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A rough hypersingular integral operator with an oscillating factor

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Abstract

We study certain hypersingular integrals $\mathcal{T}_{\Omega,\alpha,\beta} f$ defined on all test functions $f \in \mathcal{S}(\mathbb{R}^n)$, where the kernel of the operator $\mathcal{T}_{\Omega,\alpha,\beta}$ has a strong singularity $|y|^{-n-\alpha}$ $(\alpha > 0)$ at the origin, an oscillating factor $e^{i|y|^{-\beta}}$ $(\beta > 0)$ and a distribution $\Omega \in H^r(S^{n-1})$, $0 < r < 1$. We show that $\mathcal{T}_{\Omega,\alpha,\beta}$ extends to a bounded linear operator from the Sobolev space $\dot{L}^p_\gamma \cap L^p$ to the Lebesgue space L^p for $\beta/(\beta-\alpha) < p < \beta/\alpha$, if the distribution Ω is in the Hardy space $H^r(S^{n-1})$ with $0 < r = (n-1)/(n-1+\gamma)$ $(0 < \gamma \leqslant \alpha)$ and $\beta > 2\alpha > 0$.

Keywords: Singular integrals; Hardy spaces on spheres; Maximal operators; Sobolev spaces

1. Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, $n \geqslant 2$, with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $H^r(S^{n-1})$ be the Hardy space on S^{n-1}. Recall that $H^r(S^{n-1})$ are distribution spaces if

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$0 < r < 1$; $H^r(S^{n-1}) = L^r(S^{n-1})$ if $1 < r < \infty$ and $H^1(S^{n-1})$ is a proper subspace of the Lebesgue space $L^1(S^{n-1})$.

Let $\chi_{(a,b)}(t)$ stand for the characteristic function on the interval (a,b). For $\epsilon > 0$ and $\alpha \geqslant 0$, we define

$$L_\epsilon(t) = \chi_{(\epsilon,\infty)}(t) b(t) t^{-1-\alpha},$$

where $b(t)$ is a bounded function. For $f \in \mathscr{S}(\mathbb{R}^n)$, we write $f(x-y) = f_{x,t}(y')$ with $t = |y|$ and $y' = y/|y|$ if $y \neq 0$. Denote $\langle \Omega, \phi \rangle$ as the pairing between Ω and a C^∞ function ϕ on S^{n-1}. The operators $T_{\Omega,\alpha,\epsilon}$ are defined on the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ by

$$T_{\Omega,\alpha,\epsilon} f(x) = \int_0^\infty L_\epsilon(t) \langle \Omega, f_{x,t} \rangle \, dt. \tag{1}$$

The hypersingular integral operator $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha}(f)(x) = \lim_{\epsilon \to 0} T_{\Omega,\alpha,\epsilon} f(x), \tag{2}$$

where $\Omega \in H^r(S^{n-1})$, $r = (n-1)/(n-1+\gamma)$, $0 < \gamma \leqslant \alpha$, satisfies the mean value zero condition

$$\langle \Omega, Y_m \rangle = 0 \tag{3}$$

for all spherical polynomials Y_m with degrees $\leqslant [\alpha]$.

Let $\mathscr{L}_\epsilon(t) = \chi_{(\epsilon,\infty)}(t) t^{-1-\alpha} e^{it^{-\beta}}$. In this paper, we study the hypersingular integral operator $\mathscr{T}_{\Omega,\alpha,\beta}$ defined by

$$\mathscr{T}_{\Omega,\alpha,\beta}(f)(x) = \lim_{\epsilon \to 0} \mathscr{T}_{\Omega,\alpha,\beta,\epsilon} f(x), \tag{4}$$

where

$$\mathscr{T}_{\Omega,\alpha,\beta,\epsilon}(f)(x) = \left\langle \Omega, \int_0^\infty \mathscr{L}_\epsilon(t) f_{x,t} \, dt \right\rangle. \tag{5}$$

From the discussion in [2], we see that the definition of $T_{\Omega,\alpha}$ in (2) is well defined and $T_{\Omega,\alpha}(f)(x)$ exists for all $x \in \mathbb{R}^n$ because of the cancellation condition (3). Denote $T_{\Omega,\alpha}$ by T_Ω if $\alpha = 0$. For $\Omega \in L^1(S^{n-1})$, T_Ω is the well-known rough singular integral operator initially studied by Calderón and Zygmund in their pioneering papers [7,8]. In [8], using the method of rotation, Calderón and Zygmund proved that if $\Omega \in \mathrm{L}\log^+\mathrm{L}(S^{n-1})$ satisfies the mean value zero condition over S^{n-1}, then the operator T_Ω with kernel $\Omega(x')|x|^{-n}$ is a bounded operator on the Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Later on, the above results were extended and improved by many authors. Readers can view [3,9–14,17,18,20] among many other references for a good survey. Particularly, we list the following results which are related to this paper.

Theorem A. [15,16] *Suppose $\Omega \in H^1(S^{n-1})$ satisfies* (3). *If $\beta > 2\alpha > 0$, then the operator $\mathscr{T}_{\Omega,\alpha,\beta}$ is bounded on $L^p(\mathbb{R}^n)$ for $\beta/(\beta-\alpha) < p < \beta/\alpha$.*

Theorem B. [2] *Suppose $\Omega \in H^r(S^{n-1})$ with $r = (n-1)/(n-1+\alpha)$ and Ω satisfies* (3). *Then for $1 < p < \infty$,*

$$\left\| T_{\Omega,\alpha}(f) \right\|_{L^p(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{L}^p_\alpha(\mathbb{R}^n)}, \tag{6}$$

where $\dot{L}^p_\alpha(\mathbb{R}^n)$ is the homogeneous Sobolev space whose definition can be found in Section 2.

Observe that all the results in Theorems A, B and in the above mentioned references assume the cancellation condition (3). On the other hand, people are interested in the operator with an oscillating factor $e^{it^{-\beta}}$ in its kernel since it is related to the Bochner–Riesz operators (see [19]). It is clear that the oscillating factor $e^{it^{-\beta}}$ ($\beta > 0$) in the kernel of $\mathscr{T}_{\Omega,\alpha,\beta}$ eliminates the singularity at the origin that is caused by $\alpha > 0$, while the kernel has no singularity at infinity because of $\alpha > 0$. By integrating by parts, it is straightforward to check that $\mathscr{T}_{\Omega,\alpha,\beta} f(x)$ in (4) exists for each $x \in \mathbb{R}^n$ if $\beta > \alpha$, even without assuming the cancellation property (3) on Ω. This leads us to expect that the operator $\mathscr{T}_{\Omega,\alpha,\beta}$ (without the assumption (3)) may be bounded in some function spaces, like the operator $T_{\Omega,\alpha}$ in Theorem B.

