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A RELATION OF WEAK MAJORIZATION AND ITS APPLICATIONS TO CERTAIN INEQUALITIES FOR MEANS

Shan-He Wu* — Huan-Nan Shi**

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ABSTRACT. A relation of weak majorization for n-dimensional real vectors is established, the result is then used to derive some inequalities involving the power mean, the arithmetic mean and the geometric mean in n variables.

1. Introduction

Over the years, the theory of majorization as a powerful tool has widely been applied to the related research areas of pure mathematics and the applied mathematics (see [1], [15]). A good survey on the theory of majorization was given by Marshall and Olkin in [4]. Recently, the authors have given considerable attention to the applications of majorization in the field of inequalities, for details, we refer the reader to our papers [2], [3], [5]–[14] and [16]–[19].

In this paper, we shall establish a weak majorization relation for positive real numbers $x_1, x_2, \dots, x_n$ with $x_1 x_2 \cdots x_n \geq 1$, and discuss the Schur-convexity of

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the elementary symmetric function. In Section 4, the result is used to derive some inequalities involving the power mean, the arithmetic mean and the geometric mean in n variables.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbf{x} = (x_1, \dots, x_n)$ denotes n-tuple (n-dimensional real vector), the set of vectors can be written as

$$\begin{aligned} \mathbb{R}^n &= \big\{\mathbf{x} = (x_1, \dots, x_n) : \ x_i \in \mathbb{R}, \ \ i = 1, \dots, n\big\}, \\ \mathbb{R}^n_{+} &= \big\{\mathbf{x} = (x_1, \dots, x_n) : \ x_i \geq 0, \ \ i = 1, \dots, n\big\}, \\ \mathbb{R}^n_{++} &= \big\{\mathbf{x} = (x_1, \dots, x_n) : \ x_i > 0, \ \ i = 1, \dots, n\big\}. \end{aligned}$$

Definition 1. ([15]) Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $k = 1, \dots, n$ the kth elementary symmetric function is defined as follows:

$$E_k(\mathbf{x}) = E_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}.$$

The dual form of the elementary symmetric function is defined by

$$E^*_k(\mathbf{x}) = E^*_k(x_1, \dots, x_n) = \prod_{1 \leq i_1 < \ldots < i_k \leq n} \sum_{j=1}^{k} x_{i_j}.$$

Definition 2. ([4]) Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, $y_{[1]} \geq \cdots \geq y_{[n]}$ be rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order, respectively. Then

(1) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (briefly $\mathbf{x} \prec \mathbf{y}$) if $\sum\limits_{i=1}^{k} x_{[i]} \leq \sum\limits_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \dots, n-1$ and $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$;

(2) $\mathbf{x}$ is said to be weakly submajorized by $\mathbf{y}$ (briefly $\mathbf{x} \prec_w \mathbf{y}$) if $\sum\limits_{i=1}^{k} x_{[i]} \leq \sum\limits_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \dots, n$;

(3) a function $\varphi\colon \Omega \to \mathbb{R}$ is said to be increasing on $\Omega \subset \mathbb{R}^n$ if $\mathbf{x} \geq \mathbf{y}$ implies $\varphi(\mathbf{x}) \geq \varphi(\mathbf{y})$, where $\mathbf{x} \geq \mathbf{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \dots, n$;

(4) a function $\varphi\colon \Omega \to \mathbb{R}$ is said to be Schur-convex on $\Omega \subset \mathbb{R}^n$ if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$.

If $-\varphi$ is an increasing (resp. Schur-convex) function on Ω, then φ is said to be decreasing (resp. Schur-concave) on Ω.

2. Lemmas

To prove the main results stated in Sections 3 and 4, we need the following lemmas.

