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Unification of General Relativity with Quantum Field Theory *

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In the frame of quantum field theory, instead of using the action principle, we deduce the Einstein equation from purely the general covariant principle and the homogeneity of spacetime. The Einstein equation is shown to be the gauge equation to guarantee the local symmetry of spacetime translation. Gravity is an apparent force due to the curvature of spacetime resulted from the conservation of energy-momentum. In the action of quantum field theory, only electroweak-strong interactions should be considered with the curved spacetime metric determined by the Einstein equation.

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An unified physical theory of all interactions is a long pursued goal for physicists. The unification of electricity and magnetism by Maxwell was a great step in this direction. It is believed that in nature, there are four types of fundamental interactions: the electromagnetic interaction, weak interaction, strong interaction and gravity. Now the electromagnetic, weak and strong interactions are unified using the so-called standard model,[1] based on the Yang–Mills gauge field theory.[2] However, researchers are still not be able to unify gravitation with the other three interactions.

One common approach of the deduction of the Einstein equation of general relativity[3] for gravity is by using the action principle.[4,5] Using Einstein–Hilbert action and with the variation of the metric $g^{\mu\nu}$, the classic action principle leads to the Einstein equation for vacuum, with the coupling to matter added in a comprehensive way. Then the metric $g^{\mu\nu}$ should be considered as a field function in quantum field theory correspondingly. However, considering the metric $g^{\mu\nu}$ as a quantum field function poses many difficulties, such as nonrenormalizability, coupling with matter fields, etc.[6] In this Letter, we show that the Einstein equation can be deduced purely from the homogeneity of spacetime combined with the general covariant principle in the scheme of quantum field theory. The Einstein equation is the gauge equation for the action to satisfy the local symmetry of spacetime translation.

First, the action should satisfy the principle of general covariance stating that the physics, as embodied in the action, must be invariant under an arbitrary coordinate transformation. Thus, the action S is a scalar in curved spacetime. In the following, we use the path integral expression for quantum field theory. Let us begin with the Lagrange of matter $\mathcal{L}_m$. For example, the Lagrange of matter $\mathcal{L}_m$ for a scalar field in curved spacetime is

$$\mathcal{L}_m = \frac{1}{2}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2), \tag{1}$$

with the action for a scalar field in curved spacetime given by

$$S_m = \int d^4x\sqrt{-g}\frac{1}{2}(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2), \tag{2}$$

where $g^{\mu\nu}$ is the metric tensor in Riemann spacetime. For a vector field A_μ, we should use a covariant derivative

$$D_\alpha A_\mu = \partial_\alpha A_\mu - \Gamma^\nu_{\alpha\mu}A_\nu, \tag{3}$$

where $\Gamma^\nu_{\alpha\mu}$ is the Levi–Civita connection of the metric defined by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}). \tag{4}$$

The first covariant derivative of a scalar function coincides with the ordinary derivative. With the Lagrange of matter $\mathcal{L}_m$, we can define the energy-momentum tensor

$$T^{\mu\nu} \equiv -\frac{1}{\sqrt{-g}}\frac{D(\sqrt{-g}\mathcal{L}_m)}{D(D_\mu\varphi(x))}D_\nu\varphi(x) + g^{\mu\nu}\mathcal{L}_m(x). \tag{5}$$

With the energy-momentum tensor $T^{\mu\nu}$, we can construct another scalar

$$S_e = \alpha_1\int d^4x\sqrt{-g}g_{\mu\nu}T^{\mu\nu}, \tag{6}$$

where α_1 is a constant parameter. It should be noted that all the deductions and conclusions in this study are not dependent on the types of the fields contained in the matter Lagrange $\mathcal{L}_m$. For other fields, symmetric Belinfante tensors[7] may be needed to replace $T^{\mu\nu}$ in S_e. With the metric $g^{\mu\nu}$, we can also construct a scalar as follows:

$$S_g = \alpha_2\int d^4x\sqrt{-g}R, \tag{7}$$

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where α_2 is a constant parameter. R is the scalar curvature. S_g is the so-called Einstein–Hilbert action for gravity.[4,5] R is defined as $g^{\mu\nu}R_{\mu\nu}$. The Ricci tensor is defined as $R_{\mu\nu} = R^{\kappa}_{\mu\kappa\nu}$. $R^{\lambda}_{\mu\nu\kappa}$ is the Riemann curvature tensor defined by

$$R^{\lambda}_{\mu\nu\kappa} = \partial_\nu \Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa \Gamma^{\lambda}_{\mu\nu} + \Gamma^{\sigma}_{\mu\kappa}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\kappa\sigma}. \tag{8}$$

The total action S_t should be the sum of the above three parts and a constant term.

$$\begin{aligned} S_t = &\int d^4x\sqrt{-g}\mathcal{L}_m + \alpha_1 \int d^4x \sqrt{-g} g_{\mu\nu}T^{\mu\nu} \\ &+ \alpha_2 \int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\Lambda', \end{aligned} \tag{9}$$

where Λ' is the cosmologic constant. We might also construct a scalar term in the total action S_t using $R_{\mu\nu}T^{\mu\nu}$. Since we can make metric redefinition $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ with $\delta g_{\mu\nu} = \alpha_3 R_{\mu\nu}$, we can cancel off the term of $R_{\mu\nu}T^{\mu\nu}$. If all the quantum fields involved in the standard model are considered in Lagrange matter $\mathcal{L}_m$, then the action S_t will contain all the interactions, including gravity.

It should be noted that we do not consider the metric $g_{\mu\nu}$ as an independent field. Here $g_{\mu\nu}$ is a functional of field function $\varphi(x)$, $g_{\mu\nu} = g_{\mu\nu}(\varphi(x))$.

Secondly, we consider the presumption of the homogeneity of spacetime. The total action should possess the symmetry of spacetime translations locally. We transform the fields via $\varphi(x) \to \varphi(x-a)$, where a^μ is a constant four vector. For an infinitesimal translation, $\varphi(x) \to \varphi(x) - a^\nu\partial_\nu\varphi(x)$. We have $\delta\varphi(x) = -a^\nu\partial_\nu\varphi(x)$.

If we make an infinitesimal change $\varphi(x) \to \varphi(x) + \delta\varphi(x)$ in the quantum field, we will have $\sqrt{-g}\mathcal{L}_m(x) \to \sqrt{-g}\mathcal{L}_m(x) + \delta(\sqrt{-g}\mathcal{L}_m(x))$, where $\delta(\sqrt{-g}\mathcal{L}_m(x))$ is given by the chain rule,

$$\begin{aligned} \delta(\sqrt{-g}\mathcal{L}_m(x)) = &\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial\varphi(x)}\delta\varphi(x) \\ &+ \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(D_\mu\varphi(x))}D_\mu\delta\varphi(x) \\ &+ \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g_{\mu\nu}}\delta g_{\mu\nu}. \end{aligned} \tag{10}$$

$S_m = \int d^4y\sqrt{-g}\mathcal{L}_m(y)$ is the action for matter. Taking $\delta/\delta\varphi(x)$ as a functional derivative, we have

$$\begin{aligned} \frac{\delta S_m}{\delta\varphi(x)} = &\int d^4y \frac{\sqrt{-g}\mathcal{L}_m(y))}{\delta\varphi(x)} = \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial\varphi(x)} \\ &- D_\mu\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(D_\mu\varphi(x))} + \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g_{\mu\nu}}\frac{\partial g_{\mu\nu}}{\partial\varphi(x)}. \end{aligned} \tag{11}$$

