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ASYMPTOTIC BEHAVIOR OF THE SMALLEST EIGENVALUE OF MATRICES ASSOCIATED WITH COMPLETELY EVEN FUNCTIONS (mod r)

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In this paper, we present systematically analysis on the smallest eigenvalue of matrices associated with completely even functions (mod r). We obtain several theorems on the asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions (mod r). In particular, we get information on the asymptotic behavior of the smallest eigenvalue of the famous Smith matrices. Finally some examples are given to demonstrate the main results.

Keywords: Arithmetic progression; completely even function (mod r); tensor product; Dirichlet convolution; Dirichlet's theorem; Mertens' theorem; Cauchy's interlacing inequalities.

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1. Introduction and Statements of Results

For any given arithmetical function f, we denote by $f(m, r)$ the function f evaluated at the greatest common divisor (m, r) of positive integers m and r. Cohen [11] called the function $f(m, r)$ a *completely even function* (mod r). Let $1 \le x_1 < \cdots < x_n < \cdots$ be a given arbitrary strictly increasing infinite sequence of positive integers. For any integer $n \ge 1$, let $S_n = \{x_1, \ldots, x_n\}$. Let I be the function defined for any positive integer m by $I(m) := m$. In 1876, Smith [43] published his famous theorem showing that the determinant of the $n \times n$ matrix $[I(x_i, x_j)]$

on $S_n = \{1, \ldots, n\}$ is the product $\prod_{k=1}^{n} \phi(k)$, where ϕ is Euler's totient function. Smith also proved that if $S_n = \{1, \ldots, n\}$, then $\det[f(i,j)] = \prod_{k=1}^{n}(f*\mu)(k)$, where μ is the Möbius function and $f*\mu$ is the Dirichlet convolution of f and μ. In 1972, Apostol [2] extended Smith's result. In 1986, McCarthy [39] generalized Smith's and Apostol's results to the class of even functions of $m \pmod r$, where m and r are positive integers. A complex-valued function $\beta(m,r)$ is said to be an *even function of* $m \pmod r$ if $\beta(m,r) = \beta((m,r),r)$ for all values of m [10, 11]. Clearly a completely even function $\pmod r$ is an even function of $m \pmod r$, but the converse is not true. In 1993, Bourque and Ligh [5] extended the results of Smith, Apostol, and McCarthy. In 1999, Hong [18] improved the lower bounds for the determinants of matrices considered by Bourque and Ligh [5]. In 2002, Hong [19] generalized the results of Smith, Apostol, McCarthy and Bourque and Ligh to certain classes of arithmetical functions. Another kind of extension of Smith's determinant were obtained by Codecá and Nair [9] and Hilberdink [15].

Let ε be a real number. Wintner [47] proved in 1944 that $\limsup_{n\to\infty}\Lambda_n(\varepsilon) < \infty$ if and only if $\varepsilon > 1$, where $\Lambda_n(\varepsilon)$ denotes the largest eigenvalue of the matrix N_n defined as follows:

$$N_n := \left(\frac{(i,j)^{2\varepsilon}}{i^{\varepsilon}\cdot j^{\varepsilon}}\right)_{1\le i,j\le n}.$$

Let $\lambda_n(\varepsilon)$ denote the smallest eigenvalue of the matrix N_n. Lindqvist and Seip [37] in 1998 use the work of [14] about Riesz bases to investigate the asymptotic behavior of $\lambda_n(\varepsilon)$ and $\Lambda_n(\varepsilon)$ as $n \to \infty$. In particular, they got a sharp bound for $\lambda_n(\varepsilon)$ and $\Lambda_n(\varepsilon)$. In 2004, Hong and Loewy [29] made some progress in the study of asymptotic behavior of the eigenvalues of the $n \times n$ matrix $(\xi_\varepsilon(x_i, x_j))$ on S_n, where ξ_ε is defined for any positive integer m by $\xi_\varepsilon(m) := m^\varepsilon$. It was proved in [29] that if $0 < \varepsilon \le 1$ and $q \ge 1$ is any fixed integer, then the qth smallest eigenvalue of the $n \times n$ matrix $(\xi_\varepsilon(i,j))$ defined on the set $S_n = \{1, \ldots, n\}$ approaches zero as n tends to infinity. Recently, Hong and Lee [28] studied the asymptotic behavior of the eigenvalues of the reciprocal power LCM matrices and made some progress while Hong [26] got some results about asymptotic behavior of the largest eigenvalue of matrices associated with completely even functions $\pmod r$. Notice also that Bhatia [3], Bhatia and Kosaki [4] and Hong [27] considered infinite divisibility of matrices associated with multiplicative functions.

Given any set S of positive integers, we define the class $\tilde{\mathcal{C}}_S$ of arithmetical functions by

$$\tilde{\mathcal{C}}_S := \{f : (f*\mu)(d') > 0 \text{ whenever } d' \mid x, \text{ for any } x \in S\}.$$

For an arbitrary given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers, we define the class $\tilde{\mathcal{C}}$ of arithmetical functions by

$$\tilde{\mathcal{C}} := \{f : (f*\mu)(d') > 0 \text{ whenever } d' \mid x, \text{ for any } x \in \{x_i\}_{i=1}^{\infty}\}.$$

Let $S_n = \{x_1, \ldots, x_n\}$ for any integer $n \ge 1$. Then it is clear that if $f \in \tilde{\mathcal{C}}$, then $f \in \tilde{\mathcal{C}}_{S_n}$. In 1993, Bourque and Ligh [6] showed that if $f \in \tilde{\mathcal{C}}_{S_n}$, then the matrix $(f(x_i, x_j))$ (abbreviated by $(f(S_n))$) is positive definite. Hong [16] improved Bourque and Ligh's bounds for $\det(f(S_n))$ if $f \in \tilde{\mathcal{C}}_{S_n}$. In [22, 24], Hong obtained several results on the nonsingularity of the matrix $(f(S_n))$. On the other hand, the $n \times n$ matrix $(f[x_i, x_j])$ (abbreviated by $(f[S_n])$) having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its (i,j)th entry on any set $S_n = \{x_1, \ldots, x_n\}$ is not positive definite in general. It may even be singular. In fact, Hong [17] showed that for any integer $n \geq 8$, there exists a gcd-closed set $S_n = \{x_1, \ldots, x_n\}$ (i.e. $(x_i, x_j) \in S_n$ for all $1 \leq i, j \leq n$) such that the $n \times n$ matrix $(I[S_n])$ on S_n is singular. It should be remarked that Cao [8], Hong [21, 23] and Hong, Shum and Sun [30] provided several results on the nonsingularity of the $n \times n$ matrix $(\xi_\varepsilon[S_n])$, where ε is a positive integer. We note also that Li [36] and Hong and Lee [28] gave partial answers to Hong's conjecture [22] of real number power LCM matrices. From Bourque and Ligh's result [7] we can see that if S_n is a factor-closed set (i.e. it contains every divisor of x for any $x \in S_n$) and f is a multiplicative function such that $(f * \mu)(d')$ is a nonzero integer whenever $d' \,|\, \mathrm{lcm}(S_n)$, then the matrix $(f(x_i, x_j))$ divides the matrix $(f[x_i, x_j])$ in the ring $M_n(\mathbf{Z})$ of $n \times n$ matrices over the integers. Note also that Hong [20] showed that for any *multiple-closed set* S_n (i.e. $y \in S_n$ whenever $x|y|\mathrm{lcm}(S_n)$ for any $x \in S_n$, where $\mathrm{lcm}(S_n)$ means the least common multiple of all elements in S_n) and for any *divisor chain* S_n (i.e. $x_1| \cdots |x_n$), if f is a completely multiplicative function such that $(f * \mu)(d') \in \mathbf{Z}\backslash\{0\}$ whenever $d' \,|\, \mathrm{lcm}(S_n)$, then the matrix $(f(x_i, x_j))$ divides the matrix $(f[x_i, x_j])$ in the ring $M_n(\mathbf{Z})$. But such a factorization is no longer true if f is multiplicative. Some other factorization theorems about power GCD matrices and power LCM matrices are obtained by Hong [25], by Haukkanen and Korkee [13], by Hong, Zhao and Yin [31], by Feng, Hong and Zhao [12], by Tan [44], by Tan and Lin [45], by Tan, Lin and Liu [46] and by Xu and Li [48].

For any given set S of positive integers, it is natural to consider the following class of arithmetical functions:

$$\mathcal{C}_S := \{f : (f * \mu)(d') \geq 0 \text{ whenever } d' \,|\, x, \text{ for any } x \in S\}.$$

In the meantime, associated with an arbitrary given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers we define the following natural class of arithmetical functions:

$$\mathcal{C} := \{f : (f * \mu)(d') \geq 0 \text{ whenever } d' \,|\, x, \text{ for any } x \in \{x_i\}_{i=1}^\infty\}.$$

Then it is easy to see that if $f \in \mathcal{C}$, then $f \in \mathcal{C}_{S_n}$. Clearly $\tilde{\mathcal{C}}_S \subset \mathcal{C}_S$ for any given set S of positive integers, and $\tilde{\mathcal{C}} \subset \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers. Obviously for any given set S of positive integers, $\tilde{\mathcal{C}}_S$ and $\mathcal{C}_S$ are closed under addition and with respect to Dirichlet convolution, and for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers, $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are closed under addition and with respect to Dirichlet convolution. Note that $\mu \notin \mathcal{C}_S$ for any given set S of positive integers containing at least one prime, and $\mu \notin \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers containing at least one prime. However, we have the following result (Theorem 1.1 below).