Theorem 1. *Let $\Omega \in H^r(S^{n-1})$ with $0 < r = (n-1)/(n-1+\gamma)$, $\alpha \geqslant \gamma > 0$. Then*

$$\left\| \mathscr{T}_{\Omega,\alpha,\beta}(f) \right\|_{L^p(\mathbb{R}^n)} \leqslant C \|\Omega\|_{H^r(S^{n-1})} \left\{ \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)} \right\}$$

for $\beta/(\beta-\alpha) < p < \beta/\alpha$, provided that $\beta > 2\alpha$.

Moreover, if $\langle \Omega, Y_m \rangle = 0$ for all $m \leqslant [\delta]$ and $0 < \delta \leqslant \gamma$, then

$$\left\| \mathscr{T}_{\Omega,\alpha,\beta}(f) \right\|_{L^p(\mathbb{R}^n)} \leqslant C \|\Omega\|_{H^r(S^{n-1})} \left\{ \|f\|_{\dot{L}^p_\delta(\mathbb{R}^n)} + \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)} \right\}$$

for $\beta/(\beta+\delta-\alpha) < p < \beta/(\alpha-\delta)$, provided that $\beta > 2(\alpha-\delta) \geqslant 0$.

From Theorem B [2] and Theorem 1, we observe the following facts. Let $\Omega \in H^r(S^{n-1})$ with $0 < r = (n-1)/(n-1+\gamma)$, $\alpha \geqslant \gamma > 0$. If Ω satisfies the cancellation condition $\langle \Omega, Y_m \rangle = 0$ for all $m \leqslant [\gamma]$, then $\mathscr{T}_{\Omega,\alpha,\beta}$ is bounded from the homogeneous space $\dot{L}^p_\gamma(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ for all $p \in (\frac{\beta}{\beta+\gamma-\alpha}, \frac{\beta}{\alpha-\gamma})$. Without any cancellation condition on Ω, $\mathscr{T}_{\Omega,\alpha,\beta}$ is bounded from the inhomogeneous space $L^p_\gamma(\mathbb{R}^n)$ to the Lebesgue space $L^p(\mathbb{R}^n)$ for a smaller range $(\frac{\beta}{\beta-\alpha}, \frac{\beta}{\alpha})$ of p, where $L^p_\gamma(\mathbb{R}^n)$ is the set of all functions f satisfying

$$\|f\|_{L^p_\gamma(\mathbb{R}^n)} \approx \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)} < \infty.$$

The proof of Theorem 1 is different from those of Theorems A and B. It is given in Section 3, after we present some necessary background in Section 2. In Section 4, we study the operator $\mathscr{T}_{\Omega,\alpha,\beta}$ for the case $\gamma = 0$ and the case $\beta = 2(\alpha-\gamma)$. In this paper, the letter C stands for a positive constant which may vary at each occurrence. However, it is independent of any essential variable. Also we write $f(x) \approx g(x)$ if there exist some positive constants A and B such that $Af(x) \leqslant g(x) \leqslant Bf(x)$.

2. Definitions and lemmas

2.1. *The Hardy space $H^r(S^{n-1})$*

Recall that the Poisson kernel on S^{n-1} is defined by $P_{ty'}(x') = \frac{(1-t^2)}{|ty'-x'|^n}$, where $0 \leqslant t < 1$ and $x', y' \in S^{n-1}$. For any $\Omega \in \mathscr{S}'(S^{n-1})$, we define the radial maximal function $P^+\Omega(x')$ by $P^+\Omega(x') = \sup_{0 \leqslant t < 1} |\langle P_{ty'}, \Omega \rangle|$, where $\mathscr{S}'(S^{n-1})$ is the space of Schwartz distributions on S^{n-1}.

The Hardy space $H^r(S^{n-1})$, $0 < r \leqslant 1$, is the linear space of distributions $\Omega \in \mathscr{S}'(S^{n-1})$ with the finite norm $\|\Omega\|_{H^r(S^{n-1})} = \|P^+\Omega\|_{L^r(S^{n-1})} < \infty$. The space $H^r(S^{n-1})$ was studied in [4,5] (see also [6]). Note that S^1 and S^3 are compact Lie groups. For H^r on a compact Lie group, the reader can refer to [1]. An important property of $H^r(S^{n-1})$ is the atomic decomposition, which is reviewed below.

An exceptional atom $E(x)$ is an $L^\infty(S^{n-1})$ function bounded by 1. A regular (r,∞) atom is an $L^\infty(S^{n-1})$ function $a(x')$ that satisfies

$$\operatorname{supp}(a) \subset \big\{x' \in S^{n-1}\colon |x'-x'_0| < \rho\big\} \quad \text{for some } x'_0 \in S^{n-1} \text{ and } 0 < \rho \leqslant 2, \tag{7}$$

$$\int_{S^{n-1}} a(x')Y_m(x')\,d\sigma(x') = 0 \tag{8}$$

for all spherical harmonic polynomials Y_m with degrees less than or equal to $[\gamma]$, where $r = (n-1)/(n-1+\gamma)$ and

$$\|a\|_{L^\infty(S^{n-1})} \leqslant \rho^{-(n-1)/r}. \tag{9}$$

From [2], we find that any $\Omega \in H^r(S^{n-1})$ has an atomic decomposition

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j + \|\Omega\|_{H^r(S^{n-1})} A,$$

where each a_j is an (r,∞) atom and $\|A\|_{L^\infty} \leqslant 1$.

For the rest of this paper, if $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, $\xi \neq 0$, we write $\xi' = \xi/|\xi| = (\xi'_1,\dots,\xi'_n) \in S^{n-1}$.