We use the above equation to make the replacement

$$\begin{aligned} &\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial\varphi(x)} + \frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial g_{\mu\nu}}\frac{\partial g_{\mu\nu}}{\partial\varphi(x)} \\ \to &D_\mu\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(D_\mu\varphi(x))} + \frac{\delta S_m}{\delta\varphi(x)} \end{aligned} \tag{12}$$

in Eq. (10). Then we obtain[8]

$$\delta(\sqrt{-g}\mathcal{L}_m(x)) = D_\mu\frac{D(\sqrt{-g}\mathcal{L}_m)}{D(D_\mu\varphi(x))}\delta\varphi(x) + \frac{\delta S_m}{\delta\varphi(x)}\delta\varphi(x). \tag{13}$$

When we transform the fields with an infinitesimal spacetime translation a^ν, we have $\mathcal{L}_m(x) \to \mathcal{L}_m(x-a)$, and then $\delta(\sqrt{-g}\mathcal{L}_m(x)) = -a^\nu\partial_\nu(\sqrt{-g}\mathcal{L}_m(x)) = -\partial_\nu(a^\nu\sqrt{-g}\mathcal{L}_m(x))$. Also, $\delta\sqrt{-g} = 1/2\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$ and $D_\kappa g_{\mu\nu} = 0$. Combined with the first term on the right side of Eq. (13), we obtain the Noether current for the energy-momentum,[8]

$$\begin{aligned} j^\mu(x) = &\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_m)}{\partial(D_\mu\varphi(x))}(-a^\nu\partial_\nu\varphi(x)) \\ &+ a^\nu\mathcal{L}_m(x) = a_\nu T^{\mu\nu}(x). \end{aligned} \tag{14}$$

Then Eq. (13) becomes

$$\frac{\delta S_m}{\delta\varphi(x)}\delta\varphi(x) = \sqrt{-g}D_\mu j^\mu = \sqrt{-g}D_\mu(a_\nu T^{\mu\nu}(x)). \tag{15}$$

Now we consider the functional derivative for the total action S_t,

$$\begin{aligned} \frac{\delta S_t}{\delta\varphi_a(x)}\delta\varphi_a(x) = &\sqrt{-g}D_\mu(a_\nu T^{\mu\nu}(x)) \\ &+ \alpha_1\delta(g_{\mu\nu}\sqrt{-g}T^{\mu\nu}) + \alpha_2\delta(\sqrt{-g}R) \\ &+ \delta(\sqrt{-g}\Lambda'). \end{aligned} \tag{16}$$

The action should possess the symmetry of the spacetime translation locally. Under an infinitesimal spacetime translation with $\delta\varphi(x) = -a^\nu\partial_\nu\varphi(x)$, the variation of the total action should be zero. We have

$$\frac{\delta S_t}{\delta\varphi(x)} = 0. \tag{17}$$

It should be noted that Eq. (17) is different from the action principle. It can be easily confused with the classic action principle. The classical Euler–Lagrange equation of motion follows from demanding $\delta S = 0$ for any variation $\delta\varphi(x)$. However the Euler–Lagrange equation is a classical equation and is only an approximate one for quantum fields. Here we demand $\delta S = 0$ for a specific variation $\delta\varphi(x) = -a^\nu\partial_\nu\varphi(x)$ from an infinitesimal spacetime translation, which should lead to the energy-momentum conservation. In the Minkowskian action, the metric $g^{\mu\nu}$ is constant. Equation (17) cannot be satisfied generally. The classic action principle looks similar to Eq. (17) and was used mistakenly in the deduction of energy-momentum conservation. We can use the Schwinger–Dyson equation as an alternative, but the conservation of energy-momentum will become conditional. If we demand the energy-momentum conservation $\partial_\mu T^{\mu\nu}(x) = 0$ to be satisfied strictly in the Minkowski metric, then the path integral of the quantum field cannot be evaluated in an ordinary way, but with a constraint $\partial_\mu T^{\mu\nu}(x) = 0$ for quantum field $\varphi(x)$. As proposed by Einstein, spacetime with matter should be curved and the Minkowski metric is only valid for vacuum although it is a good approximate metric when the curvature effects are generally weak microscopically. When we use the Riemann spacetime metric $g^{\mu\nu}$, Eq. (17) is able to be satisfied strictly. The energy-momentum conservation is given by

$$D_\mu T^{\mu\nu}(x) = 0. \tag{18}$$

Therefore, we have

$$\alpha_1 \delta(g_{\mu\nu}\sqrt{-g}T^{\mu\nu}) + \alpha_2\delta(\sqrt{-g}R) + \delta(\sqrt{-g}\Lambda') = 0. \tag{19}$$