Let $c \geq 0$ be an integer. For any arithmetical function f, define its cth Dirichlet convolution, denoted by $f^{(c)}$, inductively as follows:

$$f^{(0)} := \delta \quad \text{and} \quad f^{(c)} := f^{(c-1)} * f \quad \text{if } c \geq 1,$$

where δ is the function defined for any positive integer m by

$$\delta(m) := \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f * \delta = f$ for any arithmetical function f and

$$f^{(c)} := \underbrace{f * \cdots * f}_{c \text{ times}}.$$

For any integer $c \geq 1$, let

$$\mathbf{Z}^c_{>0} : = \{(x_1, \ldots, x_c) : 0 < x_i \in \mathbf{Z}, \text{ for } i = 1, \ldots, c\}.$$

Theorem 1.1. *Let $c \geq 1$ and $d \geq 0$ be integers. If $f_1, \ldots, f_c$ are distinct arithmetical functions and $(l_1, \ldots, l_c) \in \mathbf{Z}^c_{>0}$, then each of the following is true.*

(i) *Let $\{x_i\}_{i=1}^\infty$ be any given strictly increasing infinite sequence of positive integers. If $f_1, \ldots, f_c \in \mathcal{C}_{S_n}$ (respectively, $f_1, \ldots, f_c \in \mathcal{C}$) and $l_1 + \cdots + l_c > d$, then we have $f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)} \in \mathcal{C}_{S_n}$ (respectively, $f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)} \in \mathcal{C}$).*

(ii) *For any prime p, we have*

$$(f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(p) = \sum_{i=1}^c l_i f_i(p) f_i(1)^{l_i - 1} \prod_{\substack{j=1 \\ j \neq i}}^c f_j(1)^{l_j} - d \prod_{i=1}^c f_i(1)^{l_i}.$$

Furthermore, if $f_1, \ldots, f_c$ are multiplicative, then we have

$$(f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(p) = \begin{cases} \displaystyle\sum_{i=1}^c l_i f_i(p) - d & \text{if } f_i(1) = 1 \quad \text{for all } 1 \leq i \leq c, \\ 0 & \text{if } f_i(1) = 0 \quad \text{for some } 1 \leq i \leq c. \end{cases}$$

We remark that if the condition $l_1 + \cdots + l_c > d$ is suppressed, then Theorem 1.1(i) fails to be true. For example, let $c = l_1 = d = 1$. Take $f_1 = \phi$. Then $\phi \in \mathcal{C}_S$ for any given set S of positive integers and $\phi \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers. But we have $f_1 * \mu = \phi * \mu \notin \mathcal{C}_S$ for any given set S of positive integers containing at least one even number and $f_1 * \mu \notin \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers containing at least one even number because $(\phi * \mu^{(2)})(2) = -1$.

Using Theorem 1.1 as well as [16, Theorem 1] and by a continuity argument, we can prove the following result.

Theorem 1.2. *Let $c \geq 1$ and $d \geq 0$ be integers and $S_n = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. If $f_1, \ldots, f_c \in \mathcal{C}_{S_n}$ are distinct and $(l_1, \ldots, l_c) \in \mathbf{Z}^c_{>0}$ satisfies $l_1 + \cdots + l_c > d$, then each of the following is true.*

(i) $$\prod_{k=1}^{n} \sum_{\substack{d' \mid x_k \\ d' \nmid x_t,\ x_t < x_k}} (f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d+1)})(d') \leq \det((f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(x_i, x_j))$$
$$\leq \prod_{k=1}^{n} (f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(x_k).$$

(ii) *The $n \times n$ matrix $((f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(x_i, x_j))$ is positive semi-definite.*

Now let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers and let $S_n = \{x_1, \ldots, x_n\}$ for any integer $n \geq 1$. Let $1 \leq q \leq n$ be a fixed integer and $c \geq 1$ and $d \geq 0$ be integers. Let $(l_1, \ldots, l_c) \in \mathbf{Z}^c_{>0}$ satisfy $l_1 + \cdots + l_c > d$ and $f_1, \ldots, f_c \in \mathcal{C}$ be distinct. In the present paper, we investigate the asymptotic behavior of the qth smallest eigenvalue of the matrix $((f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(S_n))$. Let $\lambda_n^{(1)}(l_1, \ldots, l_c, d) \leq \cdots \leq \lambda_n^{(n)}(l_1, \ldots, l_c, d)$ be the eigenvalues of the matrix $((f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. By Theorem 1.2(ii) we have

$$\lambda_n^{(q)}(l_1, \ldots, l_c, d) \geq 0.$$

But by Cauchy's interlacing inequalities (see [32] and a new proof of it, see [34]) we have

$$\lambda_{n+1}^{(q)}(l_1, \ldots, l_c, d) \leq \lambda_n^{(q)}(l_1, \ldots, l_c, d).$$

Thus the sequence $\{\lambda_n^{(q)}(l_1, \ldots, l_c, d)\}_{n=q}^{\infty}$ is a non-increasing infinite sequence of non-negative real numbers and so it is convergent. Namely, we have the following theorem.

Theorem 1.3. *Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers. Let $c \geq 1$ and $d \geq 0$ be integers and $q \geq 1$ be a given arbitrary integer. Let $f_1, \ldots, f_c \in \mathcal{C}$ be distinct and $(l_1, \ldots, l_c) \in \mathbf{Z}^c_{>0}$ satisfy $l_1 + \cdots + l_c > d$. Let $\lambda_n^{(1)}(l_1, \ldots, l_c, d) \leq \cdots \leq \lambda_n^{(n)}(l_1, \ldots, l_c, d)$ be the eigenvalues of the $n \times n$ matrix $((f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. Then the sequence $\{\lambda_n^{(q)}(l_1, \ldots, l_c, d)\}_{n=q}^{\infty}$ converges and*

$$\lim_{n \to \infty} \lambda_n^{(q)}(l_1, \ldots, l_c, d) \geq 0.$$

Let $\{y_i\}_{i=1}^{\infty}$ be a strictly increasing infinite sequence of positive integers. We say that f is *increasing on the sequence* $\{y_i\}_{i=1}^{\infty}$ if $f(y_i) \leq f(y_j)$ whenever $1 \leq i < j$. For an arbitrary strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers satisfying that $(x_i, x_j) = x$ for any $i \neq j$, where $x \geq 1$ is an integer, we have the following result.

Theorem 1.4. *Let x be a positive integer and $\{x_i\}_{i=1}^{\infty}$ be a strictly increasing infinite sequence of positive integers satisfying that for every $i \neq j, (x_i, x_j) = x$. Assume that $f \in \mathcal{C}$ and is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $(f(x_i, x_j))$ defined on the set $\{x_1, \ldots, x_n\}$. Then each of the following holds.*

(i) *If $f(x) = 0$, or $f(x) > 0$ and $f(x_1) = f(x_2)$, then $\lambda_n^{(1)} = f(x_1) - f(x)$,*
(ii) *If $f(x) > 0$ and $f(x_1) < f(x_2)$, then*

$$f(x_1) - f(x) < \lambda_n^{(1)} < f(x_1) - f(x) + \frac{f(x)}{1 + \sum_{i=2}^{n} \frac{f(x)}{f(x_i) - f(x_1)}}.$$

(iii) *If $f(x_1) = 0$ then $\lambda_n^{(1)} = 0$ for all $n \geq 1$. If $f(x_1) > 0$ and $\sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty$, then we have $\lim_{n\to\infty} \lambda_n^{(1)} = f(x_1) - f(x)$.*

From Theorem 1.4 we can deduce the following result.

Theorem 1.5. *Let x be a positive integer. Let $\{x_i\}_{i=1}^{\infty}$ be a strictly increasing infinite sequence of positive integers satisfying the following conditions.*

(i) *For every $i \neq j, (x_i, x_j) = x$;*
(ii) $\sum_{i=1}^{\infty} \frac{1}{x_i} = \infty$.

Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $(f(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. If $f \in \mathcal{C}$ and is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$ and $f(x_i) \leq Cx_i$ for all $i \geq 1$, where $C > 0$ is a constant, then we have $\lim_{n\to\infty} \lambda_n^{(1)} = f(x_1) - f(x)$.

Let $b \geq 1$ be an integer. By the well-known Dirichlet's theorem (see, for example, [1] or [35]) there are infinitely many primes in the arithmetic progression $\{1+bi\}_{i=0}^{\infty}$. In the following let

$$p_1(b) < \cdots < p_n(b) < \cdots \tag{1.1}$$

denote the primes in this arithmetic progression. Consequently, for the arithmetic progression case, we have the following result.

Theorem 1.6. *Let $a, b, c, q \geq 1$ and $d, e \geq 0$ be any given integers. Let $x_i = a + b(e + i - 1)$ for $i \geq 1$. Let $(l_1, \ldots, l_c) \in \mathbf{Z}_{>0}^c$ satisfy $l_1 + \cdots + l_c > d$. Let $f_1, \ldots, f_c \in \mathcal{C}$ be distinct, multiplicative and increasing on the sequence $\{p_i(b)\}_{i=1}^{\infty}$, where $p_i(b)(i \geq 1)$ is defined by (1.1). Let $\lambda_n^{(1)}(l_1, \ldots, l_c, d) \leq \cdots \leq \lambda_n^{(n)}(l_1, \ldots, l_c, d)$ be the eigenvalues of the $n \times n$ matrix $((f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(a+bi, a+bj))$ defined on the set $\{a + be, a + b(e + 1), \ldots, a + b(e + n - 1)\}$.*

(i) *If $(f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)})(p_i(b)) = 0$ for some $i \geq 1$, then for any large enough n we have $\lambda_n^{(q)}(l_1, \ldots, l_c, d) = 0$,*

(ii) *If* $(f_1^{(l_1)}*\cdots*f_c^{(l_c)}*\mu^{(d)})(p_i(b))\neq 0$ *for all* $i\geq 1$ *and* $\sum_{i=1}^{\infty}\frac{1}{f_1(p_i(b))+\cdots+f_c(p_i(b))}=\infty$, *then we have* $\lim_{n\to\infty}\lambda_n^{(q)}(l_1,\ldots,l_c,d)=0$.

