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Differential properties and optimality conditions for generalized weak vector variational inequalities

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Abstract In this paper, we study a generalized weak vector variational inequality, which is a generalization of a weak vector variational inequality and a Minty weak vector variational inequality. By virtue of a contingent derivative and a Φ-contingent cone, we investigate differential properties of a class of set-valued maps and obtain an explicit expression of its contingent derivative. We also establish some necessary optimality conditions for solutions of the generalized weak vector variational inequality, which generalize the corresponding results in the literature. Furthermore, we establish some unified necessary and sufficient optimality conditions for local optimal solutions of the generalized weak vector variational inequality. Simultaneously, we also show that there is no gap between the necessary and sufficient conditions under an appropriate condition.

Keywords Contingent derivative · Gap function · Generalized weak vector variational inequality · Optimality conditions

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1 Introduction

The vector variational inequality (VVI, in short) and weak vector variational inequality (WVVI, in short) were firstly introduced by Giannessi [8] in a finite-dimensional spaces. These problems have been of great interest in the academic and professional communities in the last few decades. Various kinds of variational inequalities have been discussed and a lot of important results have been established. Especially, a great deal of researches on the existence of solutions (see [4,6,19]) and the stability of the solution set map (see [13,14]) have been obtained.

To the best of our knowledge, the concept of gap function is very useful to study (VVIs). Until now, there were two kinds of gap function introduced to (VVIs). Yang and Yao [19] introduced a kind of gap function for (VVIs) as a real-valued function. In [5], Chen et al. defined another kind of gap function for (VVIs) as a set-valued map from the vector optimization point of view. Moreover, Li et al. [15] discussed the differential and sensitivity properties of the set-valued gap function for (VVIs) and (WVVIs) and obtained an explicit expression of the contingent derivative for a class of set-valued maps, and some necessary optimality conditions under some suitable coerciveness condition, respectively. In [16], Li and Zhai introduced a asymptotic second-order Φ-contingent cone, and discussed the second-order asymptotic differential properties and some necessary optimality conditions for (WVVIs).

Recently, the Minty vector variational inequality (MVVI, in short) and Minty weak vector variational inequality (MWVVI, in short) have also received extensive attentions, and many important results have been established. Giannessi [9] investigated some relationships between a solution of a (MVVI) and an efficient solution or a weakly efficient solution of a vector optimization problem under convexity and monotonicity conditions. Subsequently, Yang et al. [18] established some relations between a (MVVI) and a vector optimization problem under pseudoconvexity or pseudomonotonicity conditions, respectively. In [17], Meng and Li introduced a Φ-contingent cone and obtained an explicit expression of the contingent derivative for a class of set-valued maps without any coerciveness condition. They also defined a kind of gap function for (MVVIs) and (MWVVIs) as set-valued maps, discussed the differential and sensitivity properties of gap functions and got some necessary optimality conditions for (MVVIs) and (MWVVIs), respectively.

It is well known that the (VVI) is closely related to vector optimization problems [8,9,12], vector complementarity problems [7], vector equilibria problems [20] and so on. Moreover, there are many real world applications, such as economic or engineering problems, can be modeled by means of the (VVI) and some of its variants. Recently, the (MVVI) has been regarded as a dual form of the (VVI) based on its close relationship to the classical vector variational inequality. It has been shown, in [9,10,18], that the (MVVI) has a lot of important applications to standard optimality topics by using some generalized convexity and monotonicity assumptions. Motivated by the work reported in [9,15–17,19], we investigate a generalized weak vector variational inequality (GWVVI, in short) in this paper, which is more extensive than the (WVVI) and the (MWVVI). Given a nonempty subset K of $\mathbb{R}^n$ and a map $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, where $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the set of all linear continuous operators from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $C \subset \mathbb{R}^m$ be a pointed closed convex cone with nonempty interior $\mathrm{int}C$. The (GWVVI) is to find $x^* \in K$ such that

$$\langle F(x^*, x), x^* - x\rangle \notin \mathrm{int}C, \forall x \in K.$$

Specially,

Case (*) if $F(x, z) \equiv H(x), \forall x, z \in \mathbb{R}^n$ for some set-valued map H, then, the (GWVVI) reduces to the (WVVI).

Case (**) if $F(x, z) \equiv Q(z), \forall x, z \in \mathbb{R}^n$ for some set-valued map Q, then, the (GWVVI) reduces to the (MWVVI).

Thus, the (GWVVI) is more general than the (WVVI) and the (MWVVI). In order to establish the optimality conditions for the (GWVVI), by the similar method in [15–17], we firstly discuss the differential properties of a class of set-valued maps related to the (GWVVI) as following:

$$G(x) := \bigcup_{z\in K} \langle F(x, z), x - z\rangle,$$

and obtain an explicit expression of the contingent derivative of G. Then, we establish some necessary optimality conditions for solutions of the (GWVVI). Simultaneously, in order to obtain some unified optimality conditions, we propose the concept of local optimal solutions of the (GWVVI) and establish some necessary and sufficient optimality conditions.

The organization of this paper is as follows. In Sect. 2, we recall some basic concepts and properties. In Sect. 3, we get a general formula, which computes the contingent derivative of a class of set-valued maps and generalizes the corresponding results in [15–17] without any coerciveness condition. In Sect. 4, we introduce the concepts of a gap function and a local optimal solution for the (GWVVI), and establish some necessary and sufficient optimality conditions.

