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Modified KdV equation for solitary Rossby waves with β effect in barotropic fluids*

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This paper uses the weakly nonlinear method and perturbation method to deal with the quasi-geostrophic vorticity equation, and the modified Korteweg-de Vries(mKdV) equations describing the evolution of the amplitude of solitary Rossby waves as the change of Rossby parameter $\beta(y)$ with latitude y is obtained.

Keywords: nonlinear Rossby waves, mKdV equation, β effect, perturbation method
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1. Introduction

Rossby waves are the most important waves in the atmosphere and ocean, and are intrinsic to the large-scale systems in fluids. Theory and observation show that their basic characteristic satisfies the quasi-geostrophic and quasi-static equilibrium approximations. In barotropic fluid, Long[1] and Benney[2] discussed long waves in a homogeneous atmosphere and obtained the Korteweg-de Vries (KdV) equation, but their analysis was limited to the case where the velocity shear was small in comparison with a basic uniform zonal motion and they gave no insight pertaining to the kinds of stream-line-flow patterns accompanying these waves. Their limitation to a small shear superimposed on an order-one uniform flow avoided the special considerations required by the existence of a critical layer where the wave speed matches the zonal-flow velocity. Solitary Rossby waves were studied by Larsen[3] and Clarke,[4] but they, as well, avoided a discussion of the critical layer and did not provide any information about possible flow patterns. Redekopp[5] discussed the general theory of solitary waves in zonal, planetary shear flow. The work focused on two special atmospheric model and demonstrated that the amplitude of long Rossby wave propagating in a zonal shear flow was governed by the KdV equation or the modified Korteweg-de Vries (mKdV) equation[6] depending on the distribution of the atmospheric density stratification. Redekopp and Weidman[7] discussed the solitary Rossby waves in zonal shear flows and their interactions. A coupled pair of nonlinear evolution KdV equations were derived for describing the interaction of solitary waves propagating in a zonal shear flow and having different long-wave phase velocities. Maslowe and Redekopp[8] discussed long nonlinear waves in stratified flows. They analysed the effect of shear on long waves in a stratified flow. But they did not discussed the topography effect on the Rossby waves. Boyd[9,10] applied the method of multiple scales to the primitive equation to show that long, small amplitude Rossby waves evolved in longitude and time according to the nonlinear KdV equation or mKdV equation. Liu and Tan[11] studied Rossby waves with the change of β, and discussed the change of the Rossby parameter β with latitude and extended the β-plane approximation as $f = f_0 + \beta_0 y - \dfrac{1}{2}\gamma_0 y^2$. Luo[12,13] discussed the solitary Rossby waves with the β parameter and dipole blocking using the extended β-plane approximation. Zhao[14] investigated the dynamical influence of topography on the ultra-long Rossby waves in the long-latitude atmosphere. He concluded that the topographical forcing can lead to the instability of the ultra-long Rossby waves. Zhao *et al*[15] discussed equatorial envelope solitary Rossby waves in a shear flow. They employed a simple shallow-water model on an equatorial β-plane approximation to investigate the nonlinear equatorial solitary Rossby waves in a

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mean zonal flow with meridional shear by the asymptotic method of multiple scales. The cubic nonlinear Schrödinger equation, satisfied for large amplitude equatorial envelope solitary Rossby waves in shear flow, was derived. The effect of basic flow shear on the nonlinear solitary Rossby waves was analysed. Solutions of solitary waves play such an important role in soliton theory that many mathematicians and physicists are interested in this topic,[16−20] such as Hirota's bilinear method,[21] the Jacobi elliptic function expansion method[22,23] etc. which have been proposed and widely used.

In this paper, the β-plane approximation $f = f_0 + \beta_0 y$ (β_0 is a constant) is extended into $f = f_0 + \beta(y)y$, which includes a nonlinear function $\beta(y)$ taking the place of β in the β-plane approximation. Such approximation can depict the motion of the atmosphere and ocean more precisely, especially in the middle and high latitude regions. The mKdV equation, which describes the evolution of the amplitude of solitary Rossby waves, was derived. On the basis of it, we know that the nonlinear β effect is an important factor for the formation of solitary Rossby waves.