Lemma 1. ([15, p. 7]) *Let* $\mathbf{x} \in \mathbb{R}_+^n$, $\mathbf{y} \in \mathbb{R}^n$ *and* $\delta = \sum_{i=1}^{n}(y_i - x_i)$. *If* $\mathbf{x} \prec_w \mathbf{y}$, *then*

$$\Big(\mathbf{x}, \underbrace{\frac{\delta}{n}, \ldots, \frac{\delta}{n}}_{n}\Big) \prec \Big(\mathbf{y}, \underbrace{0, \ldots, 0}_{n}\Big).$$

Lemma 2. ([4, p. 122]) *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. *If* $\mathbf{x} \prec_w \mathbf{y}$, *then*

$$(\mathbf{x},\ x_{n+1}) \prec (\mathbf{y},\ y_{n+1}),$$

where $x_{n+1} = \min\{x_1, \ldots, x_n,\ y_1, \ldots, y_n\}$, $y_{n+1} = \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n} y_i$.

Lemma 3. ([15, p. 48]) *Let* $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *be such that* $x_i, y_i \in I \subset \mathbb{R}$, $i = 1, 2, \ldots, n$. *Then*

(1) $\mathbf{x} \prec \mathbf{y}$ *if and only if* $\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$ *holds for all convex functions* $g\colon I \to \mathbb{R}$;

(2) $\mathbf{x} \prec \mathbf{y}$ *if and only if* $\sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{n} g(y_i)$ *holds for all concave functions* $g\colon I \to \mathbb{R}$.

Lemma 4. ([15, p. 65]) *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *be such that* $x_i \in I \subset \mathbb{R}$, $i = 1, 2, \ldots, n$. *If* $g\colon I \to B \subset \mathbb{R}$ *is concave,* $\varphi\colon B^n \to \mathbb{R}$ *is increasing and Schur-concave, then* $\psi\colon I^n \to \mathbb{R}$ *given by* $\psi(\mathbf{x}) = \varphi\left(g\left(x_1\right), \ldots, g\left(x_n\right)\right)$ *is Schur-concave.*

Lemma 5. ([15, p. 59]) *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $1 \leq k \leq n$, *then the elementary symmetric function* $E_k(\mathbf{x})$ *and its dual version* $E_k^*(\mathbf{x})$ *are increasing and Schur-concave on* $\mathbb{R}_+^n$.

3. Main results and their proofs

Our main results are given in the Theorem 1 and Corollary 2 below.

Theorem 1. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $\prod_{i=1}^{n} x_i \geq 1$. *Then*

$$\Big(\underbrace{1, \dots, 1}_{n}\Big) \prec_w (x_1, \dots, x_n)\,. \tag{1}$$

P r o o f. We show the validity of majorization relation (1) by induction.

When $n = 2$, without loss of generality, we may assume that $x_1 \geq x_2$. From $x_1, x_2 > 0$ and $x_1 x_2 \geq 1$, it follows that $x_1 \geq 1$ and $x_1 + x_2 \geq 2\sqrt{x_1 x_2} \geq 2 = 1+1$. This means that $(1, 1) \prec_w (x_1, x_2)$.

We now assume that (1) holds true for $n = k$.

Let $\mathbf{x} = (x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1}_{++}$ and $\prod_{i=1}^{k+1} x_i \geq 1$. Without loss of generality, we may assume that $x_1 \geq x_2 \geq \ldots \geq x_{k+1} > 0$.

If $x_{k+1} > 1$, then $x_i > 1$ for $i = 1, \dots, k+1$. It is clear that

$$\Big(\underbrace{1, \dots, 1}_{k+1}\Big) \prec_w (x_1, \dots, x_{k+1})\,.$$

If $x_{k+1} \leq 1$, then $x_1 \geq x_2 \geq \ldots \geq x_{k-1} \geq x_k x_{k+1}$. Using the above assumption we have

$$\Big(\underbrace{1, \dots, 1}_{k}\Big) \prec_w (x_1, \dots, x_{k-1}, x_k x_{k+1})\,.$$

It follows that

$$\sum_{i=1}^{t} x_i \geq t \qquad \text{for} \quad t = 1, \dots, k-1,$$

and

$$\sum_{i=1}^{k-1} x_i + x_k x_{k+1} \geq k.$$

Thus, we have

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k-1} x_i + x_k x_{k+1} \geq k$$

and

$$\sum_{i=1}^{k+1} x_i \geq (k+1)\sqrt[k+1]{x_1 \dots x_{k+1}} \geq k+1.$$

This proves that (1) holds true for $n = k+1$, hence the proof of Theorem 1 is completed. $\square$

Remark 1. As a direct consequence of Theorem 1, we obtain the following weak majorization relations.