2 Notations and preliminaries

Throughout this paper, let K be a subset of $\mathbb{R}^n$ and $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of all linear continuous operators from $\mathbb{R}^n$ to $\mathbb{R}^m$. The symbols $0_{\mathbb{R}^n}$ and $0_{\mathbb{R}^m}$ denote the original points of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For every $L \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, we introduce the norm $\|L\|_{\mathbb{B}} = \sup\{\|L(x)\| \mid \|x\| \leq 1\}$. Since $\mathbb{R}^m$ is finite dimensional, the Banach space $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ equipped with the norm is also finite dimensional. As usual, we denote by $\mathrm{int}K$, $\mathrm{cl}K$ and $\mathrm{cone}K$ the interior, closure and cone hull of K, respectively. Let $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued map. The domain and graph of F are defined respectively by $\mathrm{dom}F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $\mathrm{gph}F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\}$.

Definition 2.1 [11] Let Q be a nonempty subset of $\mathbb{R}^m$.

(1) An element $x \in Q$ is called a weak maximal element of the set Q, if

$$(\{x\} + \mathrm{int}C) \cap Q = \emptyset.$$

(2) An element $x \in Q$ is called a local weak maximal element of the set Q, if there exists a neighborhood $U(x)$ of x such that

$$(\{x\} + \mathrm{int}C) \cap (Q \cap U(x)) = \emptyset.$$

We denote by $\mathrm{WMax}(Q, C)$ and $\mathrm{LWMax}(Q, C)$ the set of all the weak maximal elements and local weak maximal elements of Q, respectively.

Definition 2.2 [3] Let $F : \mathbb{R}^n \to \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ be a vector-valued function. F is said to be Fréchet differentiable at $x_0 \in \mathbb{R}^n$, if and only if there exists a linear continuous operator $\psi : \mathbb{R}^n \to \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - \psi(x - x_0)\|_{\mathbb{B}}}{\|x - x_0\|} = 0.$$

It is obvious that ψ is unique determined. We denote the derivative ψ of F at x_0 by $\nabla F(x_0)$. If, for every $x \in \mathbb{R}^n$, F is Fréchet differentiable at x, then F is said to be Fréchet differentiable on $\mathbb{R}^n$. Therefore, $\nabla F(\cdot) : \mathbb{R}^n \to \mathbb{B}(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m))$ is a vector-valued function, where $\mathbb{B}(\mathbb{R}^n, \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m))$ denotes the set of all linear continuous operators from $\mathbb{R}^n$ to $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$. F is said to be continuously Fréchet differentiable at x_0, if and only if $\nabla F(\cdot)$ is continuous at x_0. Clearly, if F is continuously Fréchet differentiable at x_0, then, in a neighborhood U of x_0, we have the Taylor polynomial

$$F(x) = F(x_0) + \nabla F(x_0)(x - x_0) + o(x - x_0), \forall x \in U,$$

where $o(x - x_0)$ denotes the remainder term of the Taylor polynomial with $\frac{\|o(x-x_0)\|_{\mathbb{B}}}{\|x-x_0\|} \to 0$.

Now, we recall concepts of a contingent cone and a contingent derivative for set-valued maps.

Definition 2.3 [1] Let S be a nonempty subset of $\mathbb{R}^n$ and $\hat{x} \in \mathrm{cl}S$.

(i) The contingent cone of S at $\hat{x}$ is

$$T(S, \hat{x}) := \{v \in \mathbb{R}^n \mid \exists t_n \downarrow 0, \exists v_n \to v, \text{ such that } \hat{x} + t_n v_n \in S, \forall n \in \mathbb{N}\},$$

or equivalently,

$$T(S, \hat{x}) := \left\{ v \in \mathbb{R}^n \mid \exists t_n \downarrow 0, \exists \{x_n\} \subset S \text{ with } x_n \to \hat{x}, \text{ such that } \frac{x_n - \hat{x}}{t_n} \to v \right\}.$$

(ii) The adjacent cone of S at $\hat{x}$ is

$$T^{\flat}(S, \hat{x}) := \left\{ v \in \mathbb{R}^n \mid \forall t_n \downarrow 0, \exists v_n \to v, \text{ such that } \hat{x} + t_n v_n \in S, \forall n \in \mathbb{N} \right\},$$

or equivalently,

$$T^{\flat}(S,\hat{x}):=\left\{v\in\mathbb{R}^n \mid \forall t_n\downarrow 0,\ \exists\{x_n\}\subset S \text{ with } x_n\to\hat{x},\ \text{ such that } \frac{x_n-\hat{x}}{t_n}\to v\right\}.$$

(iii) S is said to be derivative at $\hat{x}$ if and only if $T(S,\hat{x})=T^{\flat}(S,\hat{x})$.

Definition 2.4 [2] Let $F:\mathbb{R}^n\to 2^{\mathbb{R}^m}$ be a set-valued map and $(\hat{x},\hat{y})\in \mathrm{gph}F$.

(i) The contingent derivative of F at $(\hat{x},\hat{y})$ is the set-valued map $DF(\hat{x},\hat{y}):\mathbb{R}^n\to 2^{\mathbb{R}^m}$ defined by $DF(\hat{x},\hat{y})(x):=\{y\in\mathbb{R}^m \mid (x,y)\in T(\mathrm{gph}F,(\hat{x},\hat{y}))\}$.
(ii) The adjacent derivative of F at $(\hat{x},\hat{y})$ is the set-valued map $D^{\flat}F(\hat{x},\hat{y}):\mathbb{R}^n\to 2^{\mathbb{R}^m}$ defined by $D^{\flat}F(\hat{x},\hat{y})(x):=\{y\in\mathbb{R}^m \mid (x,y)\in T^{\flat}(\mathrm{gph}F,(\hat{x},\hat{y}))\}$.
(iii) F is said to be proto-differentiable at $(\hat{x},\hat{y})$ if and only if $DF(\hat{x},\hat{y})(x)=D^{\flat}F(\hat{x},\hat{y})(x),\forall x\in X$.

Next, we recall a generalized contingent cone called Φ-contingent cone introduced by Meng and Li [17].