2. Derivation of the mKdV equation

2.1. Governing equation and boundary conditions

The mathematical basics for the theory presented herein is the quasi-geostrophic form of the potential–vorticity equation[24,25] for shallow fluid (the topographic effect is neglected here, and the β effect is a nonlinear function of latitudinal variable y)

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\psi\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\psi\frac{\partial}{\partial x}\right)[\nabla^2\psi + \beta(y)y] = 0, \quad (1)$$

where $\beta(y)$ is a nonlinear function of latitude y.

The side boundary conditions are rigid-wall boundary condition

$$\psi(y_1) = \psi(y_2) = 0 \quad (2)$$

and here $y = y_1$, $y = y_2$ denote the southern and northern edges of the zonal flow where we may suppose that latitudinal boundaries exist.

It is convenient to convert Eq.(1) into the non-dimensional form by taking the following scaling rules

$$(x, y) = L_0(x^*, y^*), \quad t = \frac{L_0}{U_0}t^*,$$

$$\psi = U_0 L_0 \psi^*, \quad \beta = \frac{U_0}{L_0^2}\beta^*, \quad (3)$$

where the non-dimensional variables are marked by an asterisk. L_0 is the characteristic measure of the length of the mean zonal flow, and U_0 is the characteristic velocity scale. Substitution of Eq.(3) into Eq.(1) yields

$$\left(\frac{\partial}{\partial t^*} + \frac{\partial}{\partial x^*}\psi^*\frac{\partial}{\partial y^*} - \frac{\partial}{\partial y^*}\psi^*\frac{\partial}{\partial x^*}\right) \times [\nabla^2\psi^* + \beta^*(y^*)y^*] = 0, \quad (4)$$

where the asterisk can be dropped for simplicity, yields

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\psi\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\psi\frac{\partial}{\partial x}\right)[\nabla^2\psi + \beta(y)y] = 0. \quad (5)$$

We take the governing equation as Eq.(5), and take $y_1 = 0$, $y_2 = 1$ in the non-dimensional form, thus boundary condition Eq.(2) has the form

$$\psi(0) = \psi(1) = 0. \quad (6)$$

2.2. Perturbation method and weakly nonlinear method for the derivation of the KdV equation

We assume that the basic stream function has the form

$$\varPsi(y) = -\int [U(y) - c_0]\mathrm{d}y, \quad (7)$$

where c_0 is a constant, which we find an eigenvalue of the eigenvalue problem below. The Eq.(7) actually means that we have taken a travelling wave transformation

$$x = x - c_0 t. \quad (8)$$

It is clear that the basic flow is $U(y)$ here. We take the total stream function ψ as a disturbance stream ψ' characterized by a non-dimensional amplitude ε superimposed on the zonal flow $\varPsi(y)$. When $\varepsilon << 1$, it is a weakly nonlinear problem, which we mainly deal with in this paper. So the stream function has the form

$$\psi = \varPsi(y) + \varepsilon\psi' = -\int [U(y) - c_0]\mathrm{d}y + \varepsilon\psi', \quad (9)$$

here, and in the rest of the paper, expressions are written in the non-dimensional form and all symbols stand for dimensional quantities. Substitution of Eq.(9) into Eq.(5) yields

$$\varepsilon\frac{\partial}{\partial t}\nabla^2\psi + \varepsilon(U - c_0)\frac{\partial}{\partial x}\nabla^2\psi + \varepsilon((\beta(y)y)' - U'') \times \frac{\partial\psi}{\partial x} + \varepsilon^2 J[\psi, \nabla^2\psi] = 0, \quad (10)$$

where the apostrophe of the disturbance stream has been dropped for simplicity. $(\beta(y)y)'$ is the derivative of y, U'' is the second derivative of y and $J[A,B] = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$ is Jacobi operator.
Let us write $(\beta(y)y)' - U'' = p(y)$, simplifying Eq.(10) we obtain

$$\frac{\partial}{\partial t}\nabla^2\psi + (U - c_0)\frac{\partial}{\partial x}\nabla^2\psi + p(y)\frac{\partial\psi}{\partial x} + \varepsilon J[\psi, \nabla^2\psi] = 0. \tag{11}$$

