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An existence result for a class of p-Laplacian elliptic systems involving homogeneous nonlinearities in R^N ☆

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ABSTRACT

In the present paper, we study the existence of nontrivial solutions for a class of p-Laplacian elliptic systems in R^N. A new existence result for nontrivial solutions is obtained by means of variational methods.

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1. Introduction and main result

Consider the following p-Laplacian system:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \dfrac{1}{\mu}\dfrac{\partial F(u,v)}{\partial u} + f, & \text{in } R^N, \\ -\Delta_p v + |v|^{p-2}v = \dfrac{1}{\mu}\dfrac{\partial F(u,v)}{\partial v} + g, & \text{in } R^N, \\ u, v \in W^{1,p}(R^N), \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, $N \geq 3$, $1 < p < N$, $p < \mu < p^* = \frac{pN}{N-p}$, and $W^{1,p}(R^N)$ is the Sobolev space with the norm $\|u\|_{1,p} = \left(\int_{R^N}(|\nabla u|^p + |u|^p)\mathrm{d}x\right)^{\frac{1}{p}}$. $F \in C^1(R \times R, R^+)$ is positively homogeneous of degree μ, that is, $F(tu, tv) = t^\mu F(u, v)$ for all $(u, v) \in R \times R$ and $t > 0$, $R^+ = [0, +\infty)$, $f, g \in W^{-1,p'}(R^N) \setminus \{0\}$, where p' is the conjugate to p and $W^{-1,p'}(R^N)$ is the space dual to $W^{1,p}(R^N)$. Problem (1.1) is posed in the framework of the Sobolev space $E = W^{1,p}(R^N) \times W^{1,p}(R^N)$ with the standard norm

$$\|(u, v)\|_E = \left(\int_{R^N}(|\nabla u|^p + |u|^p)\mathrm{d}x + \int_{R^N}(|\nabla v|^p + |v|^p)\mathrm{d}x\right)^{\frac{1}{p}}.$$

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Moreover, a pair of functions $(u, v) \in E$ is said to be a weak solution of problem (1.1) if

$$\int_{R^N}(|\nabla u|^{p-2}\nabla u\nabla\varphi + |u|^{p-2}u\varphi)\mathrm{d}x + \int_{R^N}(|\nabla v|^{p-2}\nabla v\nabla\psi + |v|^{p-2}v\psi)\mathrm{d}x - \frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial u}\varphi\mathrm{d}x - \frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial v}\psi\mathrm{d}x - \langle f,\varphi\rangle_{-1,1} - \langle g,\psi\rangle_{-1,1} = 0,$$

for all $(\varphi, \psi) \in E$, where $\langle,\rangle_{-1,1}$ denotes the duality pair of $W^{-1,p'}(R^N)$ and $W^{1,p}(R^N)$. Thus, the corresponding energy functional of problem (1.1) is defined by

$$J(u,v) = \frac{1}{p}\|(u,v)\|_E^p - \frac{1}{\mu}\int_{R^N}F(u,v)\mathrm{d}x - \langle f,u\rangle_{-1,1} - \langle g,v\rangle_{-1,1}, \tag{1.2}$$

for all $(u, v) \in E$.

In recent years, there have been many papers concerned with the existence and multiplicity of nontrivial solutions for nonlinear elliptic problems in bounded domains. Results related to these problems can be found in [1–7] and the references therein. In particular, Velin [1] considered the following quasilinear elliptic system:

$$\begin{cases} -\Delta_p u = u|u|^{\alpha-1}|v|^{\beta+1} + f, & \text{in } \Omega, \\ -\Delta_q v = |u|^{\alpha+1}v|v|^{\beta-1} + g, & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < p, q < N, \alpha > -1, \beta > -1$. They obtained an existence result for when f and g are chosen small in the sense of the dual norm by using a concentration–compactness principle under the following hypotheses:

(a) $\max(p, q) < \alpha + \beta + 2$,
(b) $\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1$, where $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$.

Very recently, Wu [2] studied the following semilinear elliptic system:

$$\begin{cases} -\Delta u = \lambda f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, & \text{in } \Omega, \\ u = 0, \quad v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\alpha > 1, \beta > 1$ satisfy $2 < \alpha + \beta < 2^* = \frac{2N}{N-2}$, $1 < q < 2$, the pair of parameters $(\lambda, \mu) \in R^2 \setminus \{(0, 0)\}$ and the weight functions f, g, h satisfy the following conditions:

(A) $f, g \in L^{\frac{\alpha+\beta}{\alpha+\beta-q}}(\Omega)$, and either $f^{\pm} = \max\{\pm f, 0\} \neq 0$ or $g^{\pm} = \max\{\pm g, 0\} \neq 0$,
(B) $h \in C(\bar{\Omega})$ with $\|h\|_\infty = 1$ and $h \geq 0$.

With the help of the Nehari manifold, they proved that system (1.4) has at least two nontrivial nonnegative solutions when the pair of the parameters (λ, μ) belongs to a certain subset of R^2.

Summing up the above discussion, much attention has been paid to the existence and multiplicity of solutions for the problem in bounded domains. To the best of our knowledge, little seems to be known about the existence of nontrivial solutions of the problem (1.1), in contrast to the achievements for the problem in bounded domains.

In this paper, we consider, rather than problems (1.3) and (1.4), the p-Laplacian system in the whole space R^N. Since system (1.1) is set on R^N, it is well known that the Sobolev embedding $W^{1,p}(R^N) \hookrightarrow L^q(R^N)$ $(p \leq q < p^*)$ is not compact, and it is usually difficult to prove the Palais–Smale condition if we seek solutions of (1.1) by means of variational methods. On the other hand, in contrast to the case for problem (1.4), our work space $W^{1,p}(R^N)$ is not a Hilbert space for $p \neq 2$. We must consider the local strong convergence of the gradients of Palais–Smale sequences in $L^p(R^N)$ (Lemma 2.5). This makes the study of the problem more difficult and interesting. Motivated by some results found in [8–12], we make our principal project in this paper researching the existence of nontrivial solutions for the system (1.1). When f and g satisfy an adequate norm estimate, we obtain an existence result for problem (1.1) by showing that a minimizing sequence obtained by means of the Ekeland variational principle contains a Palais–Smale sequence and then, up to a subsequence, converges to a solution of problem (1.1).

In order to state our main result, we need some notation. To begin with, we state a proposition.

Proposition 1.1 (*[8], Remark 5*). *Suppose that $F \in C^1(R \times R, R^+)$ is positively homogeneous of degree μ with $\mu > 1$. Then:*

(i) *There exists $M_F > 0$ such that*

$$|F(u,v)| \leq M_F(|u|^\mu + |v|^\mu), \quad \forall (u,v) \in R \times R, \tag{1.5}$$

where $M_F = \max\{F(u, v) \mid u, v \in R, |u|^\mu + |v|^\mu = 1\}$.

(ii) *The following Euler identity:*

$$u\frac{\partial F(u,v)}{\partial u}+v\frac{\partial F(u,v)}{\partial v}=\mu F(u,v) \tag{1.6}$$

holds.

(iii) $\frac{\partial F}{\partial u},\frac{\partial F}{\partial v}\in C(R\times R,R)$ *are positively homogeneous of degree* $\mu-1$.

Let S be the best Sobolev constant for the embedding of $W^{1,p}(R^N)$ in $L^{\mu}(R^N)$, that is,

$$S=\inf_{u\in W^{1,p}(R^N)\setminus\{0\}}\frac{\int_{R^N}|\nabla u|^p+|u|^p\mathrm{d}x}{\left(\int_{R^N}|u|^{\mu}\mathrm{d}x\right)^{\frac{p}{\mu}}}$$

and

$$\alpha(p,\mu,\gamma,S,M_F)=\left(\frac{p\mu-p^2-(\mu-1)\gamma^p}{p}\right)^{\frac{p-1}{p}}\left(\frac{p-1}{2(\mu-1)M_F}S^{\frac{\mu}{p}}\right)^{\frac{p-1}{\mu-p}}\left(\frac{p'\gamma^{p'}}{\mu-1}\right)^{\frac{p-1}{p}},$$

where γ is a positive constant with $0<\gamma<\left(\frac{\mu-p}{\mu-1}\right)^{\frac{1}{p}}$.

Now, we can state our main result.

