≈0	0	0	0	0.000033	0.000032	**0.000017**	0.000061	0.000054	**0.000034**	0.28
300	0.8	99.467536	199.74	≈0	0	0	0	0.000045	0.000042	**0.000027**	0.000084	0.000072	**0.000045**	0.29
300	0.9	99.319287	182	≈0	0	0	0	0.000053	0.000052	**0.000035**	0.000079	0.000073	**0.000054**	0.28
300	1	99.195853	169.08	≈0	0	0	0	0.000058	0.000056	**0.00004**	0.000081	0.000085	**0.000062**	0.29
300	2.5	97.612375	75.14	≈0	0	0	0	0.000176	0.000176	**0.000156**	0.000227	0.000218	**0.000191**	0.28
300	5	94.850792	34.15	0.0000005	0	0	0	0.000548	0.000541	**0.000524**	0.000756	**0.000707**	0.000722	0.28
300	10	89.572999	14.7	0.000654	0	0	0	0.001958	0.001948	**0.001945**	**0.002394**	0.002405	0.002432	0.28
300	15	85.063158	11.65	0.001938	0	0	0	0.004377	0.004369	**0.004354**	0.006686	**0.006621**	0.006646	0.29
300	20	79.276477	6.16	0.0289215	0	0	0	0.00764	0.007642	**0.007614**	0.009832	**0.009621**	0.009637	0.3
300	25	74.214827	4.21	0.0307681	0	0	0	0.011788	**0.011784**	0.011797	0.014829	**0.014692**	**0.014692**	0.29
400	0.1	100	400	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.63
400	0.2	100	400	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.62
400	0.3	100	400	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.63
400	0.4	100	400	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	0.62
400	0.5	99.997719	398.18	0.686512	**55**	**55**	49	**0**	**0**	**0**	0.000003	**0.000002**	0.000003	0.63
400	0.6	99.812268	312.68	≈0	0	0	0	0.00002	0.000019	**0.000009**	0.000036	0.000037	**0.00002**	0.66
400	0.7	99.638296	269.88	≈0	0	0	0	0.000033	0.000032	**0.000018**	0.000049	0.000046	**0.000028**	0.67
400	0.8	99.461278	236.28	≈0	0	0	0	0.000044	0.000043	**0.000027**	**0.000064**	0.000069	**0.000043**	0.66
400	0.9	99.333396	216.55	≈0	0	0	0	0.000051	0.000051	**0.000034**	0.000073	0.000073	**0.000056**	0.66
400	1	99.194273	195.34	≈0	0	0	0	0.000058	0.000057	**0.00004**	0.000082	0.000077	**0.000057**	0.67
400	2.5	97.599712	78.98	≈0	0	0	0	0.000178	0.000173	**0.000154**	**0.000216**	0.00023	**0.000198**	0.66
400	5	94.891767	35.59	0.0000031	0	0	0	0.000549	0.000542	**0.000527**	0.000679	0.000688	**0.000678**	0.66
400	10	89.644511	15.36	0.0040239	0	0	0	0.001974	0.001969	**0.001954**	**0.002369**	0.002489	0.002489	0.67
400	15	84.294248	9.27	0.0103138	0	0	0	0.004342	0.004315	**0.004302**	0.005524	**0.005477**	0.005505	0.69
400	20	79.278985	5.29	0.0356351	0	0	0	0.007721	0.007703	**0.007686**	**0.00895**	0.008983	0.009022	0.69
400	25	74.184787	3.99	0.0521609	0	0	0	0.011805	**0.01179**	0.011792	0.01528	**0.014958**	0.015129	0.7

Table 3 (*continued*)