(iii) *In particular, if for each* $1\leq j\leq c$, *there is a positive constant* C_j *such that* $f_j(p_i(b))\leq C_j p_i(b)$ *for all* $i\geq 1$, *then we have* $\lim_{n\to\infty}\lambda_n^{(q)}(l_1,\ldots,l_c,d)=0$.

Furthermore, applying again Cauchy's interlacing inequalities, it follows from Theorems 1.3 and 1.6 that the following result holds.

Theorem 1.7. *Let* $a,b,c,q\geq 1$ *and* $d,e\geq 0$ *be any given integers. Let* $\{x_i\}_{i=1}^{\infty}$ *be any given strictly increasing infinite sequence of positive integers which contains the arithmetic progression* $\{a+bi\}_{i=e}^{\infty}$ *as its subsequence. Let* $(l_1,\ldots,l_c)\in\mathbf{Z}_{>0}^c$ *satisfy* $l_1+\cdots+l_c>d$. *Let* $f_1,\ldots,f_c\in\mathcal{C}$ *be distinct, multiplicative and increasing on the sequence* $\{p_i(b)\}_{i=1}^{\infty}$, *where* $p_i(b)(i\geq 1)$ *is defined by* (1.1). *Let* $\lambda_n^{(1)}(l_1,\ldots,l_c,d)\leq\cdots\leq\lambda_n^{(n)}(l_1,\ldots,l_c,d)$ *be the eigenvalues of the* $n\times n$ *matrix* $((f_1^{(l_1)}*\cdots*f_c^{(l_c)}*\mu^{(d)})(x_i,x_j))$ *defined on the set* $S_n=\{x_1,\ldots,x_n\}$.

(i) *If* $(f_1^{(l_1)}*\cdots*f_c^{(l_c)}*\mu^{(d)})(p_i(b))=0$ *for some* $i\geq 1$, *then for any large enough* n *we have* $\lambda_n^{(q)}(l_1,\ldots,l_c,d)=0$.

(ii) *If* $(f_1^{(l_1)}*\cdots*f_c^{(l_c)}*\mu^{(d)})(p_i(b))\neq 0$ *for all* $i\geq 1$ *and* $\sum_{i=1}^{\infty}\frac{1}{f_1(p_i(b))+\cdots+f_c(p_i(b))}=\infty$, *then we have* $\lim_{n\to\infty}\lambda_n^{(q)}(l_1,\ldots,l_c,d)=0$.

(iii) *In particular, if for each* $1\leq j\leq c$, *there is a positive constant* C_j *such that* $f_j(p_i(b))\leq C_j p_i(b)$ *for all* $i\geq 1$, *then we have* $\lim_{n\to\infty}\lambda_n^{(q)}(l_1,\ldots,l_c,d)=0$.

As a special case we have the following theorem.

Theorem 1.8. *Let* $a,b,c,q\geq 1$ *and* $d,e\geq 0$ *be any given integers such that* $c>d$. *Let* $\{x_i\}_{i=1}^{\infty}$ *be any given strictly increasing infinite sequence of positive integers which contains the arithmetic progression* $\{a+bi\}_{i=e}^{\infty}$ *as its subsequence. Let* $\lambda_n^{(1)}(c,d)\leq\cdots\leq\lambda_n^{(n)}(c,d)$ *be the eigenvalues of the* $n\times n$ *matrix* $((f^{(c)}*\mu^{(d)})(x_i,x_j))$ *defined on the set* $S_n=\{x_1,\ldots,x_n\}$. *Let* $f\in\mathcal{C}$ *be multiplicative and increasing on the sequence* $\{p_i(b)\}_{i=1}^{\infty}$, *where* $p_i(b)(i\geq 1)$ *is defined by* (1.1).

(i) *If* $(f^{(c)}*\mu^{(d)})(p_i(b))=0$ *for some* $i\geq 1$, *then for any large enough* n *we have* $\lambda_n^{(q)}(c,d)=0$.

(ii) *If* $(f^{(c)}*\mu^{(d)})(p_i(b))\neq 0$ *for all* $i\geq 1$ *and* $\sum_{i=1}^{\infty}\frac{1}{f(p_i(b))}=\infty$, *then for any given integer* $q\geq 1$, *we have* $\lim_{n\to\infty}\lambda_n^{(q)}(c,d)=0$.

(iii) *In particular, if* $f(p_i(b))\leq Cp_i(b)$ *for all* $i\geq 1$, *where* $C>0$ *is a constant, then for any given integer* $q\geq 1$, *we have* $\lim_{n\to\infty}\lambda_n^{(q)}(c,d)=0$.

Corollary 1.9. *Let* $\lambda_n^{(1)}\leq\cdots\leq\lambda_n^{(n)}$ *be the eigenvalues of the* $n\times n$ *matrix* $(f(i,j))$ *defined on the set* $S_n=\{1,\ldots,n\}$. *If* f *is an increasing multiplicative function satisfying* $(f*\mu)(y)\geq 0$ *and* $f(y)\leq Cy$ *for all positive integers* y, *where* $C>0$ *is a constant, then for any given integer* $q\geq 1$, *we have* $\lim_{n\to\infty}\lambda_n^{(q)}=0$.

This paper is organized as follows. The details of the proofs of Theorems 1.1, 1.2 and 1.4–1.6 will be given in Sec. 2. In Sec. 3 we give some examples to illustrate our results. The final section is devoted to some open questions.

Throughout this paper, we let E_n denote the $n \times n$ matrix with all entries equal to 1. For the basic facts about arithmetical functions, the readers are referred to [1, 42] or [38]. For a comprehensive review of papers related to the matrices associated with arithmetical functions not presented here, we refer to [28, 29] as well as the papers listed there.

2. The Proofs of Theorems 1.1, 1.2 and 1.4–1.6

First we prove Theorem 1.1.

Proof of Theorem 1.1. Clearly to prove Theorem 1.1 it suffices to prove that for any prime p and for any integer $l \geq 1$ and any (not necessarily distinct) arithmetical functions $g_1, \ldots, g_l$, we have

$$(g_1 * \cdots * g_l * \mu^{(d)})(p) = \sum_{i=1}^{l} g_1(1) \cdots g_{i-1}(1) g_i(p) g_{i+1}(1) \cdots g_l(1) - d g_1(1) \cdots g_l(1), \tag{2.1}$$

and if $g_1, \ldots, g_l \in \mathcal{C}_{S_n}$ (respectively, $g_1, \ldots, g_l \in \mathcal{C}$) and $l > d$, then we have

$$g_1 * \cdots * g_l * \mu^{(d)} \in \mathcal{C}_{S_n} \quad (\text{respectively, } g_1 * \cdots * g_l * \mu^{(d)} \in \mathcal{C}). \tag{2.2}$$

Furthermore, if $g_1, \ldots, g_l$ are multiplicative, then we have

$$(g_1 * \cdots * g_l * \mu^{(d)})(p) = \begin{cases} \displaystyle\sum_{i=1}^{l} g_i(p) - d & \text{if } g_i(1) = 1 \quad \text{for all } 1 \leq i \leq l, \\ 0 & \text{if } g_i(1) = 0 \quad \text{for some } 1 \leq i \leq l. \end{cases} \tag{2.3}$$

By the definition of Dirichlet convolution we have

$$\begin{aligned} &(g_1 * \cdots * g_l * \mu^{(d)})(p) \\ &\quad = \sum_{\substack{r_1 \ldots r_l \bar{r}_1 \ldots \bar{r}_d = p \\ (r_1, \ldots, r_l, \bar{r}_1, \ldots, \bar{r}_d) \in \mathbf{Z}_{>0}^{l+d}}} g_1(r_1) \cdots g_l(r_l) \mu(\bar{r}_1) \ldots \mu(\bar{r}_d) \\ &\quad = \sum_{i=1}^{l} g_1(1) \cdots g_{i-1}(1) g_i(p) g_{i+1}(1) \cdots g_l(1) \mu(1)^d + d g_1(1) \cdots g_l(1) \mu(p) \mu(1)^{d-1} \\ &\quad = \sum_{i=1}^{l} g_1(1) \cdots g_{i-1}(1) g_i(p) g_{i+1}(1) \cdots g_l(1) - d g_1(1) \cdots g_l(1). \end{aligned}$$

So (2.1) is proved. Further, if f is multiplicative, then we have $f(1)^2 = f(1)$. So we have $f(1) = 1$, or 0. Thus (2.3) follows immediately.