Corollary 1. *Let* x_1, x_2, x_3 *be positive real numbers. Then*

$$(1,1,1) \prec_w \left(\frac{x_2+x_3}{x_3+x_1}, \frac{x_3+x_1}{x_1+x_2}, \frac{x_1+x_2}{x_2+x_3}\right),$$
$$(1,1,1) \prec_w \left(\frac{x_1}{\sqrt{x_2x_3}}, \frac{x_2}{\sqrt{x_3x_1}}, \frac{x_3}{\sqrt{x_1x_2}}\right),$$
$$(1,1,1) \prec_w \left(\frac{\sqrt{x_2x_3}}{x_1}, \frac{\sqrt{x_3x_1}}{x_2}, \frac{\sqrt{x_1x_2}}{x_3}\right).$$

Corollary 2. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $\prod_{i=1}^{n} x_i \geq 1$. *Then*

$$\Big(\underbrace{1,\ldots,1}_{n}, \underbrace{A-1,\ldots,A-1}_{n}\Big) \prec \Big(x_1,\ldots,x_n, \underbrace{0,\ldots,0}_{n}\Big), \tag{2}$$

$$\Big(\underbrace{1,\ldots,1}_{n}, a\Big) \prec (x_1,\ldots,x_n,x_{n+1}), \tag{3}$$

where $A = \frac{1}{n}\sum_{i=1}^{n} x_i$, $a = \min\{x_1,\ldots,x_n,1\}$, $x_{n+1} = a - n(A-1)$.

P r o o f. Using Theorem 1, Lemma 1 and Lemma 2 the majorization relations (2) and (3) follow immediately. □

4. Some applications

In this section, we show that our results can be used to establish some new inequalities for means.

Let M_α denote the power mean which is defined for positive real numbers $x_1, x_2, \ldots, x_n$ as

$$M_\alpha = \begin{cases} \left(\frac{1}{n}\sum_{i=1}^{n} x_i^\alpha\right)^{\frac{1}{\alpha}}, & \alpha \neq 0, \\ \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}, & \alpha = 0. \end{cases}$$

Clearly, $M_1 = A$ (the arithmetic mean) and $M_0 = G$ (the geometric mean).

Theorem 2. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}, n \geq 2$ *and* $\prod_{i=1}^{n} x_i \geq 1$.

If $\alpha \geq 1$, *then*

$$M_\alpha \geq \big(1 + (A-1)^\alpha\big)^{\frac{1}{\alpha}}. \tag{4}$$

If $\alpha \geq 1$ *and* $n(A-1) \leq a$, *where* $a = \min\{x_1, \ldots, x_n, 1\}$, *then*

$$M_\alpha \geq \left(1 + \frac{a^\alpha - (a - n(A-1))^\alpha}{n}\right)^{\frac{1}{\alpha}}. \tag{5}$$

Furthermore, the inequalities (4) *and* (5) *are reversed for* $0 < \alpha < 1$.

P r o o f. When $\alpha \geq 1$, the function $f(x) = x^\alpha$ is convex on $(0, +\infty)$.

By using Lemma 3, we deduce from (2) and (3) that

$$\sum_{i=1}^{n} f(x_i) + nf(0) \geq nf(1) + nf(A-1) \tag{6}$$

and

$$\sum_{i=1}^{n} f(x_i) + f\,(a - n(A-1)) \geq nf(1) + f(a). \tag{7}$$

After a simple computation the inequalities (6) and (7) may be written in the form (4) and (5), respectively.

For $0 < \alpha < 1$, the function $f(x) = x^\alpha$ is concave on $(0, +\infty)$. Using Lemma 3 and the majorization relations (2) and (3) we obtain the reverse inequalities of (4) and (5). Theorem 2 is proved. □

Corollary 3. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$.