n	δ (%)	$\vert\mathcal{A}\vert$ (%)	n_t	Volume of $\mathcal{SB}(\pi_t,T)$	Exact solutions			Average error			Maximal error			CPU-time
					SL	SU	SM	SL	SU	SM	SL	SU	SM	
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
500	0.1	100	500	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	1.22
500	0.2	100	500	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	1.22
500	0.3	100	500	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	1.22
500	0.4	100	500	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	1.21
500	0.5	99.997587	496.98	0.5757675	44	**47**	44	**0**	**0**	**0**	0.000002	0.000002	**0.000001**	1.22
500	0.6	99.814749	370.81	≈ 0	0	0	0	0.000019	0.000019	**0.000009**	0.000032	0.000037	**0.000015**	1.28
500	0.7	99.636248	315.77	≈ 0	0	0	0	0.000032	0.000031	**0.000017**	0.000047	0.000046	**0.000017**	1.29
500	0.8	99.466629	272.5	≈ 0	0	0	0	0.000044	0.000043	**0.000026**	0.000066	0.000063	**0.000041**	1.29
500	0.9	99.333042	243.54	≈ 0	0	0	0	0.000052	0.000051	**0.000034**	0.000069	0.000068	**0.000052**	1.28
500	1	99.200858	218.58	≈ 0	0	0	0	0.000058	0.000058	**0.000042**	0.000079	0.000081	**0.000062**	1.29
500	2.5	97.591086	84.44	≈ 0	0	0	0	0.000176	0.000173	**0.000154**	0.000207	0.000222	**0.000185**	1.3
500	5	94.901523	37.12	0.000038	0	0	0	0.00055	0.000543	**0.000529**	0.000662	0.000664	**0.000639**	1.29
500	10	89.605002	16.13	0.0098597	0	0	0	0.001986	0.001978	**0.001968**	0.002333	**0.002285**	0.002295	1.31
500	15	84.333459	9.14	0.0265382	0	0	0	0.004344	0.004322	**0.004314**	0.005311	0.005293	**0.005282**	1.34
500	20	79.133066	5.82	0.0443501	0	0	0	0.007756	0.007735	**0.007715**	0.009255	0.009075	**0.009067**	1.34
500	25	74.279832	4.09	0.0582705	0	0	0	0.011949	**0.011933**	0.011937	0.014403	**0.014402**	0.014483	1.38
600	0.1	100	600	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	2.09
600	0.2	100	600	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	2.1
600	0.3	100	600	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	2.11
600	0.4	100	600	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	2.09
600	0.5	99.997440	595.47	0.4319593	22	**36**	24	**0**	**0**	**0**	0.000002	0.000002	**0.000001**	2.11
600	0.6	99.817084	428.33	≈ 0	0	0	0	0.000019	0.000019	**0.000009**	0.00003	0.000034	**0.000015**	2.22
600	0.7	99.637919	355.16	≈ 0	0	0	0	0.000034	0.000032	**0.000018**	0.000045	0.000044	**0.000026**	2.22
600	0.8	99.476361	305.76	≈ 0	0	0	0	0.000044	0.000043	**0.000026**	0.000057	0.000059	**0.000036**	2.22
600	0.9	99.326939	268.16	≈ 0	0	0	0	0.000053	0.000051	**0.000034**	0.000068	0.00007	**0.000047**	2.23
600	1	99.199354	242.3	≈ 0	0	0	0	0.000058	0.000058	**0.000041**	0.000078	0.000077	**0.000052**	2.23
600	2.5	97.600445	91.04	≈ 0	0	0	0	0.000176	0.000173	**0.000154**	0.000215	0.000215	**0.000192**	2.22
600	5	94.850590	40.24	0.0005476	0	0	0	0.000553	0.000549	**0.000532**	0.000663	0.000673	**0.000644**	2.23
600	10	89.607273	16.78	0.0188091	0	0	0	0.001951	0.001949	**0.001939**	**0.002387**	0.002401	0.002396	2.26
600	15	84.488342	9.19	0.0231484	0	0	0	0.004337	0.004328	**0.004308**	**0.005114**	0.005184	0.005115	2.26
600	20	79.247974	5.77	0.0463573	0	0	0	0.007666	0.00763	**0.007622**	0.008698	**0.008638**	0.008657	2.32
600	25	74.052354	3.81	0.0731271	0	0	0	0.011954	**0.011936**	0.011953	**0.014281**	0.014333	0.014282	2.4
700	0.1	100	700	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	3.38
700	0.2	100	700	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	3.35
700	0.3	100	700	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	3.36
700	0.4	100	700	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	3.37
700	0.5	99.997899	694.91	0.430126	**26**	36	**23**	**0**	**0**	**0**	**0.000001**	**0.000001**	**0.000001**	3.33
700	0.6	99.812712	477.36	≈ 0	0	0	0	0.00002	0.000019	**0.000009**	0.000029	0.000026	**0.000014**	3.53
700	0.7	99.634359	387.78	≈ 0	0	0	0	0.000033	0.000033	**0.000018**	0.000046	0.000043	**0.000027**	3.57
700	0.8	99.464631	330.53	≈ 0	0	0	0	0.000044	0.000043	**0.000026**	0.000057	**0.000036**	0.000039	3.57
700	0.9	99.326826	292.56	≈ 0	0	0	0	0.000051	0.00005	**0.000033**	0.000066	0.000065	**0.000043**	3.58
700	1	99.190399	260.26	≈ 0	0	0	0	0.00006	0.000059	**0.000042**	0.000079	0.000075	**0.000058**	3.59
700	2.5	97.598177	99.81	0.0000001	0	0	0	0.000176	0.000175	**0.000155**	0.000214	0.000206	**0.000183**	3.5
700	5	94.870014	44.89	0.0000951	0	0	0	0.000556	0.000551	**0.000534**	0.000651	0.000642	**0.000622**	3.53
700	10	89.623143	17.61	0.010328	0	0	0	0.001969	0.001963	**0.001955**	0.002309	0.002303	**0.002286**	3.57
700	15	84.325461	9.98	0.0260453	0	0	0	0.004371	0.004356	**0.00434**	0.00509	0.005045	**0.005005**	3.66
700	20	79.146499	5.68	0.0338831	0	0	0	0.00768	0.007649	**0.007634**	0.009204	0.009124	**0.009115**	3.59
700	25	73.947431	4.43	0.0498535	0	0	0	0.011956	**0.011942**	0.011944	0.014082	**0.013884**	0.013922	3.78
800	0.1	100	800	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	5.19