Now consider (2.2). Since the proof for the case $g_1, \ldots, g_l \in \mathcal{C}$ is completely similar to that of the case $g_1, \ldots, g_l \in \mathcal{C}_{S_n}$, we only need to show (2.2) for the case $g_1, \ldots, g_l \in \mathcal{C}_{S_n}$. In the following let $g_1, \ldots, g_l \in \mathcal{C}_{S_n}$ and $l > d$. Now for any $x \in S_n$ and any $r \mid x$, since $l \geq d+1$, we have

$$\begin{aligned}
&((g_1 * \cdots * g_l * \mu^{(d)}) * \mu)(r) \\
&\quad = (g_1 * \cdots * g_l * \mu^{(d+1)})(r) \\
&\quad = ((g_1 * \mu) * \cdots * (g_d * \mu) * (g_{d+1} * \mu) * g_{d+2} * \cdots * g_l)(r) \\
&\quad = \sum_{\substack{r_1 \ldots r_l = r \\ (r_1, \ldots, r_l) \in \mathbf{Z}_{>0}^l}} (g_1 * \mu)(r_1) \cdots (g_{d+1} * \mu)(r_{d+1}) g_{d+2}(r_{d+2}) \cdots g_l(r_l).
\end{aligned} \tag{2.4}$$

For $1 \leq i \leq d+1$, since $g_i \in \mathcal{C}_{S_n}$ and $r_i \mid x$, we have

$$(g_i * \mu)(r_i) \geq 0. \tag{2.5}$$

On the other hand, for $d+2 \leq j \leq l$, $g_j \in \mathcal{C}_{S_n}$ together with $r_j \mid x$ implies that

$$g_j(r_j) = \sum_{d' \mid r_j} (g_j * \mu)(d') \geq 0. \tag{2.6}$$

From (2.4)–(2.6) we then deduce that

$$((g_1 * \cdots * g_l * \mu^{(d)}) * \mu)(r) \geq 0.$$

Thus (2.2) holds. This completes the proof of Theorem 1.1. □

To prove Theorem 1.2 we need a result from [16].

Lemma 2.1 ([16, Theorem 1]). *Let $S_n = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. If $g \in \tilde{\mathcal{C}}_{S_n}$, then we have*

$$\det(g(x_i, x_j)) \geq \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_t,\ x_t < x_k}} (g * \mu)(d).$$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.1(i), to show Theorem 1.2 we only need to show that if $f \in \mathcal{C}_{S_n}$, then each of the following is true.

(i′) $$\prod_{k=1}^{n} \sum_{\substack{d' \mid x_k \\ d' \nmid x_t,\ x_t < x_k}} (f * \mu)(d') \leq \det(f(x_i, x_j)) \leq \prod_{k=1}^{n} f(x_k).$$

(ii′) The $n \times n$ matrix $(f(x_i, x_j))$ is positive semi-definite.

First we show the inequality on the left-hand side of (i′). Let $f \in \mathcal{C}_{S_n}$. Choose $\epsilon > 0$ and $\bar{f} \in \tilde{\mathcal{C}}_{S_n}$. Then it is easy to see that $f + \epsilon\bar{f} \in \tilde{\mathcal{C}}_{S_n}$. For an arithmetical function g and $1 \le k \le n$, let

$$\alpha_g(x_k) := \sum_{\substack{d \mid x_k \\ d \nmid x_t,\ x_t < x_k}} (g * \mu)(d).$$

By Lemma 2.1 we have

$$\det((f + \epsilon\bar{f})(x_i, x_j)) \ge \prod_{k=1}^{n} \alpha_{f+\epsilon\bar{f}}(x_k). \tag{2.7}$$

Note that both sides of (2.7) are polynomials in ϵ. Moreover, the constant coefficients of the left- and right-hand sides are, respectively, $\det(f(x_i, x_j))$ and $\prod_{k=1}^{n} \alpha_f(x_k)$. Since (2.7) holds for any $\epsilon > 0$, letting $\epsilon \to 0$ the left-hand side of (i′) is proved.

For any $1 \le l \le n$, since $f \in \mathcal{C}_{S_n}$, then the inequality on the left-hand side of (i′) implies that the determinant of any principal submatrix of $(f(x_i, x_j))$ is non-negative. This concludes part (ii′). From (ii′) the inequality on the right-hand side of (i′) follows immediately. Hence the proof of Theorem 1.2 is complete. □

The following result is known.

Lemma 2.2. *Let $n \ge 1$ be an integer and let $a_1, \ldots, a_n \in R$, where R is an arbitrary commutative ring. Then we have*

$$\det(E_n + \operatorname{diag}(a_1 - 1, \ldots, a_n - 1)) = \prod_{i=1}^{n}(a_i - 1) + \sum_{1 \le i_1 < \cdots < i_{n-1} \le n} \prod_{j=1}^{n-1}(a_{i_j} - 1).$$

In order to show Theorem 1.4 we need also the following lemma.

Lemma 2.3. *Let $\{r_i\}_{i=1}^{\infty}$ be an increasing infinite sequence of real numbers satisfying $r_1 \ge 1$ and let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $E_n + \operatorname{diag}(r_1 - 1, \ldots, r_n - 1)$. Then each of the following holds.*

(i) *If $r_1 = r_2$, then $\lambda_n^{(1)} = r_1 - 1$.*
(ii) *If $r_1 < r_2$, then*

$$r_1 - 1 < \lambda_n^{(1)} < r_1 - 1 + \frac{1}{1 + \displaystyle\sum_{i=2}^{n} \frac{1}{r_i - r_1}}.$$

(iii) *If $\sum_{i=1}^{\infty} \frac{1}{r_i} = \infty$, then $\lim_{n\to\infty} \lambda_n^{(1)} = r_1 - 1$.*

Proof. Clearly part (iii) follows immediately from parts (i) and (ii). In what follows we show parts (i) and (ii).

Write

$$F_n := E_n + \operatorname{diag}(r_1 - 1, \ldots, r_n - 1).$$

Note that F_n is positive semi-definite. Consider its characteristic polynomial $\det(\lambda I_n - F_n)$. By Lemma 2.2 we have

$$(-1)^n \det(\lambda I_n - F_n) = \det(E_n + \operatorname{diag}(r_1 - \lambda - 1, \dots, r_n - \lambda - 1))$$
$$= \prod_{i=1}^{n}(r_i - \lambda - 1) + \sum_{1 \le i_1 < \cdots < i_{n-1} \le n} \prod_{j=1}^{n-1}(r_{i_j} - \lambda - 1). \quad (2.8)$$

We then deduce that if $\lambda < r_1 - 1$, then

$$(-1)^n \det(\lambda I_n - F_n) > 0$$

and thus

$$\det(\lambda I_n - F_n) \neq 0.$$

So we have $\lambda_n^{(1)} \ge r_1 - 1$.

If $r_1 = r_2$, then by (2.8) we have

$$(\lambda - r_1 + 1) \mid \det(\lambda I_n - F_n).$$

It follows that $\lambda_n^{(1)} = r_1 - 1$ and this concludes part (i).

Now let $r_2 > r_1$. From (2.8) we deduce

$$(-1)^n \det((r_1 - 1) I_n - F_n) > 0.$$

This implies that $\lambda_n^{(1)} > r_1 - 1$. On the other hand, we have

$$F_n = (r_1 - 1) I_n + E_n + \operatorname{diag}(0, r_2 - r_1, \dots, r_n - r_1).$$

Let $\tilde{\lambda}_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $E_n + \operatorname{diag}(0, r_2 - r_1, \dots, r_n - r_1)$. Then we have

$$\lambda_n^{(1)} = r_1 - 1 + \tilde{\lambda}_n^{(1)}. \quad (2.9)$$

Since $r_1 < r_2$, the proofs of Lemma 2.2 and Corollary 2.3 of [29] yield

$$\tilde{\lambda}_n^{(1)} < \frac{1}{1 + \sum_{i=2}^{n} \frac{1}{r_i - r_1}}. \quad (2.10)$$

So the right-hand side of the inequalities in part (ii) follows immediately from (2.9) and (2.10). The proof of Lemma 2.3 is complete. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. By $f \in \mathcal{C}$ we have

$$f(x) = \sum_{d \mid x} (f * \mu)(d) \ge 0.$$

If $f(x) = 0$, then $(f(x_i, x_j)) = \operatorname{diag}(f(x_1), \ldots, f(x_n))$. Since f is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$, we have $f(x_i) \geq f(x_1)$ for $1 \leq i \leq n$. Thus $\lambda_n^{(1)} = f(x_1)$. So Theorem 1.4(i) is true in this case. Now let $f(x) \neq 0$, so $f(x) > 0$. Obviously we have

$$\frac{1}{f(x)}(f(x_i, x_j)) = E_n + \operatorname{diag}\left(\frac{f(x_1)}{f(x)} - 1, \ldots, \frac{f(x_n)}{f(x)} - 1\right). \tag{2.11}$$

For $1 \leq i \leq n$, let $r_i = \frac{f(x_i)}{f(x)}$. Since f is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$, $\{r_i\}_{i=1}^{\infty}$ is an increasing infinite sequence of real numbers. Since $x \mid x_1$ and $f \in \mathcal{C}$, we have

$$f(x_1) - f(x) = \sum_{d \mid x_1,\ d \nmid x} (f * \mu)(d) \geq 0.$$

So $f(x_1) \geq f(x)$, namely, $r_1 \geq 1$. Let $\bar{\lambda}_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ matrix $\frac{1}{f(x)}(f(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

Suppose first $f(x_1) = f(x_2)$. Then $r_1 = r_2$. Thus by (2.11) and Lemma 2.3(i) we have

$$\bar{\lambda}_n^{(1)} = \frac{f(x_1)}{f(x)} - 1.$$

Theorem 1.4(i) in this case then follows immediately from the fact that

$$\lambda_n^{(1)} = f(x) \cdot \bar{\lambda}_n^{(1)}. \tag{2.12}$$

This completes the proof of Theorem 1.4(i).