Definition 2.5 [17] Let S be a nonempty subset of $\mathbb{R}^n$ and $\hat{x}\in \mathrm{cl}S$. Consider a vector-valued map $\Phi:\mathbb{R}^n\to\mathbb{R}^m$.

(i) The Φ-contingent cone of S at $\hat{x}$ is

$$T_{\Phi}(S,\hat{x}):=\left\{v\in\mathbb{R}^m \mid \exists t_n\downarrow 0,\ \exists\{x_n\}\subset S \text{ with } x_n\to\hat{x},\ \text{ such that } \frac{\Phi(x_n)-\Phi(\hat{x})}{t_n}\to v\right\}.$$

(ii) The Φ-adjacent cone of S at $\hat{x}$ is

$$T^{\flat}_{\Phi}(S,\hat{x}):=\left\{v\in\mathbb{R}^m \mid \forall t_n\downarrow 0,\ \exists\{x_n\}\subset S \text{ with } x_n\to\hat{x},\ \text{ such that } \frac{\Phi(x_n)-\Phi(\hat{x})}{t_n}\to v\right\}.$$

(iii) S is said to be Φ-derivative at $\hat{x}$ if and only if $T_{\Phi}(S,\hat{x})=T^{\flat}_{\Phi}(S,\hat{x})$.

Obviously, $T^{\flat}(S,\hat{x})\subset T(S,\hat{x})$ and $T^{\flat}_{\Phi}(S,\hat{x})\subset T_{\Phi}(S,\hat{x})$. Moreover, if $\Phi:\mathbb{R}^n\to\mathbb{R}^n$ is an identical map, i.e., $\Phi(x)=x,\forall x\in\mathbb{R}^n$, then we have $T_{\Phi}(S,\hat{x})=T(S,\hat{x})$ and $T^{\flat}_{\Phi}(S,\hat{x})=T^{\flat}(S,\hat{x})$. Now, we collect some properties of the Φ-contingent cone.

Proposition 2.1 [17] *Let S be a nonempty subset of $\mathbb{R}^n$ and $\hat{x}\in clS$. Consider a vector-valued map $\Phi:\mathbb{R}^n\to\mathbb{R}^m$, which is continuously Fréchet differentiable at $\hat{x}$. If*

$$Null(\nabla\Phi(\hat{x}))\cap T(S,\hat{x})=\{0_{\mathbb{R}^n}\},$$

where $Null(\nabla\Phi(\hat{x}))$ denotes the null space of $\nabla\Phi(\hat{x})$, i.e., $Null(\nabla\Phi(\hat{x})):=\{x\in\mathbb{R}^n \mid \nabla\Phi(\hat{x})(x)=0_{\mathbb{R}^m}\}$. Then

$$\nabla\Phi(\hat{x})\big(T(S,\hat{x})\big)=T_{\Phi}(S,\hat{x}).$$

Proposition 2.2 [17] *Suppose that S is a nonempty compact subset of $\mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous vector-valued map. Let $\hat{x} \in clS$ and $\Lambda(\hat{x}) := \{x \in S \mid \Phi(x) = \Phi(\hat{x})\}$. Then*

$$T\big(\Phi(S), \Phi(\hat{x})\big) = \bigcup_{\bar{x}\in\Lambda(\hat{x})} T_\Phi(S, \bar{x}).$$

Proposition 2.3 *Let S be a nonempty subset of $\mathbb{R}^n$ and $\hat{x} \in clS$. Consider a vector-valued map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, which is continuously Fréchet differentiable at $\hat{x}$ with*

$$Null(\nabla\Phi(\hat{x})) \cap T(S, \hat{x}) = \{0_{\mathbb{R}^n}\}.$$

If S is derivative at $\hat{x}$, then S is Φ-derivative at $\hat{x}$.

Proof It is easy to verify that $\nabla\Phi(\hat{x})(T^\flat(S, \hat{x})) \subset T^\flat_\Phi(S, \hat{x})$. In fact, take arbitrary $v \in T^\flat(S, \hat{x})$. Then, for every sequence $t_n \downarrow 0$, there exists a sequence $v_n \to v$, such that $\hat{x} + t_n v_n \in S,\ \forall n \in \mathbb{N}$. Since Φ is continuously Fréchet differentiable at $\hat{x}$, we have

$$\frac{\Phi(\hat{x} + t_n v_n) - \Phi(\hat{x})}{t_n} = \nabla\Phi(\hat{x})(v_n) + \frac{o(t_n v_n)}{t_n} \to \nabla\Phi(\hat{x})(v),\ \text{as } n \to +\infty.$$

Together with $\hat{x} + t_n v_n \in S,\ \forall n \in \mathbb{N}$ and $\hat{x} + t_n v_n \to \hat{x}$, we can conclude that $\nabla\Phi(\hat{x})(v) \in T^\flat_\Phi(S, \hat{x})$. Thus, we have $\nabla\Phi(\hat{x})(T^\flat(S, \hat{x})) \subset T^\flat_\Phi(S, \hat{x})$. Moreover, since $Null(\nabla\Phi(\hat{x})) \cap T(S, \hat{x}) = \{0_{\mathbb{R}^n}\}$ and S is derivative at $\hat{x}$, i.e., $T(S, \hat{x}) = T^\flat(S, \hat{x})$, it follows from Proposition 2.1 that

$$T_\Phi(S, \hat{x}) = \nabla\Phi(\hat{x})\left(T(S, \hat{x})\right) = \nabla\Phi(\hat{x})\left(T^\flat(S, \hat{x})\right) \subset T^\flat_\Phi(S, \hat{x}) \subset T_\Phi(S, \hat{x}),$$

which implies

$$T_\Phi(S, \hat{x}) = T^\flat_\Phi(S, \hat{x}).$$

Therefore, S is Φ-derivative at $\hat{x}$. □

3 Differential properties of a class of set-valued maps

In remainder sections, let K be a nonempty compact subset of $\mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ be continuously Fréchet differentiable and