800	0.2	100	800	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	5.19
800	0.3	100	800	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	5.18
800	0.4	100	800	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	5.19
800	0.5	99.997847	793.15	0.3458384	22	32	**14**	**0**	**0**	**0**	0.000002	**0.000001**	**0.000001**	5.21
800	0.6	99.810156	524.62	≈ 0	0	0	0	0.00002	0.000019	**0.000009**	0.000028	0.000027	**0.000013**	5.43
800	0.7	99.633917	425.59	≈ 0	0	0	0	0.000033	0.000033	**0.000018**	0.000043	0.000043	**0.000024**	5.49
800	0.8	99.463085	356.05	≈ 0	0	0	0	0.000043	0.000042	**0.000026**	0.000058	0.000056	**0.000035**	5.49
800	0.9	99.322459	317.48	≈ 0	0	0	0	0.000052	0.000051	**0.000034**	0.000071	0.000067	**0.000047**	5.48
800	1	99.197985	283.41	≈ 0	0	0	0	0.000059	0.000058	**0.000042**	0.000077	0.00007	**0.000053**	5.51
800	2.5	97.567785	107.38	0.00000003	0	0	0	0.000179	0.000176	**0.000158**	0.000211	0.000203	**0.000185**	5.43
800	5	94.876702	47.43	0.0003798	0	0	0	0.000553	0.000548	**0.000531**	0.000672	0.000672	**0.000652**	5.37
800	10	89.589258	18.88	0.0134649	0	0	0	0.00198	0.001969	**0.00196**	0.002312	**0.002263**	0.002272	5.48
800	15	84.349193	11.13	0.0216885	0	0	0	0.00439	0.00437	**0.004357**	**0.00537**	0.00541	0.005374	5.54
800	20	79.207747	5.85	0.0178926	0	0	0	0.00769	0.00766	**0.007645**	0.009058	0.008958	**0.008955**	5.43
800	25	74.037882	4.8	0.0399213	0	0	0	0.011958	**0.011946**	0.011961	0.014022	**0.013989**	0.014092	5.67
900	0.1	100	900	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	7.45
900	0.2	100	900	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	7.42
900	0.3	100	900	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	7.45
900	0.4	100	900	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	7.43
900	0.5	99.997889	891.53	0.2683322	**24**	19	16	**0**	**0**	**0**	**0.000001**	**0.000001**	**0.000001**	7.45
900	0.6	99.813521	571.76	≈ 0	0	0	0	0.00002	0.000019	**0.000009**	0.000029	0.000027	**0.000014**	7.77
900	0.7	99.630894	458.84	≈ 0	0	0	0	0.000033	0.000032	**0.000018**	0.000047	0.000042	**0.000024**	7.84
900	0.8	99.466981	389.29	≈ 0	0	0	0	0.000044	0.000043	**0.000026**	0.000053	0.000052	**0.000032**	7.86
900	0.9	99.326368	344.49	≈ 0	0	0	0	0.000052	0.000051	**0.000034**	0.000066	0.000062	**0.000043**	7.81
900	1	99.19476	308.38	≈ 0	0	0	0	0.00006	0.000058	**0.000042**	0.000072	0.000069	**0.000051**	7.85