Lemma 2.1. *Suppose $n \geqslant 3$ and $\Omega(\cdot)$ is an (r,∞) atom on S^{n-1} supported in $S^{n-1} \cap B(\xi,\rho)$, where $B(\xi,\rho)$ is the ball with radius ρ and center $\xi \in S^{n-1}$. Let*

$$F_\Omega(s) = \big(1-s^2\big)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{S^{n-2}} \Omega\big(s, \sqrt{1-s^2}\,\tilde{y}\big)\,d\sigma(\tilde{y}).$$

Then there exist $s_o \in \mathbb{R}$ and a constant C independent of $\Omega(\cdot)$ such that

$$\operatorname{supp}(F_\Omega) \subset \big(s_o - 2r(\xi'), s_o + 2r(\xi')\big), \tag{10}$$

$$\|F_\Omega\|_\infty \leqslant C\rho^{(n-1)(1-1/r)} r(\xi')^{-1}, \tag{11}$$

$$\int_{\mathbb{R}} F_\Omega(s)s^k\,ds = 0, \quad k = 0,1,2,\dots,[\gamma], \quad \text{and} \tag{12}$$

$$\int_{\mathbb{R}} \big|F_\Omega(s)\big|\,ds \leqslant C\rho^{(n-1)(1-1/r)}, \tag{13}$$

where $r(\xi') = |A_\rho \xi'| = |\xi|^{-1}|A_\rho \xi|$ and $A_\rho \xi = (\rho^2\xi_1, \rho\xi_2, \dots, \rho\xi_n)$.

Lemma 2.2. *Suppose $n = 2$ and $\Omega(\cdot)$ is an (r,∞) supported in $S^1 \cap B(\xi,\rho)$. Let*

$$F_\Omega(s) = \big(1-s^2\big)^{-1/2} \chi_{(-1,1)}(s) \big(\Omega\big(s, \sqrt{1-s^2}\big) + \Omega\big(s, -\sqrt{1-s^2}\big)\big).$$

Then $F_\Omega(s)$ satisfies (10), (12), (13) *and*

$$\|F_\Omega\|_q \leqslant C|A_\rho \xi'|^{-1+1/q} \rho^{(1-1/r)} \quad \text{for some } q \in (1,2). \tag{14}$$

Lemmas 2.1 and 2.2 can be found in [12] (see also [13] for the case $r = 1$).

2.2. The Sobolev space $\dot{L}^p_\alpha(\mathbb{R}^n)$

Fix a radial function $\Phi \in C^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n\colon \frac{1}{2} < |x| \leqslant 2\}$, $0 \leqslant \Phi(x) \leqslant 1$ and $\Phi(x) > c > 0$ if $\frac{3}{5} \leqslant |x| \leqslant \frac{5}{3}$. Let $\Phi_j(x) = \Phi(2^j x)$. Define the function Ψ_j by $\hat{\Psi}_j(\xi) = \Phi_j(\xi)$ so that $\widehat{\Psi_j * f}(\xi) = \Phi_j(\xi)\hat{f}(\xi)$. For $1 < p < \infty$ and $\alpha \in \mathbb{R}$, the homogeneous Sobolev space $\dot{L}^p_\alpha(\mathbb{R}^n)$ is the set of all distributions f with the given norm

$$\|f\|_{\dot{L}^p_\alpha(\mathbb{R}^n)} = \left\| \left(\sum_{k\in\mathbb{Z}} \left|2^{-k\alpha}\Psi_k * f\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

It is well known that for $f \in \dot{L}^2_\alpha(\mathbb{R}^n)$,

$$\|f\|_{\dot{L}^2_\alpha(\mathbb{R}^n)} \approx \left(\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 |\xi|^{2\alpha}\, d\xi \right)^{1/2},$$

and if α is a nonnegative integer, then for any $f \in \dot{L}^p_\alpha(\mathbb{R}^n)$,

$$\|f\|_{\dot{L}^p_\alpha(\mathbb{R}^n)} \approx \sum_{|l|=\alpha} \left\|D^l f\right\|_{L^p(\mathbb{R}^n)}.$$

3. Proof of Theorem 1

In view of the results in [2], it suffices to prove the theorem by considering two cases: $\Omega(y') = a(y')$ (a regular (r,∞) atom with $r = (n-1)/(n-1+\gamma)$) and $\Omega(y') = A(y')$ (an exceptional atom). We show that there is a constant C independent of both exceptional and regular atoms such that

$$\|\mathscr{T}_{\Omega,\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \leqslant C\left\{\|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)}\right\}.$$

We will prove the theorem only for the case $n \geqslant 3$, since the proof of the case $n = 2$ is the same (with Lemma 2.2 applied instead of Lemma 2.1).

We first consider the case that Ω is a regular (r,∞) atom. If $\alpha = \gamma$, then the result comes from Theorem B. So we assume $\alpha > \gamma$. Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $(2^{j-1}, 2^{j+1})$. To be precise, we choose a radial function $\Phi \in C^\infty(\mathbb{R}^n)$ with $0 \leqslant \Phi(x) \leqslant 1$, and $\operatorname{supp}(\Phi) \subset \{x \in \mathbb{R}^n\colon 1/2 < |x| \leqslant 2\}$. We let $\Phi_j(x) = \Phi(2^j x)$ and require that $\sum_{j=-\infty}^{\infty} \Phi_j(t) = 1$, for all $t > 0$. Note that $\operatorname{supp}(\Phi_j) \subset (2^{-j-1}, 2^{-j+1})$. We write

$$\mathscr{T}_{\Omega,\alpha,\beta} f(x) = \sum_{j=-\infty}^{\infty} \mathscr{T}_{\Omega,\alpha,\beta,j} f(x),$$

where $\mathscr{T}_{\Omega,\alpha,\beta,j} f(x) = \int_{\mathbb{R}^n} e^{i|y|^{-\beta}} |y|^{-n-\alpha} \Omega(y') \Phi_j(y) f(x-y)\, dy$.

Proposition 3.1. *Let $0 < \gamma < \alpha$ and $r = (n-1)/(n-1+\gamma)$. Then there is a constant C independent of (r,∞) atoms Ω and indices j such that*

$$\|\mathscr{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \leqslant C 2^{j(\alpha-\gamma)} \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)}. \tag{15}$$

Moreover, if $\Omega \in L^1(S^{n-1})$, then

$$\|\mathscr{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \leqslant C 2^{j\alpha} \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}. \tag{16}$$

Proof. Without loss of generality, we may assume that $\operatorname{supp}(\Omega) \subset B(\mathbf{1}, \rho) \cap S^{n-1}$, where $\mathbf{1} = (1, 0, \dots, 0)$. In view of [21], for any test function f we may write $f = G_\gamma * f_\gamma$, where $\|f_\gamma\|_{L^p(\mathbb{R}^n)} \approx \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)}$, and G_γ is an $L^1(\mathbb{R}^n)$ function having the following properties:

(a) $G_\gamma \geqslant 0$,
(b) $G_\gamma(x) \leqslant C_\gamma |x|^{\gamma-n}$ if $n > \gamma$, and
(c) $|(D^\nu G_\gamma)(x)| \leqslant C_{\gamma,\nu}|x|^{\gamma-|\nu|-n}$ if $|\nu| > 0$ and $n+1 > \gamma$.