Let now $f(x_1) < f(x_2)$, i.e. $r_1 < r_2$. By (2.11) and Lemma 2.3(ii) we have

$$\frac{f(x_1)}{f(x)} - 1 < \bar{\lambda}_n^{(1)} < \frac{f(x_1)}{f(x)} - 1 + \frac{1}{1 + \sum_{i=2}^{n} \frac{f(x)}{f(x_i) - f(x_1)}}.$$

So, by (2.12) part (ii) of Theorem 1.4 follows.

Finally we show part (iii). If $f(x_1) = 0$, then we have $f(x) = 0$ because $f(x_1) \geq f(x) \geq 0$. Then by part (i) we have $\lambda_n^{(1)} = 0$ for $n \geq 1$. Thus Theorem 1.4(iii) holds in this case. If $f(x_1) > 0$ and $\sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty$, then part (iii) in this case follows immediately from parts (i) and (ii). So part (iii) of Theorem 1.4 is proved. □

Proof of Theorem 1.5. If $f(x_1) = 0$, then by Theorem 1.4(iii) we have $\lambda_n^{(1)} = 0$ for all $n \geq 1$. So Theorem 1.5 is true in this case. Now let $f(x_1) > 0$. Then $f(x_i) \geq f(x_1) > 0$ for all $i \geq 1$ because f is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$. Since $f(x_i) \leq Cx_i$ for all $i \geq 1$, we have $0 < f(x_i) \leq Cx_i$ and so $\frac{1}{f(x_i)} \geq \frac{1}{Cx_i}$ for all $i \geq 1$. But by condition (ii), $\sum_{i=1}^{\infty} \frac{1}{x_i} = \infty$. Thus we have $\sum_{i=1}^{\infty} \frac{1}{f(x_i)} = \infty$. The result in this case then follows immediately from Theorem 1.4(iii). □

Definition 2.4 ([29]). Let e and r be positive integers. Let $X = \{x_1, \ldots, x_e\}$ and $Y = \{y_1, \ldots, y_r\}$ be two sets of distinct positive integers. Then we define the *tensor product* (*set*) of X and Y, denoted by $X \odot Y$, by

$$X \odot Y := \{x_1 y_1, \ldots, x_1 y_r, x_2 y_1, \ldots, x_2 y_r, \ldots, x_e y_1, \ldots, x_e y_r\}.$$

Lemma 2.5. *Let f be a multiplicative function. Let e and r be positive integers. Let $X = \{x_1, \ldots, x_e\}$ be a set of e distinct positive integers such that for any $1 \le i \ne j \le e$, $(x_i, x_j) = 1$. Let $Y = \{y_1, \ldots, y_r\}$ be a set of r distinct positive integers such that for any $1 \le i \ne j \le r, (y_i, y_j) = 1$. Assume that for all $1 \le i \le e, 1 \le j \le r, (x_i, y_j) = 1$. Then the following equality holds*:

$$(f(X \odot Y)) = (f(X)) \otimes (f(Y)).$$

Proof. Since f is multiplicative, we have $f(1) = 0$ or $f(1) = 1$. If $f(1) = 0$, then $f(z) = 0$ for every integer $z \ge 1$ because f is multiplicative. Hence we have $(f(X \odot Y)) = (f(X)) \otimes (f(Y)) = O_{er}$, the $er \times er$ zero matrix. So the result holds in this case. Assume now that $f(1) = 1$. Then we have

$$(f(X)) = \begin{pmatrix} f(x_1) & 1 & \ldots & 1 \\ 1 & f(x_2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & f(x_e) \end{pmatrix}$$

and

$$(f(Y)) = \begin{pmatrix} f(y_1) & 1 & \ldots & 1 \\ 1 & f(y_2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & f(y_r) \end{pmatrix}.$$

Since f is multiplicative, we deduce that

$$f(x_{i_1}y_{j_1}, x_{i_2}y_{j_2}) = \begin{cases} f(x_{i_1})f(y_{j_1}) & \text{if } i_1 = i_2 \quad \text{and} \quad j_1 = j_2, \\ f(y_{j_1}) & \text{if } i_1 \ne i_2 \quad \text{and} \quad j_1 = j_2, \\ f(x_{i_1}) & \text{if } i_1 = i_2 \quad \text{and} \quad j_1 \ne j_2, \\ 1 & \text{if } i_1 \ne i_2 \quad \text{and} \quad j_1 \ne j_2. \end{cases}$$

Thus letting $Y_f = (f(Y))$ gives

$$\begin{aligned}(f(X \odot Y)) &= \begin{pmatrix} f(x_1)Y_f & Y_f & \ldots & Y_f \\ Y_f & f(x_2)Y_f & \ldots & Y_f \\ \vdots & \vdots & \ddots & \vdots \\ Y_f & Y_f & \ldots & f(x_e)Y_f \end{pmatrix} \\ &= \begin{pmatrix} f(x_1) & 1 & \ldots & 1 \\ 1 & f(x_2) & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & f(x_e) \end{pmatrix} \otimes Y_f \\ &= (f(X)) \otimes (f(Y))\end{aligned}$$

as required. $\square$

Remark 2.6. If f is *not* multiplicative, then Lemma 2.2 may fail to be true. For instance, let $X = \{1, 2\}$ and $Y = \{3, 5\}$. Then $X \odot Y = \{3, 5, 6, 10\}$. Let f be the arithmetical function defined by $f(l) = l$ for $l \neq 10$ and $f(10) = 9$. Then f is not multiplicative since $f(10) \neq f(2)f(5)$. On the other hand, we have $((f(X)) \otimes (f(Y)))_{44} = 10$ and $(f(X \odot Y))_{44} = 9$. This implies that $(f(X \odot Y)) \neq (f(X)) \otimes (f(Y))$.

We are now in a position to prove Theorem 1.6.

Proof of Theorem 1.6. First we prove part (i). For convenience we let $h := f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)}$ and $h(p_i(b)) = 0$ for some $i \geq 1$. Then $(p_i(b), a + be) = 1$ or $(p_i(b), a + b(e + 1)) = 1$. Otherwise we have $p_i(b) \,|\, (a + be)$ and $p_i(b) \,|\, (a + b(e + 1))$. It implies $p_i(b) \,|\, b$ and so $p_i(b) \leq b$. This is absurd since $p_i(b) \geq 1 + b$. We may let $(p_i(b), a + b(e + j)) = 1$, where $j = 0$, or 1. For any integer $m \geq q$, let $\nu_m^{(1)}(l_1, \ldots, l_c, d) \leq \cdots \leq \nu_m^{(m)}(l_1, \ldots, l_c, d)$ be the eigenvalues of the $m \times m$ matrix $(h(V_m))$ defined on the set

$$V_m : = \{(a + b(e + j))p_i(b), (a + b(e + j))p_i(b)p_{i+w+1}(b)), \ldots, \\ (a + b(e + j))p_i(b)p_{i+w+m-1}(b))\},$$

where $w \geq 0$ and $a + b(e + j) < p_{i+w+1}(b)) < \cdots < p_{i+w+m-1}(b)$. Clearly h is multiplicative since $f_1, \ldots, f_c$ and μ are multiplicative. For each $1 \leq l \leq m - 1$, since $p_i(b), a + b(e + j)$ and $p_{i+w+l}(b)$ are mutually coprime, and note also that $h(p_i(b)) = 0$, we have $h((a + b(e + j))p_i(b)p_{i+w+l}(b)) = h(a + b(e + j))h(p_i(b))h(p_{i+w+l}(b)) = 0$. Thus we have $(h(V)) = O_{m\times m}$, the $m\times m$ zero matrix. So we have $\nu_m^{(i)}(l_1, \ldots, l_c, d) = 0$ for all $1 \leq i \leq m$. But by Cauchy's interlacing inequalities we have for any large enough n,

$$\lambda_n^{(q)}(l_1, \ldots, l_c, d) \leq \nu_m^{(q)}(l_1, \ldots, l_c, d).$$

On the other hand, Theorem 1.2(ii) gives $\lambda_n^{(q)}(l_1, \ldots, l_c, d) \geq 0$. So we have $\lambda_n^{(q)}(l_1, \ldots, l_c, d) = 0$. This completes the proof of part (i) of Theorem 1.6.

From now on we assume that $h(p_i(b)) \neq 0$ for all $i \geq 1$. Next we prove Theorem 1.6(ii) for the case $l_1 = c = 1$ and $d = 0$. Then we have $h = f_1$.