Theorem 1.1. *For some* $0<\gamma<\left(\frac{\mu-p}{\mu-1}\right)^{\frac{1}{p}}$, *suppose that* $f,g\in W^{-1,p'}(R^N)\setminus\{0\}$ *satisfy the condition*

$$0<\|f\|_{-1,p'}+\|g\|_{-1,p'}<\alpha(p,\mu,\gamma,S,M_F). \tag{$\mathrm{f_g}$}$$

Then system (1.1) has at least one nontrivial solution $(u,v)\in E$ *with* $J(u,v)<0$.

Throughout this paper, C will denote positive constants and may be different in different places.

2. Preliminaries

In order to complete our proof, we need the following lemmas.

Lemma 2.1 (*[9], Lemma 3.1*). *Assume that* $1\le p,r<\infty$, $f\in C(R^N\times R\times R,R)$ *and*

$$f(x,u,v)\le C_0\left(|u|^{\frac{p}{r}}+|v|^{\frac{p}{r}}\right).$$

Then, for every $(u,v)\in L^p(R^N)\times L^p(R^N)$, $f(\cdot,u(\cdot),v(\cdot))\in L^r(R^N)$ *and the operator* $T:L^p(R^N)\times L^p(R^N)\to L^r(R^N):(u,v)\mapsto f(\cdot,u(\cdot),v(\cdot))$ *is continuous.*

Lemma 2.2. *The functional* J *defined by (1.2) is of class* $C^1(E,R)$ *and*

$$\begin{aligned}\langle J'(u,v),(\varphi,\psi)\rangle=&\int_{R^N}(|\nabla u|^{p-2}\nabla u\nabla\varphi+|u|^{p-2}u\varphi)\mathrm{d}x+\int_{R^N}(|\nabla v|^{p-2}\nabla v\nabla\psi+|v|^{p-2}v\psi)\mathrm{d}x\\&-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial u}\varphi\mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial v}\psi\mathrm{d}x-\langle f,\varphi\rangle_{-1,1}-\langle g,\psi\rangle_{-1,1},\end{aligned} \tag{2.1}$$

where $(u,v),(\varphi,\psi)\in E$.

Proof. We define the functional

$$\chi(u,v)=\int_{R^N}F(u,v)\mathrm{d}x.$$

It is sufficient to prove that $\chi\in C^1(E,R)$ and

$$\langle\chi'(u,v),(\varphi,\psi)\rangle=\int_{R^N}\frac{\partial F(u,v)}{\partial u}\varphi+\frac{\partial F(u,v)}{\partial v}\psi\mathrm{d}x.$$

Existence of the Gateaux derivative. By Proposition 1.1, there exists a positive constant K such that

$$\left|\frac{\partial F(u,v)}{\partial u}\right|\le K(|u|^{\mu-1}+|v|^{\mu-1}) \tag{2.2}$$

and

$$\left|\frac{\partial F(u,v)}{\partial v}\right| \le K(|u|^{\mu-1}+|v|^{\mu-1}). \tag{2.3}$$

Hence, for given $0<|t|<1$, by the mean value theorem, there exists $\lambda\in(0,1)$ such that

$$\begin{aligned}\frac{|F(u+t\varphi,v+t\psi)-F(u,v)|}{|t|} &\le \left|\frac{\partial F(u+t\lambda\varphi,v+t\lambda\psi)}{\partial u}\varphi\right|+\left|\frac{\partial F(u+t\lambda\varphi,v+t\lambda\psi)}{\partial v}\psi\right|\\ &\le 2^{\mu-2}K(|u|^{\mu-1}+|\varphi|^{\mu-1}+|v|^{\mu-1}+\psi^{\mu-1})(|\varphi|+|\psi|).\end{aligned}$$

The Hölder inequality and the Sobolev imbedding theorem imply that

$$(|u|^{\mu-1}+|\varphi|^{\mu-1}+|v|^{\mu-1}+\psi^{\mu-1})(|\varphi|+|\psi|)\in L^1(R^N).$$

It follows from the Lebesgue dominated convergence theorem that

$$\langle \chi'(u,v),(\varphi,\psi)\rangle=\int_{R^N}\frac{\partial F(u,v)}{\partial u}\varphi+\frac{\partial F(u,v)}{\partial v}\psi\mathrm{d}x.$$

Continuity of the Gateaux derivative. Assume that $(u_n,v_n)\to(u,v)$ in E. By the Sobolev imbedding theorem, $(u_n,v_n)\to(u,v)$ in $L^\mu(R^N)\times L^\mu(R^N)$. It follows from Lemma 2.1 that $\frac{\partial F(u_n,v_n)}{\partial u}\to\frac{\partial F(u,v)}{\partial u}$, $\frac{\partial F(u_n,v_n)}{\partial v}\to\frac{\partial F(u,v)}{\partial v}$ in $L^{\frac{\mu}{\mu-1}}(R^N)$. By the Hölder inequality and the Sobolev imbedding theorem, for any $(\varphi,\psi)\in E$ with $\|(\varphi,\psi)\|_E=1$, we obtain

$$\begin{aligned}\|\chi'(u_n,v_n)-\chi'(u,v)\| &= \sup_{\|(\varphi,\psi)\|_E=1}|\langle\chi'(u_n,v_n)-\chi'(u,v),(\varphi,\psi)\rangle|\\
&\le \sup_{\|(\varphi,\psi)\|_E=1}\int_{R^N}\left|\frac{\partial F(u_n,v_n)}{\partial u}-\frac{\partial F(u,v)}{\partial u}\right||\varphi|\mathrm{d}x\\
&\quad+\sup_{\|(\varphi,\psi)\|_E=1}\int_{R^N}\left|\frac{\partial F(u_n,v_n)}{\partial v}-\frac{\partial F(u,v)}{\partial v}\right||\psi|\mathrm{d}x\\
&\le \sup_{\|(\varphi,\psi)\|_E=1}\left[\left\|\frac{\partial F(u_n,v_n)}{\partial u}-\frac{\partial F(u,v)}{\partial u}\right\|_{L^{\frac{\mu}{\mu-1}}(R^N)}\|\varphi\|_{L^\mu(R^N)}\right]\\
&\quad+\sup_{\|(\varphi,\psi)\|_E=1}\left[\left\|\frac{\partial F(u_n,v_n)}{\partial v}-\frac{\partial F(u,v)}{\partial v}\right\|_{L^{\frac{\mu}{\mu-1}}(R^N)}\|\psi\|_{L^\mu(R^N)}\right]\\
&\le S^{-\frac{1}{p}}\left(\left\|\frac{\partial F(u_n,v_n)}{\partial u}-\frac{\partial F(u,v)}{\partial u}\right\|_{L^{\frac{\mu}{\mu-1}}(R^N)}+\left\|\frac{\partial F(u_n,v_n)}{\partial v}-\frac{\partial F(u,v)}{\partial v}\right\|_{L^{\frac{\mu}{\mu-1}}(R^N)}\right)\to 0,\end{aligned}$$

as $n\to\infty$. This completes the proof. □

Definition 2.1. Suppose that $c\in R$, X is a Banach space and the functional $I\in C^1(X,R)$. We say that $\{(u_n,v_n)\}\subset X$ is a Palais–Smale sequence at level c ($(\mathrm{PS})_c$-sequence, for short) for I if

$$I(u_n,v_n)\to c,\qquad I'(u_n,v_n)\to 0,$$

as $n\to\infty$. We say that I satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$-condition, for short), if every $(\mathrm{PS})_c$-sequence in X for I has a strongly convergent subsequence.