We first consider the case $0 < \gamma < 1$. By the cancellation condition on Ω and the support condition on Φ, we have

$$\left|\mathscr{T}_{\Omega,\alpha,\beta,j} f(x)\right| \leqslant C 2^{j(\alpha-\gamma)} \int_{2^{-j-1}}^{2^{-j+1}} t^{-1-\gamma} \left| \int_{S^{n-1}} \Omega(y')\big(f(x-ty') - f(x-t\mathbf{1})\big)\, d\sigma(y') \right| dt.$$

We treat $f(x-ty') - f(x-t\mathbf{1})$ with $y' \in B(\mathbf{1}, \rho)$. Using the representation $f = G_\gamma * f_\gamma$, we have

$$\begin{aligned}
\left|f(x-ty') - f(x-t\mathbf{1})\right| &\leqslant C \int_{\mathbb{R}^n} \left|f_\gamma(x-z)\right|\left|G_\gamma(z-ty') - G_\gamma(z-t\mathbf{1})\right| dz \\
&= C\left\{ \int_{|z-t\mathbf{1}|\geqslant 2t\rho} \dots\, dz + \int_{|z-t\mathbf{1}|<2t\rho} \dots\, dz \right\}. \\
&\equiv J_1 + J_2.
\end{aligned}$$

By a change of variable $z - t\mathbf{1} \to z$, we have

$$J_1 = C \int_{|z|\geqslant 2t\rho} \left|f_\gamma(x-z-t\mathbf{1})\right|\left|G_\gamma\big(z - t(\mathbf{1}-y')\big) - G_\gamma(z)\right| dz.$$

For J_1, note that $|t(y'-\mathbf{1})| \leqslant Ct\rho < C|z|/2$. By the Mean Value Theorem and by (c), we have

$$\begin{aligned}
J_1 &\leqslant C \int_{|z|\geqslant 2t\rho} t\rho\left|f_\gamma(x-z-t\mathbf{1})\right||z|^{\gamma-n-1}\, dz \\
&\leqslant C \int_{S^{n-1}} \int_{2t\rho}^{\infty} t\rho s^{\gamma-2}\left|f_\gamma(x-sz'-t\mathbf{1})\right| ds\, d\sigma(z').
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
J_1 \leqslant{}& C \int_{S^{n-1}} (t\rho)^{\gamma-1} \int_0^{2t\rho} \left|f_\gamma(x-uz'-t\mathbf{1})\right| du\, d\sigma(z') \\
&+ C \int_{S^{n-1}} \int_{2t\rho}^{\infty} (t\rho) s^{\gamma-3} \int_0^{s} \left|f_\gamma(x-uz'-t\mathbf{1})\right| du\, ds\, d\sigma(z').
\end{aligned}$$

Let $M_{z'} f(x)$ denote the maximal function $M_{z'} f(x) = \sup_{r>0}\{\frac{1}{r}\int_0^r |f_\gamma(x-uz')|\,du\}$. It is known from [20] that $\|M_{z'} f_\gamma\|_{L^p(\mathbb{R}^n)} \leqslant C\|f_\gamma\|_{L^p(\mathbb{R}^n)}$, $1<p\leqslant\infty$, where C is independent of z'. Thus we have $J_1 \leqslant C(t\rho)^\gamma \int_{S^{n-1}} M_{z'} f_\gamma(x-t\mathbf{1})\,d\sigma(z')$.

On the other hand, $J_2 \leqslant \Delta_1 + \Delta_2$, where

$$\Delta_1 = \int\limits_{|z-t\mathbf{1}|<2t\rho} \big|f_\gamma(x-z)\big| G_\gamma(z-ty')\,dz \quad\text{and}\quad \Delta_2 = \int\limits_{|z|<2t\rho} \big|f_\gamma(x-z-t\mathbf{1})\big| G_\gamma(z)\,dz.$$

Let $\tilde z = z - ty'$. Then for Δ_1, we have $|\tilde z| \leqslant |z-t\mathbf{1}| + |t\mathbf{1}-ty'| \leqslant 3t\rho$, because $y' \in \operatorname{supp}(\Omega)$. Thus by a change of variable $z-ty' \to z$ and by (b), we obtain

$$\Delta_1 \leqslant C \int\limits_{|z|\leqslant 3t\rho} \big|f_\gamma(x-z-ty')\big||z|^{\gamma-n}\,dz = C\int\limits_{S^{n-1}}\int\limits_0^{3t\rho} u^{\gamma-1}\big|f_\gamma(x-uz'-ty')\big|\,du\,d\sigma(z').$$

Integrating by parts yields $\Delta_1 \leqslant C\int_{S^{n-1}} (t\rho)^\gamma M_{z'}(f_\gamma)(x-ty')\,d\sigma(z')$.

Similarly, $\Delta_2 \leqslant C\int_{S^{n-1}}(t\rho)^\gamma\{M_{z'}(f_\gamma)(x-t\mathbf{1}) + M_{z'}(f_\gamma)(x-ty')\}\,d\sigma(z')$. Since $|f(x-ty') - f(x-t\mathbf{1})| \leqslant J_1 + J_2$, substituting the estimates of J_1 and J_2 into $\mathscr{T}_{\Omega,\alpha,\beta,j} f(x)$, we obtain

$$\begin{aligned}&\big|2^{j(\gamma-\alpha)}\mathscr{T}_{\Omega,\alpha,\beta,j} f(x)\big|\\ &\quad\leqslant C\int\limits_{S^{n-1}}\int\limits_{S^{n-1}} \big|\Omega(y')\big|\rho^\gamma\big\{M_{\mathbf{1}}\circ M_{z'} f_\gamma(x) + M_{y'}\circ M_{z'} f_\gamma(x)\big\}\,d\sigma(z')\,d\sigma(y').\end{aligned}$$