The parameter ε is a measure of the magnitude of nonlinear products. Attention is focused on systems in which nonlinearity and dispersion are of the same order of magnitude. We will look for the asymptotic solution of the weakly nonlinear problem by the multiple scale method. To this end, it is convenient to introduce long spatial and temporal time scale X and T, respectively

$$X = \varepsilon x, \quad T = \varepsilon^3 t. \tag{12}$$

So that Eq.(11) now becomes

$$\mathscr{L}_0(\psi) + \varepsilon^2\mathscr{L}_1(\psi) + \varepsilon J[\psi, \frac{\partial^2\psi}{\partial y^2}] + \varepsilon^3 J[\psi, \frac{\partial^2\psi}{\partial X^2}] = 0, \tag{13}$$

where the equation above only includes variables X and T, and variables x, t have vanished. It is convenient to introduce two linear differential operators, they are defined as

$$\mathscr{L}_0 = [(U - c_0)\frac{\partial^2}{\partial y^2} + p(y)]\frac{\partial}{\partial X}, \tag{14}$$

$$\mathscr{L}_1 = \frac{\partial}{\partial T}\frac{\partial^2}{\partial y^2} + (U - c_0)\frac{\partial^3}{\partial X^3}. \tag{15}$$

Assuming that the disturbance stream function ψ can be expressed as the asymptotic expansion[26]

$$\psi = \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \dots , \tag{16}$$

then substituting Eq.(16) into Eq.(13), we obtain the system of equations and boundary conditions.
To the order $O(\varepsilon^0)$, we have

$$\mathscr{L}_0[\psi_0] = 0, \tag{17}$$

$$\psi_0(0) = \psi_0(1) = 0, \tag{18}$$

where Eq.(17) is a linear differential equation. We assume that ψ_0 has the form

$$\psi_0 = A(X,T)\Phi_0(y). \tag{19}$$

Substitution of Eq.(19) into Eqs.(17) and (18) yields

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{p(y)}{U - c_0}\right)\Phi_0 = 0, \tag{20}$$

$$\Phi_0(0) = \Phi_0(1) = 0. \tag{21}$$

In Eq.(20), we have assumed $U - c_0 \neq 0$. Equations (20) and (21) define an eigenvalue problem for the eigenvalue c_0. Once $p(y)$ are specified, $\Phi_0(y)$ can be determined. Since $p(y)$ is nonlinear function of variable y, it is difficult to get the analytic solution of this eigenvalue problem. Additionally, in $O(\varepsilon^0)$, we have seen two facts: one is that the space structure of the wave is clear; and another is that it is a time-invariant system. However, we could not determine the evolution of the amplitude of the solitary Rossby waves.
To the order $O(\varepsilon^1)$, we have

$$\mathscr{L}_0[\psi_1] = -J\left[\psi_0, \frac{\partial^2\psi_0}{\partial y^2}\right] \equiv F_1, \tag{22}$$

$$\psi_1(0) = \psi_1(1) = 0, \tag{23}$$

where

$$F_1 = -J\left[\psi_0, \frac{\partial^2\psi_0}{\partial y^2}\right] = A\frac{\partial A}{\partial X}\left(\frac{p(y)}{U - c_0}\right)_y \Phi_0^2,$$

in which $\left(\frac{p(y)}{U - c_0}\right)_y$ is the the derivative of $\frac{p(y)}{U - c_0}$.
For non-singular neutral modes we can continue the analysis and obtain

$$\psi_1 = \frac{1}{2}A^2(X,T)\Phi_1(y). \tag{24}$$

Substituting Eq.(24) into Eq.(22), we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{p(y)}{U - c_0}\right)\Phi_1 = \left(\frac{p(y)}{U - c_0}\right)_y \frac{\Phi_0^2}{U - c_0}, \tag{25}$$

$$\Phi_1(0) = \Phi_1(1) = 0. \tag{26}$$

In order to derive the mathematical model for the amplitude of the waves, we would solve a higher order problem, such as $O(\varepsilon^2)$.
To the order $O(\varepsilon^2)$, we have

$$\mathscr{L}_0[\psi_2] = -\mathscr{L}_1[\psi_0] - J\left[\psi_0, \frac{\partial^2\psi_1}{\partial y^2}\right] - J\left[\psi_1, \frac{\partial^2\psi_0}{\partial y^2}\right] \equiv F_2, \tag{27}$$

$$\psi_2(0) = \psi_2(1) = 0, \tag{28}$$

where

$$F_2 = -\mathscr{L}_1[\psi_0] - J\left[\psi_0, \frac{\partial^2\psi_1}{\partial y^2}\right] - J\left[\psi_1, \frac{\partial^2\psi_0}{\partial y^2}\right].$$

It is clear that there is dispersion effect in longitudinal direction and there is nonlinear effect in $O(\varepsilon^2)$.