$$G(x) := \bigcup_{z\in K} \langle F(x, z), x - z\rangle.$$

In this section, we will discuss the differential properties of G. Let $\hat{F}(x) := \langle F(\hat{x}, x), \hat{x} - x \rangle$. If $(\hat{x}, \hat{y}) \in \mathrm{gph}G$, we can define a nonempty compact subset $\Omega(\hat{x}, \hat{y})$ of K by

$$\Omega(\hat{x}, \hat{y}) := \{x \in K \mid \hat{F}(x) = \hat{y}\}.$$

Theorem 3.1 *Let $(\hat{x}, \hat{y}) \in gphG$. Then, for every $x \in dom(DG(\hat{x}, \hat{y}))$, we have*

$$DG(\hat{x}, \hat{y})(x) = \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle F(\hat{x}, \bar{x}), x \rangle + \langle \nabla F(\hat{x}, \bar{x})(x, 0), \hat{x} - \bar{x} \rangle + T_{\hat{F}}(K, \bar{x}) \right).$$

Proof First, suppose that $x \in \mathrm{dom}(DG(\hat{x}, \hat{y}))$ and $y \in DG(\hat{x}, \hat{y})(x)$. Then, there exist sequences $(x_n . y_n) \to (x, y)$ and $t_n \downarrow 0$, such that $(\hat{x}, \hat{y}) + t_n(x_n, y_n) \in \mathrm{gph}G$, i.e.,

$$\hat{y} + t_n y_n \in G(\hat{x} + t_n x_n) = \bigcup_{z \in K} \langle F(\hat{x} + t_n x_n, z), \hat{x} + t_n x_n - z \rangle.$$

So, for every $n \in \mathbb{N}$, there exists a $\bar{x}_n \in K$, such that

$$\hat{y} + t_n y_n = \langle F(\hat{x} + t_n x_n, \bar{x}_n), \hat{x} + t_n x_n - \bar{x}_n \rangle. \tag{1}$$

Since K is compact, without loss of generality, we can assume that $\bar{x}_n \to \bar{x} \in K$. From the continuity of F and (1), we have $\hat{y} = \langle F(\hat{x}, \bar{x}), \hat{x} - \bar{x} \rangle$, that is $\bar{x} \in \Omega(\hat{x}, \hat{y})$ since $\bar{x} \in K$. Then, it follows from (1) that

$$\frac{\langle F(\hat{x} + t_n x_n, \bar{x}_n), \hat{x} - \bar{x}_n \rangle - \hat{F}(\bar{x})}{t_n} = y_n - \left\langle F(\hat{x} + t_n x_n, \bar{x}_n), x_n \right\rangle. \tag{2}$$

Since F is continuously Fréchet differentiable, by the Taylor polynomial, we have

$$F(\hat{x} + t_n x_n, \bar{x}_n) = F(\hat{x}, \bar{x}_n) + t_n \nabla F(\hat{x}, \bar{x}_n)(x_n, 0) + o(t_n x_n, 0). \tag{3}$$

Then, it follows from (2) and (3) that

$$\begin{aligned} y_n - \langle F(\hat{x} + t_n x_n, \bar{x}_n), x_n \rangle = {} & \frac{\hat{F}(\bar{x}_n) - \hat{F}(\bar{x})}{t_n} + \langle \nabla F(\hat{x}, \bar{x}_n)(x_n, 0), \hat{x} - \bar{x}_n \rangle \\ & + \left\langle \frac{o(t_n x_n, 0)}{t_n}, \hat{x} - \bar{x}_n \right\rangle. \end{aligned} \tag{4}$$

Since F is continuously Fréchet differentiable, it is obvious that

$$\lim_{n \to +\infty} \left(y_n - \langle F(\hat{x} + t_n x_n, \bar{x}_n), x_n \rangle \right) = y - \langle F(\hat{x}, \bar{x}), x \rangle \tag{5}$$

and

$$\lim_{n\to+\infty}\left(\langle\nabla F(\hat{x},\bar{x}_n)(x_n,0),\hat{x}-\bar{x}_n\rangle\right)=\langle\nabla F(\hat{x},\bar{x})(x,0),\hat{x}-\bar{x}\rangle. \tag{6}$$

Then, from (4), (5), (6) and Definition 2.5, we have

$$y-\langle F(\hat{x},\bar{x}),x\rangle-\langle\nabla F(\hat{x},\bar{x})(x,0),\hat{x}-\bar{x}\rangle\in T_{\hat{F}}(K,\bar{x}).$$

Therefore, we have

$$y\in\bigcup_{\bar{x}\in\Omega(\hat{x},\hat{y})}\left(\langle F(\hat{x},\bar{x}),x\rangle+\langle\nabla F(\hat{x},\bar{x})(x,0),\hat{x}-\bar{x}\rangle+T_{\hat{F}}(K,\bar{x})\right).$$

Conversely, suppose that $y=\langle F(\hat{x},\bar{x}),x\rangle+\langle\nabla F(\hat{x},\bar{x})(x,0),\hat{x}-\bar{x}\rangle+y^*$, where $\bar{x}\in\Omega(\hat{x},\hat{y})$ and $y^*\in T_{\hat{F}}(K,\bar{x})$. Then, there exist sequences $\{\bar{x}_n\}\subset K$, $\bar{x}_n\to\bar{x}$ and $t_n\downarrow 0$ such that

$$\frac{\hat{F}(\bar{x}_n)-\hat{F}(\bar{x})}{t_n}\to y^*.$$

Since F is continuously Fréchet differentiable, we can take sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n\to x$ and

$$\begin{aligned}y_n&=\langle F(\hat{x}+t_nx_n,\bar{x}_n),x_n\rangle+\frac{\langle F(\hat{x}+t_nx_n,\bar{x}_n),\hat{x}-\bar{x}_n\rangle-\langle F(\hat{x},\bar{x}_n),\hat{x}-\bar{x}_n\rangle}{t_n}\\&\quad+\frac{\langle F(\hat{x},\bar{x}_n),\hat{x}-\bar{x}_n\rangle-\langle F(\hat{x},\bar{x}),\hat{x}-\bar{x}\rangle}{t_n}.\end{aligned} \tag{7}$$