Since Ω is an (r,∞) atom supported in $B(\mathbf{1},\rho)$ with $r=(n-1)/(n-1+\gamma)$, it follows that $\int_{S^{n-1}}|\Omega(y')|\rho^\gamma\,d\sigma(y') \leqslant C$ uniformly for Ω and ρ. Thus

$$\begin{aligned}2^{j(\gamma-\alpha)}\big\|\mathscr{T}_{\Omega,\alpha,\beta,j} f(x)\big\|_{L^p(\mathbb{R}^n)} &\leqslant C\big\{\|M_{\mathbf{1}}\circ M_{z'} f_\gamma\|_{L^p(\mathbb{R}^n)} + \|M_{y'}\circ M_{z'} f_\gamma\|_{L^p(\mathbb{R}^n)}\big\}\\ &\leqslant C\|f\|_{\dot L^p_\gamma(\mathbb{R}^n)}.\end{aligned}$$

The proposition is proved for the case $0<\gamma<1$.

We now consider the case $\gamma>1$ and γ is not an integer (see [2, Lemma 4.3] for the case that γ is an integer). We write $\gamma = m+\nu$, where m is a positive integer and $0<\nu<1$. By the Taylor theorem and by the cancellation condition on Ω, we have for $y' \in \operatorname{supp}(\Omega) = B(\mathbf{1},\rho)$,

$$\begin{aligned}&\Bigg|\int\limits_{S^{n-1}} \Omega(y') f(x-ty')\,d\sigma(y')\Bigg|\\ &\quad\leqslant Ct^m\rho^m\sum_{|\beta|=m}\int\limits_0^1\int\limits_{S^{n-1}}\big|\Omega(y')\big|\big|D^\beta f\big(x-t\mathbf{1}-st(y'-\mathbf{1})\big) - D^\beta f(x-t\mathbf{1})\big|\,d\sigma(y')\,ds.\end{aligned}$$

Similar to the case $0<\gamma<1$, we write for any fixed $s\in(0,1)$,

$$\begin{aligned}&\big|D^\beta f\big(x-t\mathbf{1}-st(y'-\mathbf{1})\big) - D^\beta f(x-t\mathbf{1})\big|\\ &\quad\leqslant C\int\limits_{\mathbb{R}^n}\big|\big(D^\beta f\big)_\nu(x-z)\big|\big|G_\nu\big(z+st(y'-\mathbf{1})+t\mathbf{1}\big) - G_\nu(z+t\mathbf{1})\big|\,dz\end{aligned}$$

$$
\begin{aligned}
&= C\left\{ \int\limits_{|z+t\mathbf{1}|\geqslant 3t\rho} \ldots dz + \int\limits_{|z+t\mathbf{1}|<3t\rho} \ldots dz \right\} \\
&\equiv I_1 + I_2.
\end{aligned}
$$

By the same argument as in the proof for the case $0 < \gamma < 1$, we have

$$
\begin{aligned}
&2^{j(\gamma-\alpha)} \|\mathscr{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \\
&\quad \leqslant C \sum_{|\beta|=m} \int_0^1 \int_{S^{n-1}} \int_{S^{n-1}} \big\| M_{\mathscr{P}} \circ M_{\mathscr{Q}}\big(\big(D^\beta f\big)_\nu\big)\big\|_{L^p(\mathbb{R}^n)} \, d\sigma(y') \, d\sigma(z') \, ds \\
&\qquad + C \sum_{|\beta|=m} \int_0^1 \int_{S^{n-1}} \int_{S^{n-1}} \big\| M_{\mathscr{R}} \circ M_{\mathscr{Q}}\big(\big(D^\beta f\big)_\nu\big)\big\|_{L^p(\mathbb{R}^n)} \, d\sigma(y') \, d\sigma(z') \, ds,
\end{aligned}
$$

where

$$
M_{\mathscr{H}} f(x) = \sup_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \big| f(x) - \mathscr{H}(t) \big| t^{-1} \, dt
$$

for some polynomials $\mathscr{H}(t) = \mathscr{P}(t)$, $\mathscr{Q}(t)$ and $\mathscr{R}(t)$ from $\mathbb{R}$ to $\mathbb{R}^n$ whose coefficients may depend on z', y' and s. From [20, p. 477], there is a constant C independent of the coefficients of $\mathscr{H}$ such that $\|M_{\mathscr{H}} f\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$. Thus we obtain

$$
\|\mathscr{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \leqslant C 2^{j(\alpha-\gamma)} \sum_{|\beta|=m} \big\| \big(D^\beta f\big)_\nu \big\|_{L^p(\mathbb{R}^n)} \approx C 2^{j(\alpha-\gamma)} \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)}.
$$

Inequality (15) is proved. Inequality (16) also follows since

$$
\|\mathscr{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \leqslant C 2^{j\alpha} \|\Omega\|_{L^1(S^{n-1})} \|M_{\mathscr{H}} f\|_{L^p(\mathbb{R}^n)}
$$

for some polynomial $\mathscr{H}(t)$. Proposition 3.1 is proved. □

We now calculate the L^2 estimate of $\mathscr{T}_{\Omega,\alpha,\beta,j} f(x)$. In light of Fourier transform, we have $\widehat{\mathscr{T}_{\Omega,\alpha,\beta,j} f}(\xi) = m_j(\xi) \hat{f}(\xi)$, where

$$
\begin{aligned}
m_j(\xi) &= \int_0^\infty e^{it^{-\beta}} t^{-1-\alpha} \Phi_j(t) \bigg(\int_{S^{n-1}} \Omega(y') e^{-2\pi i t\langle y',\xi\rangle} d\sigma(y') \bigg) dt \\
&\equiv \int_{\mathbb{R}} F_\Omega(s) N_j\big(s|\xi|\big) \, ds,
\end{aligned}
$$

where F_Ω is the function in Lemma 2.1 and

$$
N_j(u) = \int_0^\infty e^{it^{-\beta}} t^{-1-\alpha} \Phi_j(t) e^{-2\pi i t u} \, dt.
$$

By a change of variable, we have

$$N_j(u) = 2^{j\alpha} \int_0^\infty e^{i2^{\beta j} t^{-\beta}} t^{-1-\alpha} \Phi(t) e^{-2^{(-j+1)}\pi i t u}\, dt.$$