Let $\{1 + bt_i\}_{i=0}^{\infty}$ be the sequence consisting of all those elements in the sequence $\{1 + bi\}_{i=0}^{\infty}$ which are coprime to $a + be$. So $(1 + bt_i, a + be) = 1$ for all $i \geq 0$. Then this is an infinite sequence because it contains the set of all primes strictly greater than $a+be$ in $\{1+bi\}_{i=1}^{\infty}$, which is infinite by Dirichlet's theorem. For the arithmetic progression $\{a + bi\}_{i=e}^{\infty}$, consider its subsequence

$$\{a + b(e + (a + be)t_i)\}_{i=0}^{\infty} = \{(a + be)(1 + bt_i)\}_{i=0}^{\infty}.$$

For any integer $m \geq 1$, let $\gamma_m^{(1)} \leq \cdots \leq \gamma_m^{(m)}$ be the eigenvalues of the $m \times m$ matrix $(f_1(W_m))$ defined on the set

$$W_m := \{a + be, (a + be)(1 + bt_1), \ldots, (a + be)(1 + bt_{m-1})\}$$

and let $\tilde{\gamma}_m^{(1)} \le \cdots \le \tilde{\gamma}_m^{(m)}$ be the eigenvalues of the $m \times m$ matrix $(f_1(\tilde{W}_m))$ defined on the set

$$\tilde{W}_m := \{1, 1 + bt_1, \ldots, 1 + bt_{m-1}\}.$$

Since f_1 is multiplicative and $(a + be, 1 + bt_i) = 1$, we have $(f_1(W_m)) = f_1(a + be)(f_1(\tilde{W}_m))$. So we have $\gamma_m^{(i)} = f_1(a + be)\tilde{\gamma}_m^{(i)}$ for $1 \le i \le m$. In particular,

$$\gamma_m^{(q)} = f_1(a + be)\tilde{\gamma}_m^{(q)}. \tag{2.13}$$

Now let m_n be the largest integer l such that

$$t_{l-1} \le \left\lfloor \frac{n-1}{a+be} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer $\le x$. Clearly $m_n \to \infty$ as $n \to \infty$. Choose n so that $m_n \ge q$.

By Cauchy's interlacing inequalities

$$\lambda_n^{(q)}(1, 0) \le \gamma_{m_n}^{(q)}, \tag{2.14}$$

and by (2.13) and (2.14),

$$\lambda_n^{(q)}(1, 0) \le f_1(a + be)\tilde{\gamma}_{m_n}^{(q)}. \tag{2.15}$$

We claim that $\lim_{m\to\infty} \tilde{\gamma}_m^{(q)} = 0$. Then we have $\lim_{n\to\infty} \tilde{\gamma}_{m_n}^{(q)} = 0$. Thus by Theorem 1.3 and (2.15) we get $\lim_{n\to\infty} \lambda_n^{(q)}(1, 0) = 0$ as desired. It remains to prove the assertion which will be done in the following.

Let $p_1 < p_2 < \cdots$ denote the primes in the sequence $\{1 + bt_i\}_{i=0}^{\infty}$. Then $\{p_i(b)\}_{i=s}^{\infty} \subset \{p_i\}_{i=1}^{\infty}$, where $p_{s-1}(b) \le a + be < p_s(b)$, $s \ge 1$ is an integer and $p_0(b) := 1$. Let

$$Q := \{p_i(b)\}_{i=1}^{\infty} \setminus \{p_i\}_{i=1}^{\infty}.$$

Then Q is a finite set. Since $f_1(p_i(b)) \ne 0$ for all $i \ge 1$, we have $\sum_{p\in Q} \frac{1}{f_1(p)} < \infty$. So by the assumption $\sum_{i=1}^{\infty} \frac{1}{f_1(p_i(b))} = \infty$ we have

$$\sum_{i=1}^{\infty} \frac{1}{f_1(p_i)} = \infty. \tag{2.16}$$

For $i \ge 1$, let $\pi_i = p_{q-1+i}$. Then $p_{q-1} < \pi_1 < \cdots$. Since q is a fixed number, it follows from (2.16) that

$$\sum_{i=1}^{\infty} \frac{1}{f_1(\pi_i)} = \infty. \tag{2.17}$$

Now let $r \ge 2$ be an arbitrary integer and let

$$P_q := \{1, p_1, \ldots, p_{q-1}\}, \quad T_r := \{1, \pi_1, \ldots, \pi_{r-1}\}.$$

It is clear that the matrices $(f_1(P_q))$ and $(f_1(T_r))$ are positive semi-definite. Consider the tensor product set $P_q \odot T_r$. Note that the entries in the set $P_q \odot T_r$ are not arranged in increasing order, but the eigenvalues of the corresponding matrix do not depend on rearranging those entries. Since f_1 is multiplicative, by Lemma 2.5 we have

$$(f_1(P_q \odot T_r)) = (f_1(P_q)) \otimes (f_1(T_r)).$$

Let $\delta_q^{(1)} \leq \cdots \leq \delta_q^{(q)}$ and $\tilde{\lambda}_r^{(1)} \leq \cdots \leq \tilde{\lambda}_r^{(r)}$ be the eigenvalues of the matrix $(f_1(P_q))$ defined on the set P_q and the matrix $(f_1(T_r))$ defined on the set T_r, respectively. Then it is known (see [33]) that the eigenvalues of the tensor product matrix $(f_1(P_q)) \otimes (f_1(T_r))$ are given by the set

$$\{\delta_q^{(i)} \cdot \tilde{\lambda}_r^{(j)}\}_{1\leq j\leq r}^{1\leq i\leq q}.$$

Notice that

$$\delta_q^{(1)} \cdot \tilde{\lambda}_r^{(1)} \leq \cdots \leq \delta_q^{(q)} \cdot \tilde{\lambda}_r^{(1)}. \tag{2.18}$$

Clearly the sequence $\{1+bt_i\}_{i=0}^{\infty}$ is closed under the usual multiplication. So the tensor product set $P_q \odot T_r \subset \{1+bt_i\}_{i=0}^{\infty}$. For any integer $r \geq 2$, define an integer m_r by

$$m_r := \frac{p_{q-1} \cdot \pi_{r-1} - 1}{b} + 1.$$

Then $P_q \odot T_r \subseteq \{1+bt_i\}_{i=0}^{m_r-1}$. Thus the matrix $(f_1(P_q \odot T_r))$ defined on $P_q \odot T_r$ is a principal submatrix of the $m_r \times m_r$ matrix $(f_1(1+bt_i, 1+bt_j))$ defined on the set $\{1, 1+bt_1, \ldots, 1+bt_{m_r-1}\}$. Let $\bar{\lambda}_{qr}^{(1)} \leq \cdots \leq \bar{\lambda}_{qr}^{(qr)}$ be the eigenvalues of $(f_1(P_q \odot T_r))$. Then by Cauchy's interlacing inequalities we have

$$\tilde{\gamma}_{m_r}^{(q)} \leq \bar{\lambda}_{qr}^{(q)}. \tag{2.19}$$

But by (2.18)

$$\bar{\lambda}_{qr}^{(q)} \leq \delta_q^{(q)} \cdot \tilde{\lambda}_r^{(1)}. \tag{2.20}$$

So it follows from (2.19) and (2.20) that

$$\tilde{\gamma}_{m_r}^{(q)} \leq \delta_q^{(q)} \cdot \tilde{\lambda}_r^{(1)}. \tag{2.21}$$

On the other hand, in Theorem 1.4, if we choose $x = x_1 = 1$ and $x_i = \pi_{i-1}$ for $i \geq 2$, then by (2.17) the conditions of Theorem 1.4 are satisfied. It then follows immediately from Theorem 1.4 that

$$\lim_{r\to\infty} \tilde{\lambda}_r^{(1)} = 0. \tag{2.22}$$

But by Theorem 1.3 we have that the subsequence $\{\tilde{\gamma}_{m_r}^{(q)}\}_{r=1}^{\infty}$ of the sequence $\{\tilde{\gamma}_m^{(q)}\}_{m=q}^{\infty}$ converges and

$$\lim_{r\to\infty} \tilde{\gamma}_{m_r}^{(q)} \geq 0. \tag{2.23}$$

Hence by (2.21)–(2.23), $\lim_{r\to\infty} \tilde{\gamma}_{m_r}^{(q)} = 0$. Finally, again by Theorem 1.2, the desired result $\lim_{m\to\infty} \tilde{\gamma}_m^{(q)} = 0$ follows immediately. The claim is proved and this completes the proof of part (ii) of Theorem 1.6 for the case $l_1 = c = 1$ and $d = 0$.

Now consider part (ii) for the general case. In the case $l_1 = c = 1$ and $d = 0$, we replace f_1 by $h = f_1^{(l_1)} * \cdots * f_c^{(l_c)} * \mu^{(d)}$. Since $f_1, \ldots, f_c \in \mathcal{C}$ and $l_1 + \cdots + l_c > d$, by Theorem 1.1(i) we have $h \in \mathcal{C}$. Thus $h(p_i(b)) \geq 0$ for all $i \geq 1$. So by the assumption $h(p_i(b)) \neq 0$ for all $i \geq 1$ we have that $h(p_i(b)) > 0$ for all $i \geq 1$. Note that h is multiplicative. On the other hand, h is increasing on the sequence $\{p_i(b)\}_{i=1}^{\infty}$ because of the formula in Theorem 1.1(ii). It remains to prove that

$$\sum_{i=1}^{\infty} \frac{1}{h(p_i(b))} = \infty. \tag{2.24}$$

But Theorem 1.1(ii) tells us

$$h(p_i(b)) = \sum_{j=1}^{c} l_j f_j(p_i(b)) - d < \sum_{j=1}^{c} l_j f_j(p_i(b)),$$

So we have

$$\sum_{i=1}^{\infty} \frac{1}{h(p_i(b))} \geq \sum_{i=1}^{\infty} \frac{1}{\sum_{j=1}^{c} l_j f_j(p_i(b))} \geq \frac{1}{l} \sum_{i=1}^{\infty} \frac{1}{\sum_{j=1}^{c} f_j(p_i(b))}, \tag{2.25}$$

where $l = \max_{1\leq j\leq c} l_j$. Hence (2.24) follows immediately from (2.25) and the condition of Theorem 1.6(ii). So Theorem 1.6(ii) for the general case follows from Theorem 1.6(ii) for the case $l_1 = c = 1$ and $d = 0$. The proof of Theorem 1.6(ii) is complete.