So we call them weak dispersion effect and weakly nonlinear effect. We take that ψ_2 has the form $\psi_2 = B(X,T)\Phi_2(y)$. Equation (27) is multiplied by Φ_0 ,which is then integrated with respect to y (for $1 \geq y \geq 0$), at the same time employing identity

$$\Phi_0 \frac{\partial^2 \Phi_2}{\partial y^2} \equiv \frac{\partial}{\partial y}\left[\Phi_0 \frac{\partial \Phi_2}{\partial y}\right] - \frac{\partial}{\partial y}\left[\Phi_2 \frac{\partial \Phi_0}{\partial y}\right] + \Phi_2 \frac{\partial^2 \Phi_0}{\partial y^2},$$

we obtain

$$\int_0^1 \Phi_0(y) \frac{F_2}{U - c_0} \mathrm{d}y = 0. \tag{29}$$

This indicates that if perturbation problem Eq.(16) has efficient solution, then secular term F_2 must satisfy Eq.(29), or else the amplitude of the wave will be infinite, this is meaningless. Substitution of F_2 and Eq.(24) into Eq.(29) yields

$$\begin{aligned}&\int_0^1 \frac{p(y)}{(U-c_0)^2}\Phi_0^2(y)\mathrm{d}y\frac{\partial A}{\partial T} \\ &+\int_0^1 \frac{1}{2(U-c_0)}\left\{3\left(\frac{p(y)}{U-c_0}\right)_y \Phi_0^2(y)\Phi_1(y) \right.\\ &\left. -\left(\frac{1}{U-c_0}\left(\frac{p(y)}{U-c_0}\right)_y\right)_y \Phi_0^4(y)\right\}\mathrm{d}yA^2\frac{\partial A}{\partial X} \\ &-\int_0^1 \Phi_0^2(y)\mathrm{d}y\frac{\partial^3 A}{\partial X^3} = 0,\end{aligned} \tag{30}$$

here $\left(\frac{p(y)}{U-c_0}\right)_y$ is the derivative of $\frac{p(y)}{U-c_0}$ with respect to y. In order to simplify Eq.(30), we introduce coefficients I_0, S_0 and R_{00}, they are defined as

$$I_0 = \int_0^1 \frac{p(y)}{(U-c_0)^2}\Phi_0^2(y)\mathrm{d}y, \tag{31}$$

$$S_0 = \frac{-\int_0^1 \Phi_0^2(y)\mathrm{d}y}{I_0}, \tag{32}$$

$$R_{00} = \frac{\int_0^1 \frac{1}{2(U-c_0)}\{3\left(\frac{p(y)}{U-c_0}\right)_y \Phi_0^2(y)\Phi_1(y) - \left(\frac{1}{U-c_0}\left(\frac{p(y)}{U-c_0}\right)_y\right)_y \Phi_0^4(y)\}\mathrm{d}y}{I_0}, \tag{33}$$

where $\Phi_0(y)$, $\Phi_1(y)$ are determined by the solution of the eigenvalue problem Eqs.(20), (21) and (25), (26) respectively. Equation (30) has the form

$$\frac{\partial A}{\partial T} + R_{00}A^2\frac{\partial A}{\partial X} + S_0\frac{\partial^3 A}{\partial X^3} = 0. \tag{34}$$

In equation (34), the amplitude of solitary Rossby waves satisfies the well-known mKdV equation. Obviously, coefficients S_0 and R_{00} are related to functions $\beta(y)$ and $U(y)$. If $U(y)$ is a constant, i.e., there is no shear flow, coefficients S_0 and R_{00} are only related to $\beta(y)$, and Eq.(34) is also a mKdV equation. The discussion above manifests that the variation of $\beta(y)$ with the latitudical variable y can induce solitary Rossby waves.

3. Concluding remarks

In this paper, the asymptotic method and weakly nonlinear method are used to investigate nonlinear Rossby waves in a zonal flow in the middle and high latitude area by employing a simple shallow-water model. The nonlinear mKdV equation was derived, which describes the evolution of the amplitude of solitary Rossby waves and also embodies the main characteristics of solitary Rossby waves in a basic flow. Both without the shear flow and with the shear flow, coefficients S_0 and R_{00} depend on the basic flow $U(y)$ and nonlinear function $\beta(y)$.

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