If $\alpha \geq 1$, *then*

$$M_\alpha \geq (G^\alpha + (A-G)^\alpha)^{\frac{1}{\alpha}} \geq G. \tag{8}$$

If $\alpha \geq 1$ *and* $b \geq n(A-G)$, *where* $b = \min\{x_1, \ldots, x_n, G\}$, *then*

$$M_\alpha \geq \left(G^\alpha + \frac{b^\alpha - (b - n(A-G))^\alpha}{n}\right)^{\frac{1}{\alpha}} \geq G. \tag{9}$$

P r o o f. For positive numbers $x_1/G,\ x_2/G,\ \ldots,\ x_n/G$, we have

$$\prod_{i=1}^{n} \frac{x_i}{G} = 1, \quad \frac{1}{n}\sum_{i=1}^{n} \frac{x_i}{G} = \frac{A}{G}, \quad \left(\frac{1}{n}\sum_{i=1}^{n} \left(\frac{x_i}{G}\right)^\alpha\right)^{\frac{1}{\alpha}} = \frac{M_\alpha}{G},$$
$$\min\left\{\frac{x_1}{G}, \ldots, \frac{x_n}{G}, 1\right\} = \frac{b}{G}.$$

Replacing $x_1, x_2 \dots, x_n$ by $x_1/G,\ x_2/G,\ \dots,\ x_n/G$, respectively, in (4) and (5) we obtain

$$\frac{M_\alpha}{G} \geq \left(1 + \left(\frac{A}{G} - 1\right)^\alpha\right)^{\frac{1}{\alpha}} \tag{10}$$

and

$$\frac{M_\alpha}{G} \geq \left(1 + \frac{\left(\frac{b}{G}\right)^\alpha - \left(\frac{b}{G} - n\left(\frac{A}{G} - 1\right)\right)^\alpha}{n}\right)^{\frac{1}{\alpha}}, \tag{11}$$

respectively, which corresponds to (8) and (9). □

Theorem 3. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$, $0 < \alpha \leq 1$ *and* $\prod\limits_{i=1}^{n} x_i \geq 1$.

If $1 \leq k \leq n$, *then*

$$E_k(\mathbf{x}^\alpha) \leq \sum_{i=0}^{k} C_n^i C_n^{k-i} (A-1)^{(k-i)\alpha}. \tag{12}$$

If $n+1 \leq k \leq 2n$, *then*

$$\prod_{l=k-n}^{n} \left(E_l^*(\mathbf{x}^\alpha)\right)^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} \left(l + (k-l)(A-1)^\alpha\right)^{C_n^l C_n^{k-l}}. \tag{13}$$

P r o o f. By Lemma 4 and Lemma 5, we conclude that $E_k(\mathbf{x}^\alpha)$ and $E_k^*(\mathbf{x}^\alpha)$ are Schur-concave on $\mathbb{R}^n_{++}$. Using the majorization relation (2) with the definition of Schur-concavity leads us to the desired inequalities (12) and (13). □

Replacing $x_1, x_2, \dots, x_n$ by $x_1/G, x_2/G, \dots, x_n/G$, respectively, in (12) and (13) we get the following corollary.

Corollary 4. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $0 < \alpha \leq 1$.

If $1 \leq k \leq n$, *then*

$$E_k(\mathbf{x}^\alpha) \leq \sum_{i=0}^{k} C_n^i C_n^{k-i} G^{(i-k+C_n^k)\alpha} (A-G)^{(k-i)\alpha}. \tag{14}$$

If $n+1 \leq k \leq 2n$, *then*

$$\prod_{l=k-n}^{n} \left(E_l^*(\mathbf{x}^\alpha)\right)^{C_n^{k-l}} \leq \prod_{l=k-n}^{n} \left(lG^\alpha + (k-l)(A-G)^\alpha\right)^{C_n^l C_n^{k-l}}. \tag{15}$$