Finally Theorem 1.6(iii) follows from parts (i) and (ii) and Mertens' theorem [40]. This concludes the proof of Theorem 1.6. □

3. Examples

In the present section we give several examples to demonstrate our main results.

Example 3.1. Let $f = \xi_\varepsilon$, where ξ_ε is defined as in Sec. 1 and ε is a real number. Then ξ_ε is increasing on any strictly increasing infinite sequence, and completely multiplicative if $\varepsilon \geq 0$. Let $J_\varepsilon := \xi_\varepsilon * \mu$. Then $J_\varepsilon(1) = 1$ and for any integer $m > 1$,

$$J_\varepsilon(m) = m^\varepsilon \prod_{p\,|\,m} \left(1 - \frac{1}{p^\varepsilon}\right) \geq 0$$

if $\varepsilon \geq 0$. Thus $\xi_\varepsilon \in \mathcal{C}_S$ for any set S of positive integers and so $\xi_\varepsilon \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers if $\varepsilon \geq 0$. For integers

$c > d \geq 0$, let $\lambda_n^{(1)}(c,d) \leq \cdots \leq \lambda_n^{(n)}(c,d)$ be the eigenvalues of the $n \times n$ matrix $((\xi_\varepsilon^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

(i) By Theorem 1.4(ii) we get: If $\varepsilon > 0$ and S_n satisfies that for every $1 \leq i \neq j \leq n$, $(x_i, x_j) = x$, then we have

$$x_1^\varepsilon - x^\varepsilon \leq \lambda_n^{(1)}(1,0) < x_1^\varepsilon - x^\varepsilon + \frac{x^\varepsilon}{1 + \sum_{i=2}^{n} \frac{x^\varepsilon}{x_i^\varepsilon - x_1^\varepsilon}}.$$

(ii) By Theorem 1.4(iii) we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ consisting of all but finitely many primes, we have $(x_i, x_j) = 1$ for every $i \neq j$, and by Mertens' theorem [40] we have $\sum_{i=1}^\infty \frac{1}{x_i^\varepsilon} = \infty$ if $\varepsilon \leq 1$. So if $0 \leq \varepsilon \leq 1$, then we have $\lim_{n\to\infty} \lambda_n^{(1)}(1,0) = x_1^\varepsilon - 1$ (see [29]).

(iii) By Theorem 1.8 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers which contains the arithmetic progression $\{a + bi\}_{i=e}^\infty$ as its subsequence, where $a,\ b \geq 1$ and $e \geq 0$ are integers, if $0 \leq \varepsilon \leq 1$, then for any given integer $q \geq 1$, we have $\lim_{n\to\infty}\lambda_n^{(q)}(c,d) = 0$.

Example 3.2. Let $f = J_\varepsilon$, where ε is a real number and J_ε is defined in Example 3.1. Note that if ε is a positive integer, then J_ε becomes Jordan's totient function (see, for example, [1, 38] or [41]). Clearly $J_\varepsilon * \mu$ is multiplicative and $(J_\varepsilon * \mu)(1) = 1$. It is easy to see that if p is an odd prime number and $\varepsilon \geq \frac{\log 2}{\log 3}$, then $(J_\varepsilon * \mu)(p) = p^\varepsilon - 2 \geq 0$. For any prime p and integer $l \geq 2$, we have $(J_\varepsilon * \mu)(p^l) = p^{(l-2)\varepsilon}(p^\varepsilon - 1)^2 > 0$. Thus $J_\varepsilon \in \mathcal{C}_S$ for any set S of positive odd numbers and so $J_\varepsilon \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive odd numbers if $\varepsilon \geq \frac{\log 2}{\log 3}$. On the other hand, if $\varepsilon \geq 0$, then for any primes $3 \leq p_1 < p_2$, we have $J_\varepsilon(p_1) = p_1^\varepsilon - 1 \leq p_2^\varepsilon - 1 = J_\varepsilon(p_2)$ and for any integer $m \geq 2$, we have $J_\varepsilon(m) \leq m^\varepsilon$. For integers $c > d \geq 0$, let $\lambda_n^{(1)}(c,d) \leq \cdots \leq \lambda_n^{(n)}(c,d)$ be the eigenvalues of the $n \times n$ matrix $((J_\varepsilon^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

(i) By Theorem 1.4(ii) we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ consisting of all but finitely many odd primes, if $\frac{\log 2}{\log x_1} \leq \varepsilon < 1$, then we have

$$x_1^\varepsilon - 2 < \lambda_n^{(1)}(1,0) < x_1^\varepsilon - 2 + \frac{1}{1 + \sum_{i=2}^{n} \frac{1}{x_i^\varepsilon - x_1^\varepsilon}},$$

Furthermore by Theorem 1.5, $\lim_{n\to\infty}\lambda_n^{(1)}(1,0) = x_1^\varepsilon - 2$,

(ii) By Theorem 1.8 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive odd numbers which contains the arithmetic progression $\{a + bi\}_{i=e}^\infty$ as its subsequence, where $a,\ b \geq 1$ and $e \geq 0$ are integers, if $\frac{\log 2}{\log 3} \leq \varepsilon < 1$, then for any given integer $q \geq 1$, we have $\lim_{n\to\infty}\lambda_n^{(q)}(c,d) = 0$.

Example 3.3. Let $f = \sigma_\varepsilon := \xi_\varepsilon * \xi_0$, where ε is a real number. Then for any positive integer m we have

$$\sigma_\varepsilon(m) = \sum_{d\,|\,m} d^\varepsilon.$$

The function $d(m) = \sigma_0(m)$ is the usual divisor function. The function $\sigma(m) = \sigma_1(m)$ gives the sum of the divisors of m. Clearly σ_ε is multiplicative. Since $\sigma_\varepsilon * \mu = \xi_\varepsilon * \xi_0 * \mu = \xi_\varepsilon$, we have $(\sigma_\varepsilon * \mu)(m) = m^\varepsilon > 0$ for any integer $m \geq 1$. So $\sigma_\varepsilon \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers. Obviously if $\varepsilon \geq 0$ and $p_1 < p_2$ are primes, then $\sigma_\varepsilon(p_1) = 1 + p_1^\varepsilon \leq 1 + p_2^\varepsilon = \sigma_\varepsilon(p_2)$. For integers $c > d \geq 0$, let $\lambda_n^{(1)}(c,d) \leq \cdots \leq \lambda_n^{(n)}(c,d)$ be the eigenvalues of the $n \times n$ matrix $((\sigma_\varepsilon^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

(i) By Theorem 1.4(ii) we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ consisting of all the primes in $\mathbf{Z}^+$ except finitely many of them, if $\varepsilon > 0$, then we have

$$x_1^\varepsilon < \lambda_n^{(1)}(1,0) < x_1^\varepsilon + \frac{1}{1 + \sum_{i=2}^{n} \frac{1}{x_i^\varepsilon - x_1^\varepsilon}}.$$

Furthermore by Theorem 1.4(iii), if $0 \leq \varepsilon \leq 1$, then we have $\lim_{n\to\infty} \lambda_n^{(1)}(1,0) = x_1^\varepsilon$.

(ii) By Theorem 1.8 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers which contains the arithmetic progression $\{a + bi\}_{i=e}^\infty$ as its subsequence, where $a,\ b \geq 1$ and $e \geq 0$ are integers, since

$$\sum_{i=1}^\infty \frac{1}{\sigma_\varepsilon(p_i(b))} = \sum_{i=1}^\infty \frac{1}{p_i(b)^\varepsilon + 1} \geq \frac{1}{2} \sum_{i=1}^\infty \frac{1}{p_i(b)^\varepsilon}$$

if $\varepsilon \geq 0$, we deduce that if $0 \leq \varepsilon \leq 1$

$$\sum_{i=1}^\infty \frac{1}{\sigma_\varepsilon(p_i(b))} = \infty.$$

Then for any given integer $q \geq 1$, if $0 \leq \varepsilon \leq 1$, we have $\lim_{n\to\infty} \lambda_n^{(q)}(c,d) = 0$.

Example 3.4. Let $f = \psi_\varepsilon$, where ε is a real number and ψ_ε is defined for any positive integer m by

$$\psi_\varepsilon(m) := \sum_{d\,|\,m} d^\varepsilon \left|\mu\left(\frac{m}{d}\right)\right|.$$

The function ψ_1 is called Dedekind's function (see, for instance, [38]). Clearly ψ_ε is multiplicative. Then for any positive integer m we have

$$\psi_\varepsilon(m) = m^\varepsilon \prod_{p\,|\,m} \left(1 + \frac{1}{p^\varepsilon}\right) = \frac{J_{2\varepsilon}(m)}{J_\varepsilon(m)}.$$

Thus for any positive integer l and any prime p, we have

$$(\psi_\varepsilon * \mu)(p^l) = \begin{cases} p^\varepsilon & \text{if } l = 1, \\ p^{(l-2)\varepsilon}(p^{2\varepsilon} - 1) & \text{if } l \geq 2. \end{cases}$$

If $\varepsilon \geq 0$, then $\psi_\varepsilon \in \mathcal{C}$ for any given strictly increasing infinite sequences $\{x_i\}_{i=1}^\infty$ of positive integers. For integers $c > d \geq 0$, let $\lambda_n^{(1)}(c,d) \leq \cdots \leq \lambda_n^{(n)}(c,d)$ be the eigenvalues of the $n \times n$ matrix $((\psi_\varepsilon^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

(i) By Theorem 1.4(ii) we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ consisting of all the primes in $\mathbf{Z}^+$ except finitely many of them, if $\varepsilon > 0$, then we have

$$x_1^\varepsilon < \lambda_n^{(1)}(1,0) < x_1^\varepsilon + \frac{1}{1 + \displaystyle\sum_{i=2}^n \frac{1}{x_i^\varepsilon - x_1^\varepsilon}}.$$

Furthermore by Theorem 1.4(iii), if $0 \leq \varepsilon \leq 1$, then we have $\lim_{n\to\infty} \lambda_n^{(1)}(1,0) = x_1^\varepsilon$.