It follows from Van der Corput's lemma that $|N_j(u)| \leqslant C 2^{j(\alpha-\beta/2)}$. Thus

$$\left|m_j(\xi)\right| \leqslant C 2^{j(\alpha-\beta/2)} \int_{\mathbb{R}} \left|F_\Omega(s)\right| ds \leqslant C 2^{j(\alpha-\beta/2)} \rho^{-\gamma}. \tag{17}$$

On the other hand, observe that F_Ω is supported in the interval $(s_o - 2r(\xi'), s_o + 2r(\xi'))$. By the cancellation property of F_Ω, we have

$$m_j(\xi) = \int_{\mathbb{R}} F_\Omega(s)\left\{N_j(u) - N_j(u_o)\right\} ds, \tag{18}$$

where $u = s|\xi|$, $u_o = s_o|\xi|$ and

$$N_j(u) - N_j(u_o) = 2^{j\alpha} \int_0^\infty e^{i2^{\beta j} t^{-\beta}} t^{-1-\alpha} \Phi(t)\left\{e^{-2^{(1-j)}\pi i t u} - e^{-2^{(1-j)}\pi i t u_o}\right\} dt.$$

Let $N = [\alpha]$ and denote

$$\Psi(s,t) = e^{-2^{(1-j)}\pi i t (s-s_o)|\xi|} - \sum_{k=0}^{N} \frac{(-i2^{(1-j)}\pi t(s-s_o)|\xi|)^k}{k!}.$$

Applying the cancellation property of F_Ω (Lemma 2.1), Eq. (18) becomes

$$m_j(\xi) = C \int_{\mathbb{R}} F_\Omega(s)\left(2^{j\alpha} \int_0^\infty e^{i(2^{\beta j} t^{-\beta} - 2^{(1-j)} t\pi u_o)} t^{-1-\alpha} \Phi(t) \Psi(s,t)\, dt\right) ds.$$

We first estimate the quantity appearing in the parentheses in the above equation. Since $\operatorname{supp}(\Phi) \subset [1/2, 2]$ and $\operatorname{supp}(F_\Omega) \subset (s_o - 2r(\xi'), s_o + 2r(\xi'))$, it follows from the definition of $r(\xi')$ in Lemma 2.1 that

$$\left|\Psi(s,t)\right| \leqslant C 2^{-j(N+1)} \rho^{N+1} |\xi|^{N+1} \quad \text{and} \quad \left|\frac{\partial \Psi(s,t)}{\partial t}\right| \leqslant C 2^{-j(N+1)} \rho^{N+1} |\xi|^{N+1}.$$

By Van der Corput's lemma, we have

$$\begin{aligned}
&\left|2^{j\alpha} \int_0^\infty e^{i(2^{\beta j} t^{-\beta} - 2^{(1-j)} t\pi u_o)} t^{-1-\alpha} \Phi(t) \Psi(s,t)\, dt\right| \\
&\quad \leqslant C 2^{j(\alpha-\beta/2)} \left\{ \sup_{t\in[1/2,2],\, s\in \operatorname{supp}(F_\Omega)} \left|\Phi(t)\Psi(s,t)\right| \right. \\
&\qquad \left. + \int_{1/2}^{2} \left(\left|\Phi'(t)\Psi(s,t)\right| + \left|\Phi(t)\frac{\partial \Psi(s,t)}{\partial t}\right|\right) dt \right\}.
\end{aligned}$$

Using the above estimate and inequality (13), we see that

$$\left|m_j(\xi)\right| \leqslant C 2^{j(\alpha-\beta/2)} 2^{-j(N+1)} \rho^{-\gamma+N+1} |\xi|^{N+1}. \tag{19}$$

Write $|m_j(\xi)| = |m_j(\xi)|^{(N+1-\gamma)/(N+1)} |m_j(\xi)|^{\gamma/(N+1)}$. Inequalities (17) and (19) imply that

$$\left|m_j(\xi)\right| \leqslant C 2^{-j(\beta/2+\gamma-\alpha)} |\xi|^{\gamma}. \tag{20}$$

Thus by Plancherel's theorem,

$$\|\mathcal{T}_{\Omega,\alpha,\beta,j} f\|_{L^2(\mathbb{R}^n)} \leqslant C 2^{-j(\beta/2+\gamma-\alpha)} \|f\|_{\dot{L}^2_\gamma(\mathbb{R}^n)}. \tag{21}$$

Interpolating between (15) and (21) yields for $j \geqslant 0$ and $1 < p \leqslant 2$,

$$\|\mathcal{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \leqslant C 2^{-j(\beta/p'+\gamma-\alpha)} \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)}, \tag{22}$$

where $p' = p/(p-1)$. Since $\alpha > \gamma$, inequalities (15) and (22) imply that

$$\|\mathcal{T}_{\Omega,\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \leqslant C \sum_{j\in\mathbb{Z}} \|\mathcal{T}_{\Omega,\alpha,\beta,j} f\|_{L^p(\mathbb{R}^n)} \leqslant C \|f\|_{\dot{L}^p_\gamma(\mathbb{R}^n)},$$

provided that $\beta/(\beta+\gamma-\alpha) < p \leqslant 2$. By using similar arguments, we also obtain the result for the range $2 \leqslant p < \beta/(\alpha-\gamma)$.

It remains to consider the case that Ω is an exceptional atom. By (17) and by the definition of F_Ω, we have $|m_j(\xi)| \leqslant C 2^{j(\alpha-\beta/2)} \|\Omega\|_{L^\infty(S^{n-1})}$, which implies that

$$\|\mathcal{T}_{\Omega,\alpha,\beta,j} f\|_{L^2(\mathbb{R}^n)} \leqslant C 2^{j(\alpha-\beta/2)} \|f\|_{L^2(\mathbb{R}^n)}. \tag{23}$$

Interpolating inequalities (16) and (23), we obtain

$$\|\mathcal{T}_{\Omega,\alpha,\beta} f\|_{L^p(\mathbb{R}^n)} \leqslant C \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for } \beta/(\beta-\alpha) < p < \beta/\alpha.$$

Combining the L^p estimates for both regular atom and exceptional atom, we obtain the first result of Theorem 1. For the remaining result of Theorem 1, observe that if Ω satisfies the moment conditions as mentioned in Theorem 1, then we can view an exceptional atom as an (r,∞) atom with $r = (n-1)/(n-1+\delta)$. Thus the last result follows from the L^p estimate on (r,∞) atom obtained in the previous case. The proof of Theorem 1 is complete.