Using the following relation

$$R = \frac{1}{1+4\beta} g_{\mu\nu}(R^{\mu\nu} + \beta g^{\mu\nu} R), \tag{20}$$

we can rewrite Eq. (19) in a more symmetric way as

$$\delta\big[g_{\mu\nu}\big(\alpha_1\sqrt{-g}T^{\mu\nu} + \alpha_2\sqrt{-g}\frac{1}{1+4\beta}(R^{\mu\nu} + \beta g^{\mu\nu} R) + \frac{1}{4}\sqrt{-g}\Lambda' g^{\mu\nu}\big)\big] = 0. \tag{21}$$

Since we can use any local coordinate frame, $g_{\mu\nu}$ can be a very general function and we expect that the terms in the bracket after $g_{\mu\nu}$ in Eq. (21) cancel off. Thus we obtain

$$\alpha_1 T^{\mu\nu} + \frac{\alpha_2}{1+4\beta}(R^{\mu\nu} + \beta g^{\mu\nu} R) + \frac{1}{4}\Lambda' g^{\mu\nu} = 0. \tag{22}$$

Using the Bianchi identity,[4] we can see $\beta = -1/2$ in order to satisfy the energy-momentum conservation equation $D_\mu T^{\mu\nu}(x) = 0$. Expressed with the gravitational constant G, we have $\alpha_1 = -8\pi G\alpha_2$. Thus we reach the Einstein equation of general relativity,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + g^{\mu\nu}\Lambda = -8\pi G T^{\mu\nu}, \tag{23}$$

where $\Lambda = 2\pi G\Lambda'/\alpha_1$. Since Einstein equation (23) guarantees that both Eqs. (18) and (19) are satisfied, Eq. (17) is satisfied automatically. Therefore, the homogeneity of spacetime is guaranteed. Instead of using an action principle, we use only the local symmetry of spacetime translation combined with a general covariant principle to deduce the Einstein equation. Since the Einstein equation is not deduced from the action principle, evidently the metric $g^{\mu\nu}$ cannot be related to a real interaction nor regraded as a quantum field function.

When we put the Einstein equation back into the total action S_t and use the parameter relation $\alpha_1 = -8\pi G\alpha_2$, we find that the terms S_e and S_g cancel off. Only the action S_m for matter remains in the total action S_t. Thus the action becomes

$$S_t = \int d^4x\sqrt{-g}\mathcal{L}_m. \tag{24}$$

The path integral for the action of Eq. (24) should be carried out for the quantum fields with the metric $g^{\mu\nu}$ in the action satisfying Einstein equation (23). Both $T^{\mu\nu}$ and $g^{\mu\nu}$ are symmetric for the index μ and ν. There are 10 independent functions for $g^{\mu\nu}$. Thus there are 10 equations in (23). Since the energy-momentum conservation equations $D_\mu T^{\mu\nu}(x) = 0$ are satisfied automatically. We have 6 independent equations for 10 functions. We then have 4 functions to make coordinate transformation for $g^{\mu\nu}$. Thus the Einstein equation uniquely determines the metric $g^{\mu\nu}$ from the field functions. On the right-hand side of Eq. (23), $T^{\mu\nu}$ is determined by Eq. (5), which is not the averaged energy-momentum tensor $\langle T^{\mu\nu}\rangle$ as formulated in the quasi-classic theory without quantizing the gravity.[9] However, we shall use the statistical average of $T^{\mu\nu}$ in the Einstein equation for a macroscopical system. It should be noted that the energy-momentum conservation is strictly guaranteed by the Einstein equation of general relativity through a curved spacetime. In Minkowski metric, the energy-momentum conservation is only guaranteed by the classic action principle, which is not strictly maintained for quantum fields. Now we have a natural unification of the general relativity with the quantum theory. In the following, we will discuss the related issues.