Theorem 4. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$, $\prod_{i=1}^{n} x_i \geq 1$ *and* $A \leq 1 + \frac{a}{n}$, *where* a *as in* Theorem 2. *If* $1 \leq k \leq n$ *and* $0 < \alpha \leq 1$, *then*

$$E_k(\mathbf{x}^\alpha) + (a - n(A-1))^\alpha E_{k-1}(\mathbf{x}^\alpha) \leq C_n^k + C_n^{k-1} a^\alpha \tag{16}$$

and

$$E_k^*(\mathbf{x}^\alpha) \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left((a - (A-1))^\alpha + \sum_{j=1}^{k-1} x_{i_j}^\alpha \right) \leq k^{C_n^k} \left(a^\alpha + k - 1\right)^{C_n^{k-1}}. \tag{17}$$

P r o o f. Since $E_k(\mathbf{x}^\alpha)$ and $E_k^*(\mathbf{x}^\alpha)$ are Schur-concave on $\mathbb{R}^n_{++}$, then the majorization relation (3) with the definition of Schur-concavity yield inequalities (16) and (17). □

Again, replacing $x_1, x_2, \ldots, x_n$ by $x_1/G, x_2/G, \ldots, x_n/G$, respectively, in (16) and (17) we get the following corollary.

Corollary 5. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, $n \geq 2$ *and* $b \geq n(A - G)$, *where* b *as in* Corollary 3. *If* $1 \leq k \leq n$ *and* $0 < \alpha \leq 1$, *then*

$$E_k(\mathbf{x}^\alpha) + (b - n(A-G))^\alpha E_{k-1}(\mathbf{x}^\alpha) \leq C_n^k G^{k\alpha} + C_n^{k-1} b^\alpha G^{(k-1)\alpha},$$

and

$$\begin{aligned} E_k^*(\mathbf{x}^\alpha) \prod_{1 \leq i_1 < \ldots < i_k \leq n} &\left((b - n(A-G))^\alpha + \sum_{j=1}^{k-1} x_{i_j}^\alpha \right) \\ &\leq k^{C_n^k} G^{\alpha C_n^k} \left(b^\alpha + (k-1)G^\alpha\right)^{C_n^{k-1}}. \end{aligned}$$

Remark 2. Theorems 2, 3, 4 and their corollaries enable us to obtain a large number of inequalities for particular choice of parameters α, n and k. For example, if we take $n = 3$, $k = 2$ in (14) and $n = 3$, $k = 5$ in (15), we get for $x_i > 0$, $i = 1, 2, 3$ and $0 < \alpha < 1$ the following interesting inequalities:

$$\left(x_1^\alpha x_2^\alpha + x_2^\alpha x_3^\alpha + x_3^\alpha x_1^\alpha\right)/3 \leq G^\alpha (A-G)^{2\alpha} + 3G^{2\alpha}(A-G)^\alpha + G^{3\alpha}, \tag{18}$$

$$\begin{aligned} \left(x_1^\alpha + x_2^\alpha + x_3^\alpha\right) &\sqrt[3]{\left(x_1^\alpha + x_2^\alpha\right)\left(x_2^\alpha + x_3^\alpha\right)\left(x_3^\alpha + x_1^\alpha\right)} \\ &\leq \left(2G^\alpha + 3(A-G)^\alpha\right)\left(3G^\alpha + 2(A-G)^\alpha\right). \end{aligned} \tag{19}$$

In particular, for $\alpha = 1$ the inequalities (18) and (19) reduce to

$$\left(x_1x_2 + x_2x_3 + x_3x_1\right)/3 \le G(A^2 + AG - G^2)$$

and

$$\left(x_1 + x_2 + x_3\right) \sqrt[3]{\left(x_1 + x_2\right)\left(x_2 + x_3\right)\left(x_3 + x_1\right)} \le (3A - G)(G + 2A).$$

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* *Department of Mathematics and Computer Science*
Longyan University
Longyan, Fujian, 364012
P. R. CHINA
E-mail: wushanhe@yahoo.com.cn

** *Department of Electric information*
Teacher's College
Beijing Union University
Beijing, 100011
P. R. CHINA
E-mail: shihuannan@yahoo.com.cn