4. Endpoints estimates

Recall that if $\delta = \gamma$, then the second result of Theorem 1 requires $\beta > 2(\alpha-\gamma) > 0$ and $\beta/(\beta+\gamma-\alpha) < p < \beta/(\alpha-\gamma)$. We now study the operator $\mathcal{T}_{\Omega,\alpha,\beta} f$ for the case $\beta = 2(\alpha-\gamma)$ with $\alpha > \gamma > 0$.

Theorem 2. *Let Ω be given as in Theorem* 1 *and satisfy the moment condition $\langle \Omega, Y_m\rangle = 0$ for all $m \leqslant [\gamma]$. If $\beta = 2(\alpha-\gamma) > 0$ with $\alpha > \gamma > 0$, then*

$$\left\|\mathcal{T}_{\Omega,\alpha,\beta}(f)\right\|_{L^2(\mathbb{R}^n)} \leqslant C \|\Omega\|_{H^r(S^{n-1})} \|f\|_{\dot{L}^2_\gamma(\mathbb{R}^n)}.$$

Proof. For simplicity, we only prove for the case $0 < \alpha < 1$ (see the proof of Theorem 1 for the treatment of the case $\alpha \geqslant 1$). Inspecting the proof of Theorem 1, we may assume that Ω is a regular (r,∞) atom supported in $S^{n-1} \cap B(\mathbf{1}, \rho)$. As in the proof of Theorem 1, we write

$$\mathcal{T}_{\Omega,\alpha,\beta} f(x) = \sum_{j=-\infty}^{\infty} \mathcal{T}_{\Omega,\alpha,\beta,j} f(x), \qquad \widehat{\mathcal{T}_{\Omega,\alpha,\beta,j}} f(\xi) = m_j(\xi) \hat{f}(\xi),$$

$$m_j(\xi) = \int_{\mathbb{R}} F_\Omega(s) N_j\big(s|\xi|\big)\, ds = \int_{\mathbb{R}} F_\Omega(s)\big\{N_j(u) - N_j(u_o)\big\}\, ds,$$

where $u = s|\xi|$, $u_o = s_o|\xi|$ and $N_j(u) = \int_0^\infty e^{it^{-\beta}} t^{-1-\alpha} \Phi_j(t) e^{-2\pi i t u}\, dt$.

We first estimate $|m_j(\xi)|$ for the case $j \leqslant 0$. By a direct integration, we have

$$\big|m_j(\xi)\big| \leqslant C 2^{j\alpha} \int_{\mathbb{R}} \big|F_\Omega(s)\big|\, ds = 2^{j\alpha} \rho^{-\gamma}. \tag{24}$$

On the other hand, by a change of variables, it follows that

$$N_j(u) - N_j(u_o) = 2^{j\alpha} \int_0^\infty e^{i(2^{\beta j} t^{-\beta} - 2^{(1-j)}\pi t u_o)} t^{-1-\alpha} \Phi(t)\big(e^{-2^{(1-j)}\pi i t(u-u_o)} - 1\big)\, dt.$$

Therefore,

$$\big|m_j(\xi)\big| \leqslant C \int_{\mathbb{R}} \big|F_\Omega(s)\big|\big|N_j\big(s|\xi|\big) - N_j\big(s_o|\xi|\big)\big|\, ds \leqslant C 2^{j(\alpha-1)} |\xi| \rho^{1-\gamma}. \tag{25}$$

Inequalities (24) and (25) imply that

$$\big|m_j(\xi)\big| \leqslant C 2^{j(\alpha-\gamma)} |\xi|^\gamma \quad \text{for } j \leqslant 0. \tag{26}$$

Now for $j > 0$, we assume that $u_o \neq 0$. Let $\theta(t) = 2^{j\beta} t^{-\beta} - 2^{(1-j)}\pi t u_o$. Then $\theta'(t) = -\beta 2^{j\beta} t^{-\beta-1} - 2^{(1-j)}\pi u_o$. There exist three positive constants c, c_1 and c_2 such that for $t \in [1/2, 2]$ we have

$$\big|\theta'(t)\big| \geqslant c 2^{j\beta} \quad \text{if } 2^j \geqslant c_2 |u_o|^{1/(\beta+1)}$$

and

$$\big|\theta'(t)\big| \geqslant c 2^{-j} |u_o| \quad \text{if } 2^j \leqslant c_1 |u_o|^{1/(\beta+1)}.$$

To see this, we may choose $c_1 = (\beta^{-1} 2^{-\beta+2})^{1/(\beta+1)}$ and $c_2 = 2(2/\beta)^{1/(\beta+1)}$.

For $s \in \operatorname{supp}(F_\Omega)$, if $2^j \geqslant c_2 |u_o|^{1/(\beta+1)}$ or if $2^j \leqslant c_1 |u_o|^{1/(\beta+1)}$, then by integrating by parts, we obtain

$$\begin{aligned}\big|N_j\big(s|\xi|\big) - N_j\big(s_o|\xi|\big)\big| &= \left|2^{j\alpha} \int_0^\infty t^{-1-\alpha} \Phi(t)\big(e^{-2^{(1-j)}\pi i t(u-u_o)} - 1\big) \frac{de^{i\theta(t)}}{\theta'(t)}\right| \\ &\leqslant C 2^{j(\alpha-\beta-1)} \rho.\end{aligned} \tag{27}$$

On the other hand, for $s \in \operatorname{supp}(F_\Omega)$ and if $c_1 |u_o|^{1/(\beta+1)} \leqslant 2^j \leqslant c_2 |u_o|^{1/(\beta+1)}$, then an application of Van der Corput's lemma yields $|N_j(s|\xi|) - N_j(s_o|\xi|)| \leqslant 2^{-j(\beta/2-\alpha)} 2^j \rho$.