From (7), we have $y_n\to y$ and

$$\hat{y}+t_ny_n=\langle F(\hat{x}+t_nx_n,\bar{x}_n),\hat{x}+t_nx_n-\bar{x}_n\rangle\in G(\hat{x}+t_nx_n).$$

So, we can conclude that

$$y\in DG(\hat{x},\hat{y})(x).$$

This completes the proof. □

From Proposition 2.1 and Theorem 3.1, we can easily get the following corollary.

Corollary 3.1 *Let $(\hat{x},\hat{y})\in gphG$ and F be continuously Fréchet differentiable. If for every $\bar{x}\in\Omega(\hat{x},\hat{y})$, $Null(\nabla\hat{F}(\bar{x}))\cap T(K,\bar{x})=\{0_{\mathbb{R}^n}\}$, then for every $x\in dom(DG(\hat{x},\hat{y}))$, we have*

$$DG(\hat{x},\hat{y})(x)=\bigcup_{\bar{x}\in\Omega(\hat{x},\hat{y})}\left(\langle F(\hat{x},\bar{x}),x\rangle+\langle\nabla F(\hat{x},\bar{x})(x,0),\hat{x}-\bar{x}\rangle+\nabla\hat{F}(\bar{x})(T(K,\bar{x}))\right).$$

Remark 3.1 For **Case** (∗), i.e., $F(x,z) \equiv H(x), \forall x, z \in \mathbb{R}^n$ for some set-valued map H, the result of Theorem 3.1 reduces to

$$DG(\hat{x},\hat{y})(x) = \langle H(\hat{x}), x\rangle + \bigcup_{\bar{x}\in\Omega(\hat{x},\hat{y})} \Big(\langle \nabla H(\hat{x})(x), \hat{x}-\bar{x}\rangle + T_{\hat{H}}(K,\bar{x})\Big),$$

where $\hat{H}(x) = \langle H(\hat{x}), \hat{x}-x\rangle$ and $\Omega(\hat{x},\hat{y}) = \{x \in K \mid \hat{H}(x) = \hat{y}\}$, which is the result of Corollary 4.1 in [16] and a generalization of Theorem 3.1 in [15].

For **Case** (∗∗), i.e., $F(x,z) \equiv Q(z), \forall x, z \in \mathbb{R}^n$ for some set-valued map Q, it follows from the proof of Theorem 3.1 that we only need that F is continuous. Simultaneously, the result of Theorem 3.1 reduces to

$$DG(\hat{x},\hat{y})(x) = \bigcup_{\bar{x}\in\Omega(\hat{x},\hat{y})} \Big(\langle Q(\bar{x}), x\rangle + T_{\hat{Q}}(K,\bar{x})\Big),$$

where $\hat{Q}(x) = \langle Q(x), \hat{x}-x\rangle$ and $\Omega(\hat{x},\hat{y}) = \{x \in K \mid \hat{Q}(x) = \hat{y}\}$, which is the result of Theorem 3.1 in [17]. Furthermore, if Q is continuously Fréchet differentiable and $Null(\nabla\hat{Q}(\bar{x})) \cap T(K,\bar{x}) = \{0_{\mathbb{R}^n}\}, \forall \bar{x} \in \Omega(\hat{x},\hat{y})$, then the result of Corollary 3.1 reduces to

$$DG(\hat{x},\hat{y})(x) = \bigcup_{\bar{x}\in\Omega(\hat{x},\hat{y})} \Big(\langle Q(\bar{x}), x\rangle + \nabla\hat{Q}(\bar{x})(T(K,\bar{x}))\Big),$$

which is the result of Corollary 3.1 in [17].

Now, we give the following example to illustrate Theorem 3.1.

Example 3.1 Let $K = [0,1]$ and $F : \mathbb{R}\times\mathbb{R} \to \mathbb{B}(\mathbb{R},\mathbb{R})$ with $F(x,z) = xz, \forall x, z \in \mathbb{R}$. Take $\hat{x} = 1$, $\hat{y} = 0$. Then, for every $x \in \mathbb{R}$, we have

$$G(x) = \bigcup_{z\in K} \langle F(x,z), x-z\rangle = \bigcup_{z\in[0,1]} xz(x-z) = \left[\min\{x^2-x, 0\}, \max\left\{\frac{1}{4}x^3, 0\right\}\right].$$

It is obvious that $(\hat{x},\hat{y}) \in \mathrm{gph}G$ and

$$T(\mathrm{gph}G, (\hat{x},\hat{y})) = \bigcup_{x\in\mathbb{R}} \left(x \times (\{x^-\} + \mathbb{R}_+)\right),$$

where $x^- = \min\{x, 0\}$. Therefore, for every $x \in dom(DG(\hat{x},\hat{y})) = \mathbb{R}$, we have

$$DG(\hat{x},\hat{y})(x) = \{x^-\} + \mathbb{R}_+. \tag{8}$$

On the other hand, we have

$$\hat{F}(x) = \langle F(\hat{x}, x), \hat{x}-x\rangle = x(1-x)$$

and

$$\Omega(\hat{x}, \hat{y}) = \{0, 1\}.$$

Denote $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. By Definition 2.5 and directly calculating, we have