Lemma 2.3. *If $\{(u_n,v_n)\}\subset E$ is a $(\mathrm{PS})_c$-sequence for J, then $\{(u_n,v_n)\}$ is bounded in E.*

Proof. Let $\{(u_n,v_n)\}$ be a $(\mathrm{PS})_c$-sequence in E, that is $J(u_n,v_n)=c+o_n(1)$ and $J'(u_n,v_n)=o_n(1)$. Since $F\in C^1(R\times R,R^+)$ is positively homogeneous of degree μ, by the Hölder inequality, the Young inequality and Proposition 1.1(ii), for any $0<\theta_1<\left(\frac{\mu-p}{\mu-1}\right)^{\frac{1}{p}}$, we have

$$\begin{aligned}c+\|(u_n,v_n)\|_E+o_n(1) &\ge J(u_n,v_n)-\frac{1}{\mu}\langle J'(u_n,v_n),(u_n,v_n)\rangle\\
&=\left(\frac{1}{p}-\frac{1}{\mu}\right)\|(u_n,v_n)\|_E^p-\left(1-\frac{1}{\mu}\right)\langle f,u_n\rangle_{-1,1}-\left(1-\frac{1}{\mu}\right)\langle g,v_n\rangle_{-1,1}\\
&\ge\left(\frac{1}{p}-\frac{1}{\mu}\right)\|(u_n,v_n)\|_E^p-\left(1-\frac{1}{\mu}\right)\|f\|_{-1,p'}\|u_n\|_{1,p}-\left(1-\frac{1}{\mu}\right)\|g\|_{-1,p'}\|v_n\|_{1,p}\end{aligned}$$

$$
\begin{aligned}
&\geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\|(u_n,v_n)\|_E^p-\left(1-\frac{1}{\mu}\right)\frac{1}{p'}\theta_1^{-p'}\|f\|_{-1,p'}^{p'}-\left(1-\frac{1}{\mu}\right)\frac{1}{p}\theta_1^p\|u_n\|_{1,p}^p\\
&\quad-\left(1-\frac{1}{\mu}\right)\frac{1}{p'}\theta_1^{-p'}\|g\|_{-1,p'}^{p'}-\left(1-\frac{1}{\mu}\right)\frac{1}{p}\theta_1^p\|v_n\|_{1,p}^p\\
&\geq \left(\frac{1}{p}-\frac{1}{\mu}-\frac{\mu-1}{p\mu}\theta_1^p\right)\|(u_n,v_n)\|_E^p-\left(1-\frac{1}{\mu}\right)\frac{1}{p'}\theta_1^{-p'}(\|f\|_{-1,p'}^{p'}+\|g\|_{-1,p'}^{p'}).
\end{aligned}
$$

Since $\frac{1}{p}-\frac{1}{\mu}-\frac{\mu-1}{p\mu}\theta_1^p>0$ and $p>1$, this implies that $\{(u_n,v_n)\}$ is bounded in E. □

Lemma 2.4 ([10]). *There exist constants C_1, C_2, C_3 and C_4 such that for all $x, y\in R^N$, $N\geq 1$,*

$$
\begin{aligned}
&(|x|^{p-2}x-|y|^{p-2}y,x-y)\geq C_1(|x|+|y|)^{p-2}|x-y|^2,\quad \text{for } 1<p\leq 2,\\
&|\,|x|^{p-2}x-|y|^{p-2}y|\leq C_2|x-y|^{p-1},\quad \text{for } 1<p\leq 2,\\
&(|x|^{p-2}x-|y|^{p-2}y,x-y)\geq C_3|x-y|^p,\quad \text{for } p\geq 2,\\
&|\,|x|^{p-2}x-|y|^{p-2}y|\leq C_4(|x|+|y|)^{p-2}|x-y|,\quad \text{for } p\geq 2.
\end{aligned}
$$

Lemma 2.5. *For any given function $\varphi,\psi\in C_0^\infty(R^N)$, if $\{(u_n,v_n)\}\subset E$ is a $(\mathrm{PS})_c$-sequence for J, then we have that, up to a subsequence, there exists $(u,v)\in E$ such that*

$$\lim_{n\to\infty}\int_{R^N}|\nabla u_n-\nabla u|^p\varphi \mathrm{d}x=0$$

and

$$\lim_{n\to\infty}\int_{R^N}|\nabla v_n-\nabla v|^p\psi \mathrm{d}x=0.$$

Proof. By Lemma 2.3, $\{(u_n,v_n)\}$ is bounded in E; we can assume, up to a subsequence, that

$$u_n\rightharpoonup u,\qquad v_n\rightharpoonup v,\quad \text{in } W^{1,p}(R^N), \tag{2.4}$$

$$u_n\to u,\qquad v_n\to v,\quad \text{in } L^q_{\mathrm{loc}}(R^N),\ p\leq q<p^*, \tag{2.5}$$

$$u_n\to u,\qquad v_n\to v,\quad \text{a.e. in } R^N. \tag{2.6}$$

Since $\varphi(u_n-u)$ and $\psi(v_n-v)$ are bounded in $W^{1,p}(R^N)$, by the reflexivity of the space $W^{1,p}(R^N)$, we can affirm, up to a subsequence, that

$$\varphi(u_n-u)\rightharpoonup 0,\qquad \psi(v_n-v)\rightharpoonup 0,\quad \text{in } W^{1,p}(R^N). \tag{2.7}$$

It follows from (2.7) and Lemma 2.2 that

$$
\begin{aligned}
\langle J'(u,v),(\varphi(u_n-u),0)\rangle &= \int_{R^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)\varphi \mathrm{d}x+\int_{R^N}|\nabla u|^{p-2}\nabla u\nabla\varphi(u_n-u)\mathrm{d}x\\
&\quad+\int_{R^N}|u|^{p-2}u(u_n-u)\varphi \mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial u}(u_n-u)\varphi \mathrm{d}x-\langle f,\varphi(u_n-u)\rangle_{-1,1}\to 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle J'(u,v),(0,\psi(v_n-v))\rangle &= \int_{R^N}|\nabla v|^{p-2}\nabla v\nabla(v_n-v)\psi \mathrm{d}x+\int_{R^N}|\nabla v|^{p-2}\nabla v\nabla\psi(v_n-v)\mathrm{d}x\\
&\quad+\int_{R^N}|v|^{p-2}v(v_n-v)\psi \mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial v}(v_n-v)\psi \mathrm{d}x\\
&\quad-\langle g,\psi(v_n-v)\rangle_{-1,1}\to 0,
\end{aligned}
$$

as $n\to\infty$. On the other hand, since $\{(u_n,v_n)\}\subset E$ is a $(\mathrm{PS})_c$-sequence for J, and using Lemma 2.2, it is easy to obtain that

$$
\begin{aligned}
\langle J'(u_n,v_n),(\varphi(u_n-u),0)\rangle &= \int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\varphi \mathrm{d}x+\int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi(u_n-u)\mathrm{d}x\\
&\quad+\int_{R^N}|u_n|^{p-2}u_n(u_n-u)\varphi \mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u_n,v_n)}{\partial u}(u_n-u)\varphi \mathrm{d}x\\
&\quad-\langle f,\varphi(u_n-u)\rangle_{-1,1}\to 0
\end{aligned}
$$

and

$$\begin{aligned}\langle J'(u_n,v_n),(0,\psi(v_n-v))\rangle &= \int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla(v_n-v)\psi\,\mathrm{d}x+\int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla\psi(v_n-v)\mathrm{d}x\\&\quad+\int_{R^N}|v_n|^{p-2}v_n(v_n-v)\psi\,\mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u_n,v_n)}{\partial v}(v_n-v)\psi\,\mathrm{d}x\\&\quad-\langle g,\psi(v_n-v)\rangle_{-1,1}\to 0,\end{aligned}$$

as $n\to\infty$. Hence, using that $\varphi,\psi\in C_0^\infty(R^N)$, (2.7) and the Hölder inequality, we have

$$\begin{aligned}\int_{R^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)\varphi\mathrm{d}x &= -\int_{R^N}|\nabla u|^{p-2}\nabla u\nabla\varphi(u_n-u)\mathrm{d}x-\int_{R^N}|u|^{p-2}u(u_n-u)\varphi\mathrm{d}x\\&\quad+\frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial u}(u_n-u)\varphi\mathrm{d}x+\langle f,\varphi(u_n-u)\rangle_{-1,1}+o_n(1)\to 0,\end{aligned}\tag{2.8}$$

$$\begin{aligned}\int_{R^N}|\nabla v|^{p-2}\nabla v\nabla(v_n-v)\psi\,\mathrm{d}x &= -\int_{R^N}|\nabla v|^{p-2}\nabla v\nabla\psi(v_n-v)\mathrm{d}x-\int_{R^N}|v|^{p-2}v(v_n-v)\psi\,\mathrm{d}x\\&\quad+\frac{1}{\mu}\int_{R^N}\frac{\partial F(u,v)}{\partial v}(v_n-v)\psi\,\mathrm{d}x+\langle g,\psi(v_n-v)\rangle_{-1,1}+o_n(1)\to 0,\end{aligned}\tag{2.9}$$

$$\begin{aligned}\int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\varphi\mathrm{d}x &= -\int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi(u_n-u)\mathrm{d}x-\int_{R^N}|u_n|^{p-2}u_n(u_n-u)\varphi\mathrm{d}x\\&\quad+\frac{1}{\mu}\int_{R^N}\frac{\partial F(u_n,v_n)}{\partial u}(u_n-u)\varphi\mathrm{d}x+\langle f,\varphi(u_n-u)\rangle_{-1,1}+o_n(1)\to 0,\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}\int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla(v_n-v)\psi\,\mathrm{d}x &= -\int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla\psi(v_n-v)\mathrm{d}x-\int_{R^N}|v_n|^{p-2}v_n(v_n-v)\psi\,\mathrm{d}x\\&\quad+\frac{1}{\mu}\int_{R^N}\frac{\partial F(u_n,v_n)}{\partial v}(v_n-v)\psi\,\mathrm{d}x+\langle g,\psi(v_n-v)\rangle_{-1,1}+o_n(1)\to 0,\end{aligned}\tag{2.11}$$

as $n\to\infty$. Define

$$P_1(x)=(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u)(x)$$

and

$$P_2(x)=(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v,\nabla v_n-\nabla v)(x).$$