(ii) By Theorem 1.8 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers which contains the arithmetic progression $\{a+bi\}_{i=e}^\infty$ as its subsequence, where a, $b \geq 1$ and $e \geq 0$ are integers, in a same way as in Example 3.3, we can check that for $\varepsilon \geq 0$, ψ_ε is increasing on the sequence $\{p_i(b)\}_{i=1}^\infty$ and if $0 \leq \varepsilon \leq 1$

$$\sum_{i=1}^\infty \frac{1}{\psi_\varepsilon(p_i(b))} = \infty.$$

Then for any given integer $q \geq 1$, if $0 \leq \varepsilon \leq 1$, we have $\lim_{n\to\infty} \lambda_n^{(q)}(c,d) = 0$.

Example 3.5. Let $f = \phi$, Euler's totient function. Clearly ϕ and $\phi * \mu$ are multiplicative, and $\phi(1) = (\phi*\mu)(1) = 1$. For any prime p we have $(\phi*\mu)(p) = \phi(p) - 1 = p - 2 \geq 0$, and for any integer $l \geq 2$ we have

$$(\phi * \mu)(p^l) = \sum_{i=1}^l \phi(p^i)\mu(p^{l-i}) = \phi(p^l) - \phi(p^{l-1}) = p^{l-2}(p-1)^2 > 0.$$

Thus $\phi \in \mathcal{C}_S$ for any set S of positive integers and so $\phi \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ of positive integers. Note that $\phi(p) = p - 1 \leq p$. So for any primes $p_1 < p_2$, $\phi(p_1) < \phi(p_2)$. For integers $c > d \geq 0$, let $\lambda_n^{(1)}(c,d) \leq \cdots \leq \lambda_n^{(n)}(c,d)$ be the eigenvalues of the $n \times n$ matrix $((\phi^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

(i) By Theorem 1.4(ii) we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^\infty$ consisting of all the primes in $\mathbf{Z}^+$ except finitely many of them, we have

$$x_1 - 2 < \lambda_n^{(1)}(1,0) < x_1 - 2 + \frac{1}{1 + \displaystyle\sum_{i=2}^n \frac{1}{x_i - x_1}}.$$

Furthermore, by Theorem 1.5 we have $\lim_{n\to\infty} \lambda_n^{(1)}(1,0) = x_1 - 2$.

(ii) By Theorem 1.8 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers which contains the arithmetic progression $\{a+bi\}_{i=e}^{\infty}$ as its subsequence, where $a,\ b \geq 1$ and $e \geq 0$ are integers, and for any given integer $q \geq 1$, we have $\lim_{n\to\infty}\lambda_n^{(q)}(c,d) = 0$.

Example 3.6. Let $f_1 = \xi_\varepsilon$ and $f_2 = \phi$ be as in Examples 3.1 and 3.5, respectively. Clearly ξ_ε and ϕ are distinct and multiplicative. Note that ξ_ε is increasing on any strictly increasing infinite sequence of positive integers if $\varepsilon \geq 0$ and ϕ is increasing on any subsequence of strictly increasing infinite sequence consisting of all the primes in $\mathbf{Z}^+$. By Examples 3.1 and 3.5 we know that $\xi_\varepsilon \in \mathcal{C}$ and $\phi \in \mathcal{C}$ for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers if $\varepsilon \geq 0$. For integers $c_1 > 0, c_2 > 0$ and $d \geq 0$, let $\lambda_n^{(1)}(c_1,c_2,d) \leq \cdots \leq \lambda_n^{(n)}(c_1,c_2,d)$ be the eigenvalues of the $n \times n$ matrix $((\xi_\varepsilon^{(c_1)} * \phi^{(c_2)} * \mu^{(d)})(x_i,x_j))$ defined on the set $S_n = \{x_1,\ldots,x_n\}$. Since for any prime p, we have $\phi(p) \leq p$ and $\xi_\varepsilon(p) \leq p$ if $\varepsilon \leq 1$, then by Theorem 1.7 we get: For any given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers which contains the arithmetic progression $\{a+bi\}_{i=e}^{\infty}$ as its subsequence, where $a,\ b \geq 1$ and $e \geq 0$ are integers, if $0 \leq \varepsilon \leq 1$ and $c_1 + c_2 > d$, then for any given integer $q \geq 1$, we have $\lim_{n\to\infty}\lambda_n^{(q)}(c_1,c_2,d) = 0$.

4. Open Questions

Let $\{x_i\}_{i=1}^{\infty}$ be an arbitrary strictly increasing infinite sequence of positive integers. For an integer $n \geq 1$, let $S_n = \{x_1,\ldots,x_n\}$. Let $c,q \geq 1$ and $d \geq 0$ be given integers. Let $\lambda_n^{(1)}(c,d) \leq \cdots \leq \lambda_n^{(n)}(c,d)$ be the eigenvalues of the matrix $((f^{(c)} * \mu^{(d)})(x_i,x_j))$ defined on the set S_n. It follows from Theorem 1.4 that if $\{x_i\}_{i=1}^{\infty}$ is a strictly increasing infinite sequence of positive integers satisfying that for every $i \neq j$, $(x_i,x_j) = x_1$ and $f \in \mathcal{C}$ is increasing on the sequence $\{x_i\}_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}\frac{1}{f(x_i)} = \infty$, then $\lim_{n\to\infty}\lambda_n^{(1)}(1,0) = 0$. Then by Cauchy's interlacing inequalities and Theorem 1.3 we know that for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers which contains a subsequence $\{x'_i\}_{i=1}^{\infty}$ satisfying that for every $i \neq j$, $(x'_i,x'_j) = x'_1$, if $f \in \mathcal{C}$ (with respect to the whole sequence $\{x_i\}_{i=1}^{\infty}$) and f is increasing on the sequence $\{x'_i\}_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}\frac{1}{f(x'_i)} = \infty$, then $\lim_{n\to\infty}\lambda_n^{(1)}(1,0) = 0$. (Note that this holds when some $f(x'_i)$ is 0.) On the other hand, by Theorem 1.8 we know that for any given strictly increasing infinite sequence $\{x_i\}_{i=1}^{\infty}$ of positive integers containing the arithmetic progression $\{a+bi\}_{i=e}^{\infty}$ as its subsequence, if $c > d \geq 0$ and $f \in \mathcal{C}$ is multiplicative and increasing on the sequence $\{p_i(b)\}_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}\frac{1}{f(p_i(b))} = \infty$, where $p_i(b)(i \geq 1)$ is defined as in (1.1), then for any given integer $q \geq 1$, we have $\lim_{n\to\infty}\lambda_n^{(q)}(c,d) = 0$. First we would like to understand for what sequences $\{x_i\}_{i=1}^{\infty}$, $\lim_{n\to\infty}\lambda_n^{(1)}(c,d) = 0$. Namely, we have the following question.

Question 4.1. Given any multiplicative function f, and given non-negative integers c and d such that $c > d$, characterize all strictly increasing infinite sequences

$\{x_i\}_{i=1}^{\infty}$ of positive integers so that $\lim_{n\to\infty}\lambda_n^{(1)}(c,d) = 0$, where, as before, $\lambda_n^{(1)}(c,d)$ is the smallest eigenvalue of the matrix $((f^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

Consequently, we raise a further problem.

Question 4.2. The same as the previous question, with $\lambda_n^{(1)}(c,d)$ is replaced by $\lambda_n^{(q)}(c,d)$, where, as before, $\lambda_n^{(q)}(c,d)$ is the qth smallest eigenvalue of the matrix $((f^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$.

In concluding this paper we propose the following question and conjecture.

Question 4.3. Let $c > d \geq 0$ be given integers and $\{x_i\}_{i=1}^{\infty}$ be an arbitrary strictly increasing infinite sequence of positive integers. Let $\lambda_n^{(1)}(c,d)$ be the smallest eigenvalue of the $n \times n$ matrix $((f^{(c)} * \mu^{(d)})(x_i, x_j))$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. Assume that $f \in \mathcal{C}$ is multiplicative. Are the following true:

(i) If f satisfies that $f(x_i) \geq Cx_i^{\varepsilon}$ for all $i \geq 1$, where $\varepsilon > 1$ and $C > 0$ are constants, do we have $\lim_{n\to\infty}\lambda_n^{(1)}(c,d) > 0$?

(ii) If f satisfies that $\sum_{i=1}^{\infty} \frac{1}{f(x_i)} < \infty$, do we have $\lim_{n\to\infty}\lambda_n^{(1)}(c,d) > 0$?

Conjecture 4.4. *Let $\varepsilon > 0$ and $\{x_i\}_{i=1}^{\infty}$ be an arbitrary given strictly increasing infinite sequence of positive integers. Let $\lambda_n^{(1)}$ be the smallest eigenvalue of the $n \times n$ power GCD matrix $((x_i, x_j)^{\varepsilon})$ defined on the set $S_n = \{x_1, \ldots, x_n\}$. If $\sum_{i=1}^{\infty} \frac{1}{x_i^{\varepsilon}} < \infty$, then we have $\lim_{n\to\infty}\lambda_n^{(1)} > 0$.*

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