$$T_{\hat{F}}(K, \bar{x}_1) = T_{\hat{F}}(K, \bar{x}_2) = \mathbb{R}_+.$$

Then, for every $x \in \mathbb{R}$, we have

$$\begin{aligned} DG(\hat{x}, \hat{y})(x) &= \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \big(\langle F(\hat{x}, \bar{x}), x\rangle + \langle \nabla F(\hat{x}, \bar{x})(x, 0), \hat{x} - \bar{x}\rangle + T_{\hat{F}}(K, \bar{x})\big) \\ &= \big(\langle F(\hat{x}, \bar{x}_1), x\rangle + \langle \nabla F(\hat{x}, \bar{x}_1)(x, 0), \hat{x} - \bar{x}_1\rangle + T_{\hat{F}}(K, \bar{x}_1)\big) \\ &\bigcup \big(\langle F(\hat{x}, \bar{x}_2), x\rangle + \langle \nabla F(\hat{x}, \bar{x}_2)(x, 0), \hat{x} - \bar{x}_2\rangle + T_{\hat{F}}(K, \bar{x}_2)\big) \\ &= \mathbb{R}_+ \bigcup (\{x\} + \mathbb{R}_+) \\ &= \{x^-\} + \mathbb{R}_+. \end{aligned} \tag{9}$$

So, it follows from (8) and (9) that Theorem 3.1 holds.

4 Optimality conditions for the (GWVVI)

In this section, we establish optimality conditions for the (GWVVI). At first, we introduce the local optimal solution for the (GWVVI).

We call that x^* is a local optimal solution of the (GWVVI) if there exists a neighborhood $U(x^*)$ of x^* such that

$$\langle F(x^*, x), x^* - x\rangle \notin \mathrm{int}C, \forall x \in K \cap U(x^*).$$

Motivated by the definition in [5], we define a gap function for the (GWVVI) as a set-valued map.

Definition 4.1 The set-valued map W defined from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be a gap function for the (GWVVI) if and only if

(a) $0_{\mathbb{R}^m} \in W(\hat{x})$ if and only if $\hat{x}$ solves the (GWVVI);
(b) $W(x) \cap (-\mathrm{int}C) = \emptyset, \forall x \in K$.

Consider the set-valued map $N : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ defined by $N(x)$=LWMax$(G(x), C)$, and the set-valued map $W : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ defined by $W(x)$=WMax$(G(x), C)$. We have the following theorems.

Theorem 4.1 *The set-valued map $W(x)$ is a gap function for the* (*GWVVI*).

Proof Suppose that $0_{\mathbb{R}^m} \in W(\hat{x})$, that is, $0_{\mathbb{R}^m} \in$ WMax$(G(\hat{x}), C)$, then $G(\hat{x}) \cap \mathrm{int}C = \emptyset$. We have $\forall x \in K, \langle F(\hat{x}, x), \hat{x} - x\rangle \notin \mathrm{int}C$. Therefore, $\hat{x}$ solves the

(GWVVI). Conversely, if $\hat{x}$ solves the (GWVVI), we have $\forall x \in K$, $\langle F(\hat{x}, x), \hat{x} - x\rangle \notin \text{int}C$. Then, $G(\hat{x}) \cap \text{int}C = \emptyset$ and we can conclude that $0_{\mathbb{R}^m} \in W(\hat{x})$ since $0_{\mathbb{R}^m} \in G(\hat{x})$.

On the other hand, for every $x \in K$ and $y \in W(x)$, i.e., $y \in \text{WMax}(G(x), C)$, we have $(\{y\} + \text{int}C) \cap G(x) = \emptyset$. Obviously, $0_{\mathbb{R}^m} \in G(x)$. Then, $0_{\mathbb{R}^m} \notin \{y\} + \text{int}C$ and $y \notin -\text{int}C$. So, we have $W(x) \cap (-\text{int}C) = \emptyset$. This completes the proof. □

Theorem 4.2 *For the set-valued map $N : \mathbb{R}^n \to 2^{\mathbb{R}^m}$, we have*

(1) *If $0_{\mathbb{R}^m} \in N(\hat{x})$, then $\hat{x}$ is a local optimal solution of the* (*GWVVI*).
(2) *If $\hat{x}$ is a local optimal solution of the* (*GWVVI*) *and $\Omega(\hat{x}, 0_{\mathbb{R}^m}) = \{\hat{x}\}$, then $0_{\mathbb{R}^m} \in N(\hat{x})$.*

Proof (1) Let $0_{\mathbb{R}^m} \in N(\hat{x})$, that is $0_{\mathbb{R}^m} \in \text{LWMax}(G(\hat{x}), C)$. Assume that $\hat{x}$ is not a local optimal solution of the (GWVVI), then there exists a sequence $\{x_n\} \subset K, x_n \neq \hat{x}$ such that $x_n \to \hat{x}$ and $\langle F(\hat{x}, x_n), \hat{x} - x_n\rangle \in \text{int}C$. Let $y_n = \langle F(\hat{x}, x_n), \hat{x} - x_n\rangle$, then $y_n \in G(\hat{x})$, $y_n \in \text{int}C$. Since F is continuous and $x_n \to \hat{x}$, we have $y_n \to 0_{\mathbb{R}^m}$. This is a contradiction to $0_{\mathbb{R}^m} \in \text{LWMax}(G(\hat{x}), C)$.