Without loss of generality, we can assume that $\varphi\geq 0$, $\psi\geq 0$. By Lemma 2.4, when $p\geq 2$, we have

$$\begin{aligned}&\int_{R^N}|\nabla u_n-\nabla u|^p\varphi\mathrm{d}x+\int_{R^N}|\nabla v_n-\nabla v|^p\psi\,\mathrm{d}x\\&\leq C\left(\int_{R^N}P_1(x)\varphi\mathrm{d}x+\int_{R^N}P_2(x)\psi\,\mathrm{d}x\right)\\&\leq C\left(\int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\varphi\mathrm{d}x-\int_{R^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)\varphi\mathrm{d}x\right.\\&\quad\left.+\int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla(v_n-v)\psi\,\mathrm{d}x-\int_{R^N}|\nabla v|^{p-2}\nabla v\nabla(v_n-v)\psi\,\mathrm{d}x\right).\end{aligned}\tag{2.12}$$

When $1<p<2$, we have

$$\begin{aligned}&\int_{R^N}|\nabla u_n-\nabla u|^p\varphi\mathrm{d}x+\int_{R^N}|\nabla v_n-\nabla v|^p\psi\,\mathrm{d}x\\&\leq\int_{R^N}P_1(x)^{\frac{p}{2}}(|\nabla u_n|+|\nabla u|)^{\frac{p(2-p)}{2}}\varphi\mathrm{d}x+C\int_{R^N}P_2(x)^{\frac{p}{2}}(|\nabla v_n|+|\nabla v|)^{\frac{p(2-p)}{2}}\psi\,\mathrm{d}x\\&\leq C\left(\int_{R^N}P_1(x)\varphi\mathrm{d}x\right)^{\frac{p}{2}}\left(\int_{R^N}(|\nabla u_n|^p+|\nabla u|^p)\varphi\mathrm{d}x\right)^{\frac{2-p}{2}}+C\left(\int_{R^N}P_2(x)\psi\,\mathrm{d}x\right)^{\frac{p}{2}}\left(\int_{R^N}(|\nabla v_n|^p+|\nabla v|^p)\psi\,\mathrm{d}x\right)^{\frac{2-p}{2}}\\&\leq C\left(\int_{R^N}P_1(x)\varphi\mathrm{d}x\right)^{\frac{p}{2}}+C\left(\int_{R^N}P_2(x)\psi\,\mathrm{d}x\right)^{\frac{p}{2}}\end{aligned}$$

$$\leq C\left(\int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\varphi \mathrm{d}x-\int_{R^N}|\nabla u|^{p-2}\nabla u\nabla(u_n-u)\varphi \mathrm{d}x\right)^{\frac{p}{2}}$$
$$+C\left(\int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla(v_n-v)\psi \mathrm{d}x-\int_{R^N}|\nabla v|^{p-2}\nabla v\nabla(v_n-v)\psi \mathrm{d}x\right)^{\frac{p}{2}}. \tag{2.13}$$

From (2.8)–(2.13), one has

$$\lim_{n\to\infty}\int_{R^N}|\nabla u_n-\nabla u|^p\varphi \mathrm{d}x+\int_{R^N}|\nabla v_n-\nabla v|^p\psi \mathrm{d}x=0.$$

That is,

$$\lim_{n\to\infty}\int_{R^N}|\nabla u_n-\nabla u|^p\varphi \mathrm{d}x=0$$

and

$$\lim_{n\to\infty}\int_{R^N}|\nabla v_n-\nabla v|^p\psi \mathrm{d}x=0.$$

Therefore Lemma 2.5 is proved. □

Lemma 2.6. *If $\{(u_n, v_n)\}\subset E$ is a $(\mathrm{PS})_c$-sequence for J with $(u_n, v_n)\rightharpoonup(u, v)$, then $J'(u, v)=0$.*

Proof. For all $(\varphi,\psi)\in E$, it is sufficient to prove that

$$\langle J'(u_n, v_n), (\varphi,\psi)\rangle\to\langle J'(u, v), (\varphi,\psi)\rangle,$$

as $n\to\infty$. In fact, note that

$$\langle J'(u_n, v_n), (\varphi,\psi)\rangle=\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi+|u_n|^{p-2}u_n\varphi)\mathrm{d}x+\int_{R^N}(|\nabla v_n|^{p-2}\nabla v_n\nabla\psi+|v_n|^{p-2}v_n\psi)\mathrm{d}x$$
$$-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u_n, v_n)}{\partial u}\varphi \mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u_n, v_n)}{\partial v}\psi \mathrm{d}x-\langle f,\varphi\rangle_{-1,1}-\langle g,\psi\rangle_{-1,1} \tag{2.14}$$

and

$$\langle J'(u, v), (\varphi,\psi)\rangle=\int_{R^N}(|\nabla u|^{p-2}\nabla u\nabla\varphi+|u|^{p-2}u\varphi)\mathrm{d}x+\int_{R^N}(|\nabla v|^{p-2}\nabla v\nabla\psi+|v|^{p-2}v\psi)\mathrm{d}x$$
$$-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u, v)}{\partial u}\varphi \mathrm{d}x-\frac{1}{\mu}\int_{R^N}\frac{\partial F(u, v)}{\partial v}\psi \mathrm{d}x-\langle f,\varphi\rangle_{-1,1}-\langle g,\psi\rangle_{-1,1}. \tag{2.15}$$

Hence, the next step is to prove that

$$\int_{R^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi \mathrm{d}x\to\int_{R^N}|\nabla u|^{p-2}\nabla u\nabla\varphi \mathrm{d}x, \tag{2.16}$$
$$\int_{R^N}|\nabla v_n|^{p-2}\nabla v_n\nabla\psi \mathrm{d}x\to\int_{R^N}|\nabla v|^{p-2}\nabla v\nabla\psi \mathrm{d}x, \tag{2.17}$$
$$\int_{R^N}|u_n|^{p-2}u_n\varphi \mathrm{d}x\to\int_{R^N}|u|^{p-2}u\varphi \mathrm{d}x, \tag{2.18}$$
$$\int_{R^N}|v_n|^{p-2}v_n\psi \mathrm{d}x\to\int_{R^N}|v|^{p-2}v\psi \mathrm{d}x, \tag{2.19}$$
$$\int_{R^N}\frac{\partial F(u_n, v_n)}{\partial u}\varphi \mathrm{d}x\to\int_{R^N}\frac{\partial F(u, v)}{\partial u}\varphi \mathrm{d}x \tag{2.20}$$

and

$$\int_{R^N}\frac{\partial F(u_n, v_n)}{\partial v}\psi \mathrm{d}x\to\int_{R^N}\frac{\partial F(u, v)}{\partial v}\psi \mathrm{d}x, \tag{2.21}$$

as $n\to\infty$. First, for any $\varepsilon>0$ small enough, since $C_0^\infty(R^N)$ is dense in $W^{1,p}(R^N)$, then there exists an element φ_ε in $C_0^\infty(R^N)$ such that

$$\|\varphi-\varphi_\varepsilon\|_{1,p}\leq\varepsilon.$$

Hence, by Lemma 2.4, when $1 < p \leq 2$, we have

$$\begin{aligned}
&\left|\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla\varphi \mathrm{d}x\right| \\
&\leq \left|\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla\varphi_\varepsilon \mathrm{d}x\right| + \left|\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla\varphi - \nabla\varphi_\varepsilon)\mathrm{d}x\right| \\
&\leq C\int_{R^N}|\nabla u_n - \nabla u|^{p-1}|\nabla\varphi_\varepsilon|\mathrm{d}x + C\int_{R^N}|\nabla u_n - \nabla u|^{p-1}|\nabla\varphi - \nabla\varphi_\varepsilon|\mathrm{d}x \\
&\leq C\int_{\mathrm{supp}\ \varphi_\varepsilon}|\nabla u_n - \nabla u|^{p-1}|\nabla\varphi_\varepsilon|\mathrm{d}x + C\left(\int_{R^N}|\nabla u_n - \nabla u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{R^N}|\nabla\varphi - \nabla\varphi_\varepsilon|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq C\left(\int_{\mathrm{supp}\ \varphi_\varepsilon}|\nabla u_n - \nabla u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathrm{supp}\ \varphi_\varepsilon}|\nabla\varphi_\varepsilon\ |^p\ \mathrm{d}x|\right)^{\frac{1}{p}} + C\varepsilon.
\end{aligned}$$