The energy-momentum conservation means that matter is conserved during a physical process. Thus in any matter movement, energy-momentum conservation should be maintained naturally. If we restrict ourself to one specific metric, such as the Minkowski metric, the energy-momentum current described by $T^{\mu\nu}$ is not necessarily conserved unless we impose an energy-momentum conservation equation $D_\mu T^{\mu\nu}(x) = 0$. In this way, the energy-momentum conservation becomes an extrinsic condition to restrict the motion of matter. The spacetime metric is the geometric structure dressed by matter in its movement. Therefore the spacetime metric should have a structure implicating the conservation of matter naturally.

Although any metrics guaranteeing the energy-momentum conservation are equivalent, we have some preference in selecting the metrics. Two types of metrics play important roles. One is the local Minkowski metric, which is useful when matter is not very condensed. The other is the static metric, which does not change with time. Not all metrics can be transformed into static metrics. However, fortunately, the Einstein equation has static metric solutions for a macroscopic matter system.[10,11] The static metrics are also related to the equilibrium states in macroscopic scale. Thus we can have stable structures such as stars.

Gravity is only an emergent property of the curved spacetime metric for energy-momentum conservation. How can gravity attract matter to congregate and form, for example, stable stars? For a spherical star, the density of mass is larger in the middle. The outer layer is less dense than the inner layer. The energy cannot remain static due to temperature, the uncertainty principle or the Pauli exclusive principle. There should be momentum associated with the energy. For a spherical structure, the outer layer, in spite of being less dense with small momentum, has a larger volume for energy to contribute. Therefore, an equilibrium state for energy-momentum current can be reached and the star has a static metric. The curvature of the static metric contributes the apparent attraction interaction with other matter near the star.

In addition to the static metric solutions for the Einstein equation, there are also wave metric solutions for the Einstein equation. However this wave is just a kind of matter wave. We do not have a graviton in our theory. Thus the gravitational wave, if its exists, should not be a graviton wave. It is just an energy wave of matter. At present, there is no experimental evidence of the graviton wave.

Now let us turn to the cosmological constant. The observed cosmological constant is very small. However, it is believed that a series of spontaneous symmetries break as the universe cools, generating a large vacuum energy, which leads to a very large cosmological constant. This poses a significant discrepancy between the value expected from quantum field theory with the observed one. The cosmological constant problem could be solved in our unified frame in a possible way. Since $\Lambda = 2\pi G\Lambda'/\alpha_1$, Λ in the Einstein equation is proportional to Λ' in the action with a ratio of $2\pi G/\alpha_1$. G is small and the parameter α_1 should be large for the action S_e to be effective. Then the cosmological constant Λ in the Einstein equation can be small while the cosmological constant Λ' in the action of the quantum fields is large.

In summary, we have shown that the Einstein equation can be deduced purely from the homogeneity of spacetime combined with the general covariant principle in the scheme of quantum field theory. The Einstein equation is the gauge equation of the curved metric $g^{\mu\nu}$ for the action to satisfy the local symmetry of spacetime translation. We show that the metric $g^{\mu\nu}$ is not a quantum field and only has a geometric meaning. Gravity is shown to be an apparent force due to the curvature of spacetime. The path integral for the action of matter should be carried out for quantum fields in curved spacetime with the metric $g^{\mu\nu}$ in the action satisfying the Einstein equation. The Minkowski metric is shown to be inappropriate and approximate in the conservation of energy-momentum. The two basic principles, the general covariant principle and the homogeneity of spacetime, can be merged into one basic principle: any Riemann spacetime metric guaranteeing that the energy-momentum conservation are equivalent, which can be called the conserved general covariant principle.

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