(2) Assume that $0_{\mathbb{R}^m} \notin N(\hat{x})$, that is, $0_{\mathbb{R}^m} \notin \text{LWMax}(G(\hat{x}), C)$. Then there exists a sequence $y_n \in G(\hat{x}) \cap \text{int}C$ such that $y_n \to 0_{\mathbb{R}^m}$. By the definition of G, we have $\forall n \in N$, $\exists x_n \in K$ such that $y_n = \langle F(\hat{x}, x_n), \hat{x} - x_n\rangle$. Since K is compact, there exist a subsequence (without loss of generality, we can denote the same) $\{x_n\}$ and $\bar{x} \in K$ such that $x_n \to \bar{x}$. Then we have $\langle F(\hat{x}, \bar{x}), \hat{x} - \bar{x}\rangle = 0_{\mathbb{R}^m}$ since $y_n \to 0_{\mathbb{R}^m}$ and F is continuous. So, $\bar{x} \in \Omega(\hat{x}, 0_{\mathbb{R}^m}) = \{\hat{x}\}$, that is $\bar{x} = \hat{x}$. Then, we can conclude that $\{x_n\} \subset K, x_n \to \hat{x}$ and $y_n = \langle F(\hat{x}, x_n), \hat{x} - x_n\rangle \in \text{int}C$, that is, $\hat{x}$ is not a local optimal solution of the (GWVVI). This is a contradiction. □

Next, we discuss the optimality conditions for the (GWVVI). Firstly, we establish a necessary optimality condition for solutions of the (GWVVI), and then give some sufficient and necessary optimality conditions for local optimal solutions of the (GWVVI).

Theorem 4.3 *Let $\hat{x} \in K$ be a solution of the* (*GWVVI*)*, then $DG(\hat{x}, 0_{\mathbb{R}^m})(0_{\mathbb{R}^n}) \cap \text{int}C = \emptyset$.*

Proof It is obvious that

$$\langle F(\hat{x}, x), \hat{x} - x\rangle \notin \text{int}C, \quad \forall x \in K \tag{10}$$

since $\hat{x}$ is a solution of the (GWVVI). By the definition of $\hat{F}(x)$, we have $\hat{F}(x) \notin \text{int}C, \forall x \in K$. Clearly, $\hat{F}(\hat{x}) = 0_{\mathbb{R}^m}$. Thus, (10) is equivalent to

$$y \notin \text{int}C, \quad \forall y \in \text{clcone}\left(\hat{F}(K) - \{0_{\mathbb{R}^m}\}\right).$$

By Definition 2.3, it is clear that $T(\hat{F}(K), 0_{\mathbb{R}^m}) \subset \text{clcone}\big(\hat{F}(K) - \{0_{\mathbb{R}^m}\}\big)$, we have

$$T(\hat{F}(K), 0_{\mathbb{R}^m}) \cap \text{int}C = \emptyset. \tag{11}$$

From Proposition 2.2,

$$\bigcup_{\bar{x}\in\Omega(\hat{x},0_{\mathbb{R}^m})} T_{\hat{F}}(K,\bar{x}) = T(\hat{F}(K),0_{\mathbb{R}^m}). \tag{12}$$

It follows from Theorem 3.1 that

$$DG(\hat{x},0_{\mathbb{R}^m})(0_{\mathbb{R}^n}) = \bigcup_{\bar{x}\in\Omega(\hat{x},0_{\mathbb{R}^m})} T_{\hat{F}}(K,\bar{x}), \tag{13}$$

and this completes the proof from (11), (12) and (13). □

Remark 4.1 For **Case** $(*)$, Theorem 4.3 reduces to Theorem 5.2(1) in [16] and is a generalization of Theorem 5.1 in [15]. For **Case** $(**)$, Theorem 4.3 reduces to Theorem 5.1 in [17].

In the following, we discuss the necessary and sufficient optimality conditions for local optimal solutions of the (GWVVI).

Theorem 4.4 *Let $\hat{x}\in K$ be a local optimal solution of the (GWVVI) and $\Omega(\hat{x},0_{\mathbb{R}^m}) = \{\hat{x}\}$, then $DG(\hat{x},0_{\mathbb{R}^m})(0_{\mathbb{R}^n})\cap intC = \emptyset$.*

Proof Assume that $DG(\hat{x},0_{\mathbb{R}^m})(0_{\mathbb{R}^n})\cap \text{int}C \neq \emptyset$, then $\exists y \in DG(\hat{x},0_{\mathbb{R}^m})(0_{\mathbb{R}^n})\cap \text{int}C$. From Theorem 3.1 and $\Omega(\hat{x},0_{\mathbb{R}^m}) = \{\hat{x}\}$, we have

$$DG(\hat{x},0_{\mathbb{R}^m})(0_{\mathbb{R}^n}) = \bigcup_{\bar{x}\in\Omega(\hat{x},0_{\mathbb{R}^m})} T_{\hat{F}}(K,\bar{x}) = T_{\hat{F}}(K,\hat{x}).$$

So, $y\in T_{\hat{F}}(K,\hat{x})$. By Definition 2.5, there exists a sequence $\{x_n\}\subset K$, $x_n\to\hat{x}$ and $t_n\downarrow 0$, such that

$$\frac{\hat{F}(x_n)-\hat{F}(\hat{x})}{t_n} \to y.$$

Since $y\in \text{int}C$ and $\hat{F}(\hat{x}) = 0_{\mathbb{R}^m}$, we can conclude that for sufficiently large $n\in\mathbb{N}$,

$$\frac{\hat{F}(x_n)}{t_n} \in \text{int}C.$$

Then, $y_n := \hat{F}(x_n) = \langle F(\hat{x},x_n),\hat{x}-x_n\rangle \in \text{int}C$, for sufficiently large $n\in\mathbb{N}$. Moreover, for sufficiently large $n\in\mathbb{N}$, we have $y_n\in G(\hat{x})\cap\text{int}C$ and $y_n\to 0_{\mathbb{R}^m}$ since $x_n\to\hat{x}$, that is, $0_{\mathbb{R}^m}\notin N(\hat{x})$. This is a contradiction from Theorem 4.2(2). □

Remark 4.2 Obviously, the results in Theorems 4.3 and 4.4 are the same. However, their assumptions are different. In fact, the condition $\Omega(\hat{x},0_{\mathbb{R}^m}) = \{\hat{x}\}$ is not necessary in Theorem 4.3, but essential in Theorem 4.4. The following examples illustrate the cases.