When $p > 2$, we have

$$\begin{aligned}
&\left|\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla\varphi \mathrm{d}x\right| \\
&\leq \left|\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla\varphi_\varepsilon \mathrm{d}x\right| + \left|\int_{R^N}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla\varphi - \nabla\varphi_\varepsilon)\mathrm{d}x\right| \\
&\leq C\int_{R^N}(|\nabla u_n| + |\nabla u|)^{p-2}|\nabla u_n - \nabla u|\,|\nabla\varphi_\varepsilon|\mathrm{d}x + C\int_{R^N}(|\nabla u_n| + |\nabla u|)^{p-2}|\nabla u_n - \nabla u|\,|\nabla\varphi - \nabla\varphi_\varepsilon|\mathrm{d}x \\
&\leq C\int_{\mathrm{supp}\ \varphi_\varepsilon}(|\nabla u_n| + |\nabla u|)^{p-2}|\nabla u_n - \nabla u|\,|\nabla\varphi_\varepsilon|\mathrm{d}x \\
&\quad + C\left(\int_{R^N}(|\nabla u_n| + |\nabla u|)^p\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{R^N}|\nabla u_n - \nabla u|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{R^N}|\nabla\varphi - \nabla\varphi_\varepsilon|^p\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq C\left(\int_{\mathrm{supp}\ \varphi_\varepsilon}(|\nabla u_n| + |\nabla u|)^p\mathrm{d}x\right)^{\frac{p-2}{p}}\left(\int_{\mathrm{supp}\ \varphi_\varepsilon}|\nabla u_n - \nabla u|^p\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathrm{supp}\ \varphi_\varepsilon}|\nabla\varphi_\varepsilon|^p\mathrm{d}x\right)^{\frac{1}{p}} + C\varepsilon.
\end{aligned}$$

Hence, by the arbitrariness of ε and Lemma 2.5, (2.16) holds. Taking into account (2.5) and Lemma 2.5, the verification of (2.17)–(2.19) can be done in a similar way. In what follows, we will prove that (2.20) holds. Indeed, by (2.2), (2.5) and Lemma 2.1, we have

$$\frac{\partial F(u_n, v_n)}{\partial u} \to \frac{\partial F(u, v)}{\partial u}, \quad \text{in } L^{\mu/\mu-1}_{\mathrm{loc}}(R^N), \tag{2.22}$$

as $n \to \infty$. Furthermore, for each fixed $\varphi \in W^{1,p}(R^N)$, one has that for any $\varepsilon_1 > 0$, there exists $r_0 > 0$ such that

$$\left(\int_{R^N\setminus B_{r_0}(0)}|\varphi(x)|^\mu\mathrm{d}x\right)^{\frac{1}{\mu}} < \varepsilon_1. \tag{2.23}$$

Hence, for large n, it follows from (2.2), (2.22), (2.23) and the Hölder inequality that

$$\begin{aligned}
\left|\int_{R^N}\left(\frac{\partial F(u_n, v_n)}{\partial u} - \frac{\partial F(u, v)}{\partial u}\right)\varphi \mathrm{d}x\right| &\leq \int_{R^N}\left|\frac{\partial F(u_n, v_n)}{\partial u} - \frac{\partial F(u, v)}{\partial u}\right|\,|\varphi|\mathrm{d}x \\
&\leq \left(\int_{B_{r_0}(0)}\left|\frac{\partial F(u_n, v_n)}{\partial u} - \frac{\partial F(u, v)}{\partial u}\right|^{\frac{\mu}{\mu-1}}\mathrm{d}x\right)^{\frac{\mu-1}{\mu}}\left(\int_{B_{r_0}(0)}|\varphi|^\mu\mathrm{d}x\right)^{\frac{1}{\mu}} \\
&\quad + \int_{R^N\setminus B_{r_0}(0)}[K(|u_n|^{\mu-1} + |v_n|^{\mu-1}) + K(|u|^{\mu-1} + |v|^{\mu-1})]|\varphi|\mathrm{d}x \\
&\leq \left(\int_{B_{r_0}(0)}\left|\frac{\partial F(u_n, v_n)}{\partial u} - \frac{\partial F(u, v)}{\partial u}\right|^{\frac{\mu}{\mu-1}}\mathrm{d}x\right)^{\frac{\mu-1}{\mu}}\left(\int_{B_{r_0}(0)}|\varphi|^\mu\mathrm{d}x\right)^{\frac{1}{\mu}} \\
&\quad + \varepsilon_1 K(\|u_n\|^{\mu-1}_{L^\mu(R^N)} + \|v_n\|^{\mu-1}_{L^\mu(R^N)}) + \varepsilon_1 K(\|u\|^{\mu-1}_{L^\mu(R^N)} + \|v\|^{\mu-1}_{L^\mu(R^N)}).
\end{aligned}$$

Therefore,

$$\left|\int_{R^N}\left(\frac{\partial F(u_n,v_n)}{\partial u}-\frac{\partial F(u,v)}{\partial u}\right)\varphi \mathrm{d}x\right|\to 0,\quad \text{as } n\to\infty.$$

The verification of (2.21) can be done in a similar way. This completes the proof of Lemma 2.6. □

Lemma 2.7. *Let $\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n)\rightharpoonup(u,v)$ weakly in E. Then we have*

$$\int_{R^N}F(u_n,v_n)\mathrm{d}x=\int_{R^N}F(u_n-u,v_n-v)\mathrm{d}x+\int_{R^N}F(u,v)\mathrm{d}x+o_n(1).$$

Proof. We will follow the approach presented in [8,11] to give the proof of this lemma. Using the mean value theorem, for given $0<|\lambda|<1$, it follows from (2.2) and (2.3) that

$$\begin{aligned}|F(u_n,v_n)-F(u_n-u,v_n-v)| &= |\nabla F(u_n-u+\lambda u,v_n-v+\lambda v)\cdot(u,v)|\\ &\le K(|u_n-u+\lambda u|^{\mu-1}+|v_n-v+\lambda v|^{\mu-1})|u|\\ &\quad +K(|u_n-u+\lambda u|^{\mu-1}+|v_n-v+\lambda v|^{\mu-1})|v|\\ &\le C(|u_n-u|^{\mu-1}|u|+|u|^{\mu}+|v_n-v|^{\mu-1}|u|+|v|^{\mu-1}|u|\\ &\quad +|u_n-u|^{\mu-1}|v|+|u|^{\mu-1}|v|+|v_n-v|^{\mu-1}|v|+|v|^{\mu})\\ &\le C(|u_n-u|^{\mu-1}|u|+|v_n-v|^{\mu-1}|v|+|u_n-u|^{\mu-1}|v|+|v_n-v|^{\mu-1}|u|\\ &\quad +|u|^{\mu}+|v|^{\mu}+|u|^{\mu-1}|v|+|v|^{\mu-1}|u|).\end{aligned}$$

Hence, for any given $\varepsilon_2>0$, applying the Young inequality to the last inequality, there exists $C_{\varepsilon_2}>0$ such that

$$|F(u_n,v_n)-F(u_n-u,v_n-v)|\le\varepsilon_2(|u_n-u|^{\mu}+|v_n-v|^{\mu})+C_{\varepsilon_2}(|u|^{\mu}+|v|^{\mu}).\tag{2.24}$$