900	2.5	97.582469	114.83	0.1638453	0	0	0	0.000178	0.000177	**0.000158**	0.000210	0.000199	**0.000185**	7.75
900	5	94.879256	51.89	0.0008543	0	0	0	0.000547	0.000543	**0.000527**	0.00065	0.00062	**0.000606**	7.75
900	10	89.594369	20.84	0.008963	0	0	0	0.001985	0.001979	**0.00197**	0.002239	0.002234	**0.002221**	7.81
900	15	84.441491	11.79	0.0211951	0	0	0	0.004410	0.004401	**0.004383**	0.005062	0.005033	**0.005027**	7.88
900	20	79.170046	6.25	0.0485055	0	0	0	0.007677	0.007665	**0.007639**	0.008909	0.008787	**0.008785**	7.75
900	25	74.009042	4.89	0.0645754	0	0	0	0.012083	**0.012062**	0.012077	0.013801	0.013728	**0.013701**	8.08
1000	0.1	100	1000	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	10.32
1000	0.2	100	1000	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	10.23
1000	0.3	100	1000	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	10.27
1000	0.4	100	1000	1	**100**	**100**	**100**	**0**	**0**	**0**	**0**	**0**	**0**	10.34
1000	0.5	99.99754	988.05	0.2294523	11	**13**	8	**0**	**0**	**0**	**0.000001**	**0.000001**	0.000002	10.26
1000	0.6	99.811059	616.57	≈ 0	0	0	0	0.00002	0.00002	**0.000009**	0.000027	0.000031	**0.000013**	10.74
1000	0.7	99.63005	492.37	≈ 0	0	0	0	0.000033	0.000033	**0.000018**	0.000041	0.000041	**0.000024**	10.83
1000	0.8	99.466907	422.94	≈ 0	0	0	0	0.000044	0.000043	**0.000027**	0.000054	0.000054	**0.000036**	10.79
1000	0.9	99.316611	364.57	≈ 0	0	0	0	0.000052	0.000051	**0.000034**	0.000062	0.000064	**0.000041**	10.82
1000	1	99.193185	330.76	≈ 0	0	0	0	0.000060	0.000058	**0.000042**	0.000071	0.000072	**0.000051**	10.77
1000	2.5	97.597558	123.26	0.0000003	0	0	0	0.000178	0.000177	**0.000157**	0.000208	0.000205	**0.000179**	10.59
1000	5	94.866468	55.1	0.001244	0	0	0	0.000553	0.00055	**0.000533**	0.000625	0.000617	**0.000589**	10.6
1000	10	89.593481	22.34	0.0117973	0	0	0	0.001966	0.001951	**0.001942**	0.002256	0.00225	**0.002218**	10.71
1000	15	84.361189	12.77	0.0255009	0	0	0	0.004353	0.004348	**0.004331**	0.004843	0.004865	**0.004837**	10.66
1000	20	79.139962	6.49	0.0451776	0	0	0	0.007725	0.0077	**0.00768**	0.008922	0.008924	**0.008877**	10.54
1000	25	74.052452	4.87	0.0517923	0	0	0	0.012042	**0.01203**	0.012031	**0.013497**	0.013567	0.013577	11.08
The worst results					4	2	4	92	15	3	68	38	11	
The best results					39	42	38	46	53	135	52	56	116	
The best results when $(\mathcal{J},\mathcal{A})$ is not complete					3	6	2	10	17	99	16	20	80	