Example 4.1 Consider Example 3.1. If we take $\hat{x} = \hat{y} = 0$ and $C = \mathbb{R}_+$, then $\Omega(\hat{x}, 0) = K$, that is, $\Omega(\hat{x}, 0) = \{\hat{x}\}$ is not satisfied. Moreover,

$$\langle F(\hat{x}, x), \hat{x} - x\rangle \equiv 0, \ \forall x \in [0, 1].$$

So, $\hat{x}$ is a solution of the (GWVVI). By Theorem 3.1 and Definition 2.5, we have

$$DG(\hat{x}, 0)(0) = \bigcup_{\bar{x} \in K} T_{\hat{F}}(K, \bar{x}) = \{0\}.$$

Thus, $DG(\hat{x}, 0)(0) \cap \mathrm{int}C = \emptyset$ and Theorem 4.3 holds.

Example 4.2 Let $K = [0, 3]$ and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{B}(\mathbb{R}, \mathbb{R})$ with $F(x, z) = x - z + 2, \forall x, z \in \mathbb{R}$. Take $\hat{x} = 0$ and $C = \mathbb{R}_+$, then $\hat{F}(x) = \langle F(0, x), 0 - x\rangle = (2 - x)(0 - x) = 0$ implies $x = 0, 2$, that is, $\Omega(\hat{x}, 0) = \{0, 2\}$. So $\Omega(\hat{x}, 0) = \{\hat{x}\}$ is not satisfied. On the one hand,

$$\langle F(0, x), 0 - x\rangle = x(x - 2), \ \forall x \in [0, 3].$$

So, it is obvious that $\hat{x} = 0$ is a local optimal solution of the (GWVVI). On the other hand, for every $x \in \mathbb{R}$, we have

$$G(x) = \bigcup_{z \in K} \langle F(x, z), x - z\rangle = \bigcup_{z \in [0,3]} (x - z + 2)(x - z).$$

Then, from Theorem 3.1 and Definition 2.5, we have

$$\begin{aligned} DG(\hat{x}, 0)(0) &= \bigcup_{\bar{x} \in \Omega(\hat{x}, 0)} T_{\hat{F}}(K, \bar{x}) \\ &= T_{\hat{F}}(K, 0) \cup T_{\hat{F}}(K, 2) \\ &= \mathbb{R}_- \cup \mathbb{R} \\ &= \mathbb{R}. \end{aligned}$$

Thus, $DG(\hat{x}, 0)(0) \cap \mathrm{int}C = \mathrm{int}\mathbb{R}_+ \neq \emptyset$ and the necessary condition in Theorem 4.4 does not hold.

Theorem 4.5 *Let $\hat{x} \in K$ and $DG(\hat{x}, 0_{\mathbb{R}^m})(0_{\mathbb{R}^n}) \cap C = \{0_{\mathbb{R}^m}\}$, then $\hat{x}$ is a local optimal solution of the (GWVVI).*

Proof By Theorem 4.2(1), we only need to prove that $0_{\mathbb{R}^m} \in N(\hat{x})$. If not, there exists a sequence $\{y_n\} \subset G(\hat{x}) \cap \mathrm{int}C$ such that $y_n \to 0_{\mathbb{R}^m}$. Then, by the definition of G, we have $\forall n \in \mathbb{N}, \exists x_n \in K$ such that $y_n = \langle F(\hat{x}, x_n), \hat{x} - x_n\rangle = \hat{F}(x_n) \in \mathrm{int}C$, that is, $\hat{F}(x_n) \neq 0_{\mathbb{R}^m}$ and $\hat{F}(x_n) \to 0_{\mathbb{R}^m}$. Denote $t_n := \|\hat{F}(x_n)\|$, then we have $t_n \downarrow 0$ and there exists a subsequence (without loss of generality, we can denote the same) of $\{\frac{\hat{F}(x_n)}{t_n}\}$ such that

$$\frac{\hat{F}(x_n)}{t_n} \to y \in S^1 \cap T(\hat{F}(K), 0_{\mathbb{R}^m}) \cap C \tag{14}$$

since $\hat{F}(x_n) \in \text{int}C$, $\{\frac{\hat{F}(x_n)}{t_n}\}$ is bounded and $\mathbb{R}^m$ is finite-dimensional. By Theorem 3.1 and Proposition 2.2, we have

$$DG(\hat{x}, 0_{\mathbb{R}^m})(0_{\mathbb{R}^n}) = \bigcup_{\bar{x} \in \Omega(\hat{x}, 0_{\mathbb{R}^m})} T_{\hat{F}}(K, \bar{x}) = T(\hat{F}(K), 0_{\mathbb{R}^m}). \tag{15}$$

From (14) and (15), we can conclude that $y \neq 0_{\mathbb{R}^m}$ and

$$y \in DG(\hat{x}, 0_{\mathbb{R}^m})(0_{\mathbb{R}^n}) \cap C.$$

This is a contradiction to assumptions. □

Remark 4.3 From Theorems 4.4 and 4.5, we know that there is no gap between sufficient and necessary optimality conditions for the local optimal solution of the (GWVVI) under an appropriate condition, i.e., $\Omega(\hat{x}, 0_{\mathbb{R}^m}) = \{\hat{x}\}$.

Now, we give an example to explain Theorem 4.5.

Example 4.3 Consider Examples 3.1 and 4.1. If we take $\hat{x} = \hat{y} = 0$ and $C = \mathbb{R}_+$, then

$$DG(\hat{x}, 0)(0) = \bigcup_{\bar{x} \in K} T_{\hat{F}}(K, \bar{x}) = \{0\}.$$

Thus, $DG(\hat{x}, 0)(0) \cap C = \{0\}$ and $\hat{x} = 0$ is a local optimal solution of the (GWVVI) by Theorem 4.5.

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