Now we define the functions

$$f_n=|F(u_n,v_n)-F(u_n-u,v_n-v)-F(u,v)|$$

and

$$g_n=f_n-\varepsilon_2(|u_n-u|^{\mu}+|v_n-v|^{\mu}).$$

Then

$$f_n\le\varepsilon_2(|u_n-u|^{\mu}+|v_n-v|^{\mu})+C_{\varepsilon_2}(|u|^{\mu}+|v|^{\mu})+|F(u,v)|$$

and

$$\begin{aligned}g_n &\le |F(u,v)|+C_{\varepsilon_2}(|u|^{\mu}+|v|^{\mu})\\ &\le M_F(|u|^{\mu}+|v|^{\mu})+C_{\varepsilon_2}(|u|^{\mu}+|v|^{\mu})\\ &\le (M_F+C_{\varepsilon_2})(|u|^{\mu}+|v|^{\mu})\in L^1(R^N).\end{aligned}$$

Since $(u_n,v_n)\rightharpoonup(u,v)$ in E, we can assume that $u_n\to u$, $v_n\to v$ a.e. in R^N. Thus, $g_n\to 0$ a.e. in R^N as $n\to\infty$. The Lebesgue dominated convergence theorem implies that

$$\lim_{n\to\infty}\int_{R^N}g_n(x)\mathrm{d}x=0.\tag{2.25}$$

Therefore, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\int_{R^N}f_n(x)\mathrm{d}x &\le \limsup_{n\to\infty}\int_{R^N}g_n(x)+\varepsilon_2(|u_n-u|^{\mu}+|v_n-v|^{\mu})\mathrm{d}x\\ &\le \limsup_{n\to\infty}\int_{R^N}g_n(x)\mathrm{d}x+\varepsilon_2\limsup_{n\to\infty}\int_{R^N}(|u_n-u|^{\mu}+|v_n-v|^{\mu})\mathrm{d}x\\ &\le C\varepsilon_2.\end{aligned}$$

By the arbitrariness of ε_2, one has

$$\lim_{n\to\infty}\int_{R^N}f_n(x)\mathrm{d}x\to 0,\quad \text{as } n\to\infty.$$

This completes the proof. □

3. The Nehari manifold

As the energy functional J is not bounded below on E, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N} = \{(u, v) \in E \setminus \{(0, 0)\} | \langle J'(u, v), (u, v)\rangle = 0\}.$$

Thus, $(u, v) \in \mathcal{N}$ if and only if

$$\langle J'(u, v), (u, v)\rangle = \|(u, v)\|_E^p - \int_{R^N} F(u, v)\mathrm{d}x - \langle f, u\rangle_{-1,1} - \langle g, v\rangle_{-1,1} = 0.$$

Define

$$\begin{aligned}
&\Phi(u, v) = \langle J'(u, v), (u, v)\rangle,\\
&\mathcal{N}^+ = \{(u, v) \in \mathcal{N} | \langle \Phi'(u, v), (u, v)\rangle > 0\},\\
&m = \inf_{(u,v)\in\mathcal{N}} J(u, v), \qquad m^+ = \inf_{(u,v)\in\mathcal{N}^+} J(u, v).
\end{aligned}$$

Then for all $(u, v) \in \mathcal{N}$,

$$\begin{aligned}
\langle \Phi'(u, v), (u, v)\rangle &= p\|(u, v)\|_E^p - \mu \int_{R^N} F(u, v)\mathrm{d}x - \langle f, u\rangle_{-1,1} - \langle g, v\rangle_{-1,1}\\
&= (p - \mu)\|(u, v)\|_E^p - (1 - \mu)(\langle f, u\rangle_{-1,1} + \langle g, v\rangle_{-1,1})\\
&= (p - \mu) \int_{R^N} F(x, u, v)\mathrm{d}x - (1 - p)(\langle f, u\rangle_{-1,1} + \langle g, v\rangle_{-1,1})\\
&= (p - 1)\|(u, v)\|_E^p - (\mu - 1) \int_{R^N} F(u, v)\mathrm{d}x. \qquad (3.1)
\end{aligned}$$

We have the following results.

Lemma 3.1. *The energy functional J is bounded below on $\mathcal{N}$.*

Proof. If $(u, v) \in \mathcal{N}$, then by the Hölder inequality and the Young inequality, for any $\theta_2 > 0$, we have

$$\begin{aligned}
J(u, v) &= \frac{1}{p}\|(u, v)\|_E^p - \frac{1}{\mu} \int_{R^N} F(u, v)\mathrm{d}x - \langle f, u\rangle_{-1,1} - \langle g, v\rangle_{-1,1}\\
&= \frac{1}{p}\|(u, v)\|_E^p - \frac{1}{\mu}\|(u, v)\|_E^p + \frac{1}{\mu}\langle f, u\rangle_{-1,1} + \frac{1}{\mu}\langle g, v\rangle_{-1,1} - \langle f, u\rangle_{-1,1} - \langle g, v\rangle_{-1,1}\\
&= \left(\frac{1}{p} - \frac{1}{\mu}\right)\|(u, v)\|_E^p - \left(1 - \frac{1}{\mu}\right)\langle f, u\rangle_{-1,1} - \left(1 - \frac{1}{\mu}\right)\langle g, v\rangle_{-1,1}\\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|(u, v)\|_E^p - \left(1 - \frac{1}{\mu}\right)\|f\|_{-1,p'}\|u\|_{1,p} - \left(1 - \frac{1}{\mu}\right)\|g\|_{-1,p'}\|v\|_{1,p}\\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|(u, v)\|_E^p - \left(1 - \frac{1}{\mu}\right)\frac{1}{p'}\theta_2^{-p'}\|f\|_{-1,p'}^{p'} - \left(1 - \frac{1}{\mu}\right)\frac{1}{p}\theta_2^p\|u\|_{1,p}^p\\
&\quad - \left(1 - \frac{1}{\mu}\right)\frac{1}{p'}\theta_2^{-p'}\|g\|_{-1,p'}^{p'} - \left(1 - \frac{1}{\mu}\right)\frac{1}{p}\theta_2^p\|v\|_{1,p}^p\\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|(u, v)\|_E^p - \left(1 - \frac{1}{\mu}\right)\frac{1}{p}\theta_2^p\|(u, v)\|_E^p - \left(1 - \frac{1}{\mu}\right)\frac{1}{p'}\theta_2^{-p'}(\|f\|_{-1,p'}^{p'} + \|g\|_{-1,p'}^{p'}).
\end{aligned}$$

By the arbitrariness of θ_2, we can choose $\theta_2 = \left(\frac{\mu-p}{\mu-1}\right)^{\frac{1}{p}}$. Consequently, for every $(u, v) \in \mathcal{N}$, we have

$$J(u, v) \geq -\left(1 - \frac{1}{\mu}\right)\frac{1}{p'}\theta_2^{-p'}(\|f\|_{-1,p'}^{p'} + \|g\|_{-1,p'}^{p'}). \qquad (3.2)$$

Hence, we have shown that J is bounded below on $\mathcal{N}$. □

Lemma 3.2. *$m \leq m^+ < 0$.*

Proof. Indeed, for all $(u, v) \in \mathcal{N}^+$, by (3.1), we obtain

$$\frac{p-1}{\mu-1}\|(u,v)\|_E^p > \int_{R^N} F(u,v)\mathrm{d}x$$

and

$$\begin{aligned} J(u,v) &= \left(\frac{1}{p}-1\right)\|(u,v)\|_E^p + \left(1-\frac{1}{\mu}\right)\int_{R^N} F(u,v)\mathrm{d}x \\ &< \left[\left(\frac{1}{p}-1\right)+\left(1-\frac{1}{\mu}\right)\frac{p-1}{\mu-1}\right]\|(u,v)\|_E^p \\ &= \left(\frac{1-p}{p}-\frac{1-p}{\mu}\right)\|(u,v)\|_E^p \\ &< 0. \end{aligned}$$