Columns 6–8 represent the numbers of instances (from 100 ones in a series) for which each permutation with the largest relative volume of a stability box generated by Algorithm SL, Algorithm SU or Algorithm SM, respectively, is optimal for the problem $1|p^*|\sum w_iC_i$ with the actual processing times $p^*=(p_1^*,p_2^*,\ldots,p_n^*)\in T$. From the experiments, it follows that condition (4) of Theorem 3 holds for all instances with a relative error $\delta\%\in\{0.1\%,0.2\%,0.3\%,0.4\%\}$ and for all instances with a relative error $\delta\%=0.5\%$ if $n\in\{10,20,30\}$. For each instance of such a series, the generated permutation with the largest relative volume of a stability box was optimal for problem $1|p^*|\sum w_iC_i$ (columns 6–8), the measure of the instance uncertainty was equal to 100% (column 3) and maximal relative volume of a stability box is equal to one (column 5).

The average (maximum) error Δ of the value $\gamma_{p^*}^k$ of the objective function $\gamma=\sum_{i=1}^n w_iC_i$ obtained for the permutation π_k with the largest relative volume of a stability box are presented in columns 9–11 (columns 12–14) for Algorithm SL, Algorithm SU and Algorithm SM, respectively. For all series, the average (maximum) error Δ of the value $\gamma_{p^*}^k$ of the objective function $\gamma=\sum_{i=1}^n w_iC_i$ obtained for the permutation π_k with the largest relative volume of a stability box was not greater than 0.012042 (not greater than 0.046049). The worst average (maximum) error Δ were obtained for the series of instances with $n=1000$ and $\delta\%=0.25\%$ (with $n=10$ and $\delta\%=0.25\%$). The maximum error Δ of the value $\gamma_{p^*}^k$ obtained for the permutation π_k with the largest relative volume of a stability box was not greater than 0.017492 for all series with $n\in\{200,300,\ldots,1000\}$.

The best (worst) results obtained by Algorithms SL, SU or SM are printed in bold face (are underlined) in columns 6–14. The numbers of the worst and the best results obtained by Algorithms SL, SU or SM are given in the last three rows of Tables 2 and 3. The worst result of the algorithm for a concrete series of instances means that the other two algorithms outperform this algorithm in this series of instances. The last row of Tables 2 and 3 presents the numbers of the best results obtained by Algorithms SL, SU or SM for the series of instances with the measure of their uncertainty $|\mathcal{A}|:((n(n-1))/2)\cdot 100\%$ less than 100%.

Our experiments show that all three permutations $\pi_l\in S$, $\pi_u\in S$ and $\pi_m\in S$ with the largest relative volume of their stability boxes generated (in the experiments) rather good objective function value $\gamma_{p^*}^k$ where $k\in\{l,u,m\}$. Among the three algorithms, Algorithm SM considerably outperforms both Algorithms SL and SU in the number of exact solutions (columns 6–8), in the average error Δ (columns 9–11) and in the maximal error Δ (columns 12–14). Algorithm SU slightly outperforms Algorithms SL.

The CPU-time (column 15) was practically the same for each of the Algorithms SL, SU and SM. The CPU-time was less than 0.02 s for each instance from a series with $n\in\{10,20,\ldots,100\}$ and it was no more than 11.08 s for each instance from a series with $n\in\{200,300,\ldots,1000\}$.

8. Concluding remarks

We showed that a minimal dominant set $S(T)\subseteq S$ of job permutations is unique for problem $1|p_i^L\le p_i\le p_i^U|\sum w_iC_i$ if we treat only one job from a subset of jobs with fixed processing times and the same weight-to-process ratio. We introduced the notion of a stability box of a permutation $\pi_k\in S$ and derived a formula for characterizing the stability box, which runs in $O(n\log n)$ time. We derived an $O(n^2)$-algorithm for finding a permutation with the largest volume of a stability box. The efficiency of the permutation with the largest relative volume of a stability box (how close it is to a factually optimal permutation) and the efficiency of the developed software (average CPU-time used for an instance) were demonstrated on a large set of randomly generated instances of problem $1|p_i^L\le p_i\le p_i^U|\sum w_iC_i$ with $10\le n\le 1000$.

The whole results that we obtained may be directly generalized to problem $1|prec,p_i^L\le p_i\le p_i^U|\sum w_iC_i$, where the precedence constraints are given a priori on the set of jobs $\mathcal{J}$. If a deterministic problem $1|prec|\sum w_iC_i$ for a particular type of precedence constraints (e.g., one defined by in-tree, out-tree or series-parallel digraph) is polynomially solvable, then the above results may be used for the uncertain counterpart $1|prec,p_i^L\le p_i\le p_i^U|\sum w_iC_i$. In such a more general scheduling problem, the digraph $(\mathcal{J},\mathcal{A})$ contains the arc (J_u,J_v) only if this arc does not violate the precedence constraint given between the jobs J_u and J_v a priori.

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