Similarly, by defining $\Theta(t) = 2^{j\beta} t^{-\beta} - 2^{(1-j)}\pi t s|\xi|$, we have

$$\big|N_j\big(s|\xi|\big)\big| = \left|2^{j\alpha} \int_0^\infty t^{-1-\alpha} \Phi(t) \frac{de^{i\Theta(t)}}{\Theta'(t)}\right| \leqslant C 2^{j(\alpha-\beta)} \tag{28}$$

if $2^j \geqslant c_2 |u|^{1/(\beta+1)}$ or if $2^j \leqslant c_1 |u|^{1/(\beta+1)}$; and

$$\big|N_j\big(s|\xi|\big)\big| \leqslant 2^{-j(\beta/2-\alpha)} \quad \text{if } c_1 |u|^{1/(\beta+1)} \leqslant 2^j \leqslant c_2 |u|^{1/(\beta+1)}. \tag{29}$$

Let E be the set of positive integers j which satisfy either $c_1|u|^{1/(\beta+1)} \leqslant 2^j \leqslant c_2|u|^{1/(\beta+1)}$ or $c_1|u_o|^{1/(\beta+1)} \leqslant 2^j \leqslant c_2|u_o|^{1/(\beta+1)}$. Then E is a finite set that contains at most $[3\log(c_2/c_1)]+1$ positive integers j and this number $[3\log(c_2/c_1)]+1$ is independent of u and u_o.

If $j \in E$, we write

$$\left|m_j(\xi)\right| = \left|\int_{\mathbb{R}} F_\Omega(s) N_j\left(s|\xi|\right) ds\right|^{1-\gamma} \left|\int_{\mathbb{R}} F_\Omega(s)\left\{N_j\left(s|\xi|\right) - N_j\left(s_o|\xi|\right)\right\} ds\right|^{\gamma}. \tag{30}$$

Recall that $\beta/2-\alpha+\gamma=0$. Inequalities (27), (29) and (30) imply that

$$\left|m_j(\xi)\right| \leqslant C2^{-j(\beta/2-\alpha+\gamma)} = C. \tag{31}$$

If $j \notin E$ and if $j > 0$, inequalities (27), (28) and (30) yield

$$\left|m_j(\xi)\right| \leqslant C2^{-j(\beta+\gamma-\alpha)}. \tag{32}$$

Consequently, by (26), (31) and (32), we obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \left|m_j(\xi)\right| &\leqslant C \sum_{j=-\infty}^{0} 2^{j(\alpha-\gamma)}|\xi|^\gamma + \sum_{j\in E}\left|m_j(\xi)\right| + C \sum_{j>0,\, j\notin E} 2^{-j(\beta+\gamma-\alpha)}|\xi|^\gamma \\ &\leqslant C\left(1+|\xi|^\gamma\right).\end{aligned}$$

Theorem 2 is proved. □

For the case $\gamma = 0$, we have the following theorem.

Theorem 3. *Let $\Omega \in L^1(S^{n-1})$. If $\beta = 2\alpha > 0$, then $\|\mathcal{T}_{\Omega,\alpha,\beta}(f)\|_{L^2(\mathbb{R}^n)} \leqslant C\|f\|_{L^2(\mathbb{R}^n)}$. If $\beta > 2\alpha > 0$, then $\|\mathcal{T}_{\Omega,\alpha,\beta}(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}$ for $\beta/(\beta-\alpha) < p < \beta/\alpha$.*

Proof. The proof of the L^2 boundedness of $\mathcal{T}_{\Omega,\alpha,\beta}(f)$ is similar to the proof of Theorem 2. The proof of the L^p boundedness of $\mathcal{T}_{\Omega,\alpha,\beta}(f)$ is the same as the proof for exceptional atoms in Theorem 1. We omit the details. □

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References

[1] B. Blank, D. Fan, Hardy spaces on compact Lie groups, Ann. Fac. Sci. Toulouse Math. 6 (1997) 429–479.
[2] J. Chen, D. Fan, Y. Ying, Certain operators with rough singular kernels, Canad. J. Math. 55 (3) (2003) 504–532.
[3] L.-K. Chen, On a singular integral, Studia Math. 85 (1) (1986) 61–72.
[4] L. Colzani, Hardy spaces on sphere, PhD thesis, Washington University, St. Louis, 1982.
[5] L. Colzani, Hardy spaces on unit sphere, Boll. Unione Mat. Ital., Analisi Funzionale e Applicazioni, Ser. VI IV-C (1985) 219–244.
[6] L. Colzani, M.H. Taibleson, G. Weiss, Maximal estimates for Cesàro and Riesz means on spheres, Indiana Univ. Math. J. 33 (6) (1984) 873–889.
[7] A.P. Calderón, A. Zygmund, On the existence of certain singular integrals, Acta Math. 88 (1952) 85–139.
[8] A.P. Calderón, A. Zygmund, On singular integrals, Amer. J. Math. 78 (1956) 289–309.
[9] J. Duoandikoetxea, J.L.R. de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math. 84 (3) (1986) 541–561.

[10] R. Fefferman, A note on singular integrals, Proc. Amer. Math. Soc. 74 (1979) 266–270.
[11] D. Fan, Y. Pan, Singular integral operators with rough kernels supported by subvarieties, Amer. J. Math. 119 (1997) 799–839.
[12] D. Fan, Y. Pan, L^2 boundedness of a singular integral operator, Publ. Math. 41 (1997) 317–333.
[13] D. Fan, Y. Pan, A singular integral operator with rough kernel, Proc. Amer. Math. Soc. 125 (1997) 3695–3703.
[14] L. Grafakos, Stefanov, Convolution Calderón–Zygmund singular operators with rough kernels, in: Analysis of Divergence, Orono, ME, 1997, in: Appl. Numer. Anal., Birkhäuser Boston, Boston, MA, 1999, pp. 119–143.
[15] H.V. Le, Hypersingular integral operators along surfaces, Integral Equations Operator Theory 44 (4) (2002) 451–465.
[16] H.V. Le, Singular integral operators along surfaces of revolution, J. Math. Anal. Appl. 274 (2) (2002) 608–625.
[17] J. Namazi, A singular integral, PhD thesis, Indiana Univ., Bloomington, 1984.
[18] F. Ricci, G. Weiss, A characterization of $H^1(\sum_{n-1})$, in: S. Wainger, G. Weiss (Eds.), Harmonic Analysis in Euclidean Spaces, in: Proc. Sympos. Pure Math., vol. 35, 1979, pp. 289–294.
[19] P. Sjólin, Convolution with oscillating kernels on H^p spaces, J. London Math. Soc. 23 (2) (1981) 442–454.
[20] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, 1993.
[21] R.L. Wheeden, On hypersingular integrals and Lebesgue spaces of differentiable functions (II), Trans. Amer. Math. Soc. (1969) 37–53.