Hence, from the definition of m and m^+, we can deduce that $m \le m^+ < 0$. □

Lemma 3.3. *Let $\{(u_n, v_n)\}$ be a minimizing sequence for J on $\mathcal{N}$, that is $\{(u_n, v_n)\} \subset \mathcal{N}$ such that $J(u_n, v_n) \to m$ whenever $n \to \infty$. Suppose that $f, g \in W^{-1,p'}(R^N) \setminus \{0\}$ satisfy the condition (f_g). Then there exists $\delta_0 > 0$ such that*

$$|\langle \Phi'(u_n, v_n), (u_n, v_n)\rangle| \ge \delta_0 > 0.$$

Proof. If not, there exists a subsequence of $\{(u_n, v_n)\}$ (still denoted by $\{(u_n, v_n)\}$) such that $|\langle \Phi'(u_n, v_n), (u_n, v_n)\rangle| \to 0$ as $n \to \infty$. Then using (3.1), we have

$$\begin{aligned} \theta_n &:= \langle \Phi'(u_n,v_n),(u_n,v_n)\rangle \\ &= p\|(u_n,v_n)\|_E^p - \mu\int_{R^N} F(u_n,v_n)\mathrm{d}x - \langle f,u_n\rangle_{-1,1} - \langle g,v_n\rangle_{-1,1} \\ &= (p-1)\|(u_n,v_n)\|_E^p - (\mu-1)\int_{R^N} F(u_n,v_n)\mathrm{d}x \\ &= (p-\mu)\|(u_n,v_n)\|_E^p - (1-\mu)(\langle f,u_n\rangle_{-1,1} + \langle g,v_n\rangle_{-1,1}) \to 0, \end{aligned}$$

as $n \to \infty$. Hence, we obtain

$$(p-1)\|(u_n,v_n)\|_E^p = (\mu-1)\int_{R^N} F(u_n,v_n)\mathrm{d}x + \theta_n, \tag{3.3}$$

$$(\mu-p)\|(u_n,v_n)\|_E^p = (\mu-1)(\langle f,u_n\rangle_{-1,1} + \langle g,v_n\rangle_{-1,1}) - \theta_n. \tag{3.4}$$

By (3.3), Proposition 1.1 and the Sobolev inequality,

$$\begin{aligned} (p-1)\|(u_n,v_n)\|_E^p &= (\mu-1)\int_{R^N} F(u_n,v_n)\mathrm{d}x + \theta_n \\ &\le (\mu-1)\int_{R^N} M_F(|u_n|^\mu + |v_n|^\mu)\mathrm{d}x + \theta_n \\ &\le (\mu-1)M_F S^{-\frac{\mu}{p}}(\|u_n\|_{1,p}^\mu + \|v_n\|_{1,p}^\mu) + \theta_n \\ &\le 2(\mu-1)M_F S^{-\frac{\mu}{p}}\|(u_n,v_n)\|_E^\mu + \theta_n. \end{aligned} \tag{3.5}$$

Using the Young inequality and (3.4), for any $0 < \gamma < \left(\frac{\mu-p}{\mu-1}\right)^{\frac{1}{p}}$, we have

$$\begin{aligned} (\mu-p)\|(u_n,v_n)\|_E^p &= (\mu-1)(\langle f,u_n\rangle_{-1,1} + \langle g,v_n\rangle_{-1,1}) - \theta_n \\ &\le (\mu-1)\|f\|_{-1,p'}\|u_n\|_{1,p} + (\mu-1)\|g\|_{-1,p'}\|v_n\|_{1,p} - \theta_n \\ &\le (\mu-1)\frac{1}{p'\gamma^{p'}}\|f\|_{-1,p'}^{p'} + (\mu-1)\frac{\gamma^p}{p}\|u_n\|_{1,p}^p(\mu-1)\frac{1}{p'\gamma^{p'}}\|g\|_{-1,p'}^{p'} + (\mu-1)\frac{\gamma^p}{p}\|v_n\|_{1,p}^p - \theta_n \\ &= (\mu-1)\frac{1}{p'\gamma^{p'}}\|f\|_{-1,p'}^{p'} + (\mu-1)\frac{1}{p'\gamma^{p'}}\|g\|_{-1,p'}^{p'} + (\mu-1)\frac{\gamma^p}{p}\|(u_n,v_n)\|_E^p - \theta_n. \end{aligned}$$

Hence, we deduce

$$\left(\frac{p\mu - p^2 - (\mu-1)\gamma^p}{p}\right)\|(u_n,v_n)\|_E^p \le (\mu-1)\frac{1}{p'\gamma^{p'}}(\|f\|_{-1,p'}^{p'} + \|g\|_{-1,p'}^{p'}) - \theta_n. \tag{3.6}$$

By (3.5), we obtain

$$\frac{1}{2(\mu-1)M_F}S^{\frac{\mu}{p}}\left(p-1-\frac{\theta_n}{\|(u_n,v_n)\|_E^p}\right)\le \|(u_n,v_n)\|_E^{\mu-p}. \tag{3.7}$$

For n sufficiently large, we claim that

$$\frac{1}{\|(u_n,v_n)\|_E^p}\le \bar{K},$$

where $\bar{K}$ is a positive constant. In fact, suppose the contrary, $\|(u_n,v_n)\|_E\to 0$, as $n\to\infty$. We conclude that $J(u_n,v_n)\to 0$, as $n\to\infty$. This implies that $m=0$, which is impossible according to Lemma 3.2. From this, we obtain the inequality

$$\left(\frac{1}{2(\mu-1)M_F}S^{\frac{\mu}{p}}\right)^{\frac{1}{\mu-p}}(p-1-\bar{K}|\theta_n|)^{\frac{1}{\mu-p}}\le \|(u_n,v_n)\|_E. \tag{3.8}$$

By (3.8), (3.6) becomes

$$\left[\left(\frac{p\mu-p^2-(\mu-1)\gamma^p}{p}\right)\left(\frac{1}{2(\mu-1)M_F}S^{\frac{\mu}{p}}\right)^{\frac{p}{\mu-p}}(p-1-\bar{K}|\theta_n|)^{\frac{p}{\mu-p}}+\theta_n\right]^{\frac{p-1}{p}}\left(\frac{p'\gamma^{p'}}{\mu-1}\right)^{\frac{p-1}{p}}$$
$$\le \|f\|_{-1,p'}+\|g\|_{-1,p'}.$$

Letting $n\to\infty$, we get

$$\left(\frac{p\mu-p^2-(\mu-1)\gamma^p}{p}\right)^{\frac{p-1}{p}}\left(\frac{p-1}{2(\mu-1)M_F}S^{\frac{\mu}{p}}\right)^{\frac{p-1}{\mu-p}}\left(\frac{p'\gamma^{p'}}{\mu-1}\right)^{\frac{p-1}{p}}\le \|f\|_{-1,p'}+\|g\|_{-1,p'}.$$

This is a contradiction. The proof is completed. □

Lemma 3.4. *Suppose that $f,g\in W^{-1,p'}(R^N)\setminus\{0\}$ satisfy the condition (f_g). Then there exists a minimizing sequence $\{(u_n.v_n)\}\subset\mathcal{N}$ such that*

$$J(u_n,v_n)=m+o_n(1),\qquad J'(u_n,v_n)=o_n(1)\quad \text{in } E^{-1}.$$

Proof. By Lemma 3.1, J is bounded below on $\mathcal{N}$. Hence, the Ekeland variational principle ensures the existence of a sequence $\{(u_n,v_n)\}$ in $\mathcal{N}$ satisfying

$$J(u_n,v_n)\to m,\qquad J'_{|\mathcal{N}}(u_n,v_n)\to 0.$$

We now claim that

$$J'(u_n,v_n)\to 0\quad \text{in } E^{-1}. \tag{3.9}$$

In fact, for some $\lambda_n\in R$, by the theory of Lagrange multipliers, we have

$$J'(u_n,v_n)=J'_{|\mathcal{N}}(u_n,v_n)-\lambda_n\Phi'(u_n,v_n). \tag{3.10}$$

Consequently

$$\langle J'(u_n,v_n),(u_n,v_n)\rangle=\langle J'_{|\mathcal{N}}(u_n,v_n),(u_n,v_n)\rangle-\lambda_n\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle.$$

Since $(u_n,v_n)\in\mathcal{N}$, we have $\langle J'(u_n,v_n),(u_n,v_n)\rangle=0$. By Lemma 3.3, $\langle\Phi'(u_n,v_n),(u_n,v_n)\rangle<-\delta_0$. Thus $\lambda_n\to 0$ as $n\to\infty$. By the boundedness of $\{(u_n,v_n)\}$, $\Phi'(u_n,v_n)$ is bounded. This implies that $\lambda_n\Phi'(u_n,v_n)\to 0$. So (3.9) follows from (3.10). This completes the proof. □

4. Proof of Theorem 1.1

First, let us introduce the functional I defined in the space E by

$$I(u,v)=\frac{1}{p}\|(u,v)\|_E^p-\frac{1}{\mu}\int_{R^N}F(u,v)\mathrm{d}x.$$

Λ is the subset of E defined by

$$\Lambda=\{(u,v)\in E\setminus\{(0,0)\}|\langle I'(u,v),(u,v)\rangle=0\}.$$

We set

$$m_0 = \inf_{(u,v)\in\Lambda} I(u, v).$$

Then, we have the following results.

Lemma 4.1. $m_0 = \inf_{(u,v)\in\Lambda} I(u, v) > 0$.

Proof. For all $(u, v) \in E \setminus \{(0, 0)\}$, by Proposition 1.1 and the Sobolev inequality, we have

$$\begin{aligned} I(u, v) &= \frac{1}{p}\|(u, v)\|_E^p - \frac{1}{\mu}\int_{R^N} F(u, v)\mathrm{d}x \\ &\geq \frac{1}{p}\|(u, v)\|_E^p - \frac{1}{\mu}2M_F S^{-\frac{\mu}{p}}\|(u, v)\|_E^{\mu}, \end{aligned}$$

which implies that 0 is a strictly local minimum of I. Thus there exist $\rho > 0$, $\alpha > 0$ such that $I(u, v) \geq \alpha$ for $\|(u, v)\|_E = \rho$. Since $\mu > p$, it is easy to see that

$$m_0 = \inf_{(u,v)\in\Lambda} I(u, v) = \inf_{(u,v)\neq(0,0)} \max_{t\geq 0} I(tu, tv) \geq \alpha > 0.$$

This completes the proof. □

Lemma 4.2. *J satisfies the $(\mathrm{PS})_m$-condition, where $m = \inf_{(u,v)\in\mathcal{N}} J(u, v)$.*

Proof. Let $\{(u_n, v_n)\} \subset E$ be a $(\mathrm{PS})_m$-sequence for J, that is,

$$J(u_n, v_n) = m + o_n(1), \qquad J'(u_n, v_n) = o_n(1) \quad \text{in } E^{-1}. \tag{4.1}$$

Then by Lemma 2.3 and the compact imbedding theorem, there exist a subsequence of $\{(u_n, v_n)\}$ (still denoted by $\{(u_n, v_n)\}$) and $(u, v) \in E$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } W^{1,p}(R^N), \\ &v_n \rightharpoonup v \quad \text{in } W^{1,p}(R^N), \\ &u_n \to u \quad \text{in } L^q_{\mathrm{loc}}(R^N),\ p \leq q < p^* = \frac{Np}{N-p}, \\ &v_n \to v \quad \text{in } L^q_{\mathrm{loc}}(R^N),\ p \leq q < p^* = \frac{Np}{N-p}, \\ &u_n \to u \quad \text{a.e. in } R^N, \\ &v_n \to v \quad \text{a.e. in } R^N. \end{aligned}$$

This implies that

$$\langle f, u_n\rangle_{-1,1} + \langle g, v_n\rangle_{-1,1} \to \langle f, u\rangle_{-1,1} + \langle g, v\rangle_{-1,1}. \tag{4.2}$$

By Lemma 2.6, we know that the pair (u, v) gives a critical point of J. In what follows, we will prove that (u, v) is nontrivial. Noting that

$$\begin{aligned} m &= J(u_n, v_n) + o_n(1) \\ &= J(u_n, v_n) - \frac{1}{\mu}\langle J'(u_n, v_n), (u_n, v_n)\rangle + o_n(1) \\ &= \frac{\mu - p}{p\mu}\|(u_n, v_n)\|_E^p - \frac{\mu - 1}{\mu}(\langle f, u_n\rangle_{-1,1} + \langle g, v_n\rangle_{-1,1}) + o_n(1), \end{aligned}$$

we obtain

$$\begin{aligned} J(u, v) &= J(u, v) - \frac{1}{\mu}J'(u, v) \\ &= \frac{\mu - p}{p\mu}\|(u, v)\|_E^p - \frac{\mu - 1}{\mu}(\langle f, u\rangle_{-1,1} + \langle g, v\rangle_{-1,1}) \\ &\leq \liminf_{n\to\infty}\left[\frac{\mu - p}{p\mu}\|(u_n, v_n)\|_E^p - \frac{\mu - 1}{\mu}(\langle f, u_n\rangle_{-1,1} + \langle g, v_n\rangle_{-1,1})\right] \\ &= m < 0. \end{aligned}$$

Thus, (u, v) is nontrivial. Now we prove that

$$u_n \to u \quad \text{strongly in } W^{1,p}(R^N)$$

and

$$v_n \to v \quad \text{strongly in } W^{1,p}(R^N).$$

In fact, suppose that $\tilde{u}_n = u_n - u$, $\tilde{v}_n = v_n - v$. Then by the Brezis–Lieb Lemma (Lemma 1.32 in [12]), we obtain

$$\|(\tilde{u}_n, \tilde{v}_n)\|_E^p \to \|(u_n, v_n)\|_E^p - \|(u, v)\|_E^p, \quad n \to \infty. \tag{4.3}$$

and by Lemma 2.7, one has

$$\int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x = \int_{R^N} F(u_n, v_n)\mathrm{d}x - \int_{R^N} F(u, v)\mathrm{d}x + o_n(1). \tag{4.4}$$

It follows from (4.1)–(4.4) that

$$I(\tilde{u}_n, \tilde{v}_n) = \frac{1}{p}\|(\tilde{u}_n, \tilde{v}_n)\|_E^p - \frac{1}{\mu}\int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x = m - J(u, v) + o_n(1) \tag{4.5}$$

and

$$\|(\tilde{u}_n, \tilde{v}_n)\|_E^p - \int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x = o_n(1).$$

Hence, we may assume that

$$\|(\tilde{u}_n, \tilde{v}_n)\|_E^p \to l, \qquad \int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x \to l. \tag{4.6}$$

If $l = 0$, the proof is complete. Assuming that $l > 0$, we will consider that t_n is such that

$$s(t_n) = \langle I'(t_n\tilde{u}_n, t_n\tilde{v}_n), (t_n\tilde{u}_n, t_n\tilde{v}_n)\rangle = t_n^p\|(\tilde{u}_n, \tilde{v}_n)\|^p - t_n^\mu \int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x = 0.$$

Without loss of generality, we can assume that $\tilde{u}_n \neq 0$ and $\tilde{v}_n \neq 0$. Then, an easy computation shows that

$$t_n = \left(\frac{\|(\tilde{u}_n, \tilde{v}_n)\|_E^p}{\int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x}\right)^{\frac{1}{\mu-p}} > 0.$$

By (4.6), it is clear that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty}\left(\frac{\|(\tilde{u}_n, \tilde{v}_n)\|_E^p}{\int_{R^N} F(\tilde{u}_n, \tilde{v}_n)\mathrm{d}x}\right)^{\frac{1}{\mu-p}} = 1. \tag{4.7}$$

Hence, it follows from (4.5) and (4.7) that

$$m - J(u, v) = \lim_{n\to\infty} I(\tilde{u}_n, \tilde{v}_n) = \lim_{n\to\infty} I(t_n\tilde{u}_n, t_n\tilde{v}_n) \geq m_0. \tag{4.8}$$

Since $J'(u, v) = 0$, we know that $(u, v) \in \mathcal{N}$. Hence, $J(u, v) \geq m$ and $m_0 \leq 0$, which contradicts $m_0 > 0$. Consequently, we have $l = 0$. This completes the proof. □

Proof of Theorem 1.1. By Lemma 3.4, there exists a $(\mathrm{PS})_m$-sequence $\{(u_n, v_n)\} \subset E$ for J. From Lemmas 3.2 and 4.2, J satisfies the $(\mathrm{PS})_m$ condition and $m < 0$. Using Lemma 2.3, we have that $\{(u_n, v_n)\}$ is bounded in E. Therefore, there exists a subsequence still denoted by $\{(u_n, v_n)\}$, together with $(u, v) \neq (0, 0)$, such that $(u_n, v_n) \to (u, v)$ in E and $J(u, v) = m < 0$. Hence, we have that (u, v) is a nontrivial solution of problem (1.1). This completes the proof. □

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