Available online at www.sciencedirect.com
SCIENCE DIRECT®
Neural Networks 17 (2004) 1415–1425
Neural Networks
www.elsevier.com/locate/neunet

Global eponential stability of cellular neural networks with time-varying coefficients and delays

Haijun Jiang*, Zhidong Teng

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China

Received 12 March 2004; accepted 12 March 2004

Abstract

In this paper, a class of cellular neural networks with time-varying coefficients and delays is considered. By constructing a suitable Liapunov functional and utilizing the technique of matrix analysis, some new sufficient conditions on the global exponential stability of solutions are obtained. The results obtained in this paper improve and extend some of the previous results.

Keywords: Global exponential stability; Liapunov functional; Time-varying coefficient; Time-varying delay; Cellular neural network

1. Introduction

Cellular neural networks (CNNs) represent a class of recurrent neural networks with local interneuron connections. As dynamic system with a special structure, CNNs have many interesting properties that deserve theoretical studies. In recent years, autonomous CNNs have been extensively studied and successfully applied to signal processing system, especially in static image treatment, and to solve nonlinear algebraic equations, such application rely on the qualitative properties of stability. During hardware implementation, time delays occur due to finite switching speed of the amplifiers and communication time. Time delay may lead to an oscillation and furthermore, to instability of networks. Therefore, the study of stability of CNNs with delay is practically required. However, the nonautonomous phenomenon often occurs in many realistic systems. Particularly, when we consider a long-time dynamical behavior of a system. The parameters of the system usually will arise change along with time. Thus the research on the nonautonomous CNNs is very important like on the autonomous CNNs.

For delayed autonomous CNNs, many important results have been obtained on the existence of equilibrium points, global asymptotic stability, global exponential stability, bifurcation and the existence of periodic solutions and almost periodic solutions (Arik, 2000; Arik & Tavsanoglu, 2000; Cao, 1999, 2000a,b, 2001; Cao & Wang, 2002, 2003; Chu, 2001; Lu, 2001; Mohamad, 2001; Mohamad & Gopalsamy, 2000; Shayer & Campbell, 2000; Takahashi, 2000; Van Den Driessche & Zou, 1998; Wei & Ruan, 1999; Zhang & Jin, 2000; Zhang, Pheng, & Kwong, 2001; Zhou & Cao, 2002 and references cited therein). Particularly, in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001), the authors applied the technique of matrix analysis and the method of Liapunov functional to discuss the global asymptotic stability of the equilibrium point. However, we see that in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001) the global exponential stability of the equilibrium point have not been studied. Therefore, one of the main purpose in this paper is to discuss the global exponential stability by improving and extending the technique of matrix analysis and the method of Liapunov functional given in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001).

In this paper, we will consider more general CNNs than that given in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001). That is the following CNNs with time-varying coefficients and delays

$$\frac{dx_i(t)}{dt} = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(x_j(t-\tau_j(t))) + I_i(t), \quad i = 1,2,\dots,n.$$

* Corresponding author. Tel.: +86-991-8580535.
E-mail address: jianghai@xju.edu.cn (H. Jiang).

0893-6080/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.neunet.2004.03.002

This system can be transformed into the following vector form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -C(t)x(t) + A(t)f(x(t)) + B(t)g(x(t-\tau(t))) + I(t), \tag{1}$$

where $t \in R_+ = [0,\infty)$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$, $C(t) = \mathrm{diag}(c_1(t), c_2(t), \ldots, c_n(t))$, $A(t) = (a_{ij}(t))_{n\times n}$, $B(t) = (b_{ij}(t))_{n\times n}$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^{\mathrm{T}}$, $g(x(t-\tau(t))) = (g_1(x_1(t-\tau_1(t))), g_2(x_2(t-\tau_2(t))), \ldots, g_n(x_n(t-\tau_n(t))))^{\mathrm{T}}$ and $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^{\mathrm{T}}$. Our main purpose in this paper is to study the global exponential stability for system (1). Like in Jiang, Li, and Teng (2003) and Jiang and Teng (2003), in this paper we will not require that system (1) has any equilibrium point and also not require that all nonlinear response functions $f_i(u)$ and $g_i(u)$ in system (1) are bounded on R_+. By constructing new Liapunov functional and using the technique of matrix analysis, we will establish new criteria on the global exponential stability of system (1). We will see that the results obtained in this paper are different completely from some well-known results on the global exponential stability obtained in Cao (1999, 2000a,b), Cao and Wang (2002), Chu (2001), Huang, Cao, and Wang (2002), Lu (2001), Mohamad (2001) and Mohamad and Gopalsamy (2000), and also are a very good improvement and extension of the method and result given in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001) to the CNNs with time-varying coefficients and delays.

We see that the CNNs with time-varying coefficients and delays have been studied (for example, Chen & Cao, 2003; Chen, Huang, & Cao, 2003; Dong, Matsui, & Huang, 2002; Jiang et al., 2003; Jiang & Teng, 2003; Liang & Cao, 2003). In Jiang et al. (2003), under the assumptions that the systems may not have any equilibrium points and the nonlinear response functions may be unbounded, by using Liapunov functional method and the technique of inequality analysis, the authors established a series of criteria on the boundedness, global exponential stability and the existence of periodic solutions for CNNs with time-varying coefficients and finite delay. In Jiang and Teng (2003), under the assumptions that the systems may not have any equilibrium points and the nonlinear response functions may be unbounded, by using Liapunov functional method and the technique of matrix analysis, the authors established a series of criteria on the boundedness, global asymptotic stability and the existence of periodic solutions for CNNs with time-varying coefficients and finite delay. In Dong et al. (2002), the existence and stability of periodic solutions for a class of periodic Hopfield neural networks are obtained by using the continuation theorem and Liapunov functional method. In Chen and Cao (2003) and Chen et al. (2003), under the assumptions that response functions are bounded, by using the Banach fixed point theorem and constructing suitable Liapunov functional, the authors established some sufficient conditions to ensure the existence, uniqueness and global stability of almost periodic solution for the delayed BAM neural networks and CNNs with almost periodic variable coefficients. In Liang and Cao (2003), the problems of boundedness and stability for a general class of nonautonomous recurrent neural networks with variable coefficients and time-varying delays are analyzed via employing Young inequality technique and Liapunov method. Some simple sufficient conditions are given for boundedness and stability of the solutions for recurrent neural networks.

This paper is organized as follows. In Section 2, we will given the definitions and assumptions. In Section 3, we will establish new sufficient conditions for the global exponential stability of all solutions for system (1) by constructing new Liapunov functionals and utilizing the technique of matrix analysis. In Section 4, we will obtain a series of corollaries and remarks. In Section 5, two examples are given to illustrate the theory. In Section 6, we given some concluding remarks of the results.

2. Definitions and assumptions

Firstly, in order to simplify our description, we introduce some notations as follows. Let P be real symmetric matrix, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are the maximal eigenvalue and the minimal eigenvalue of P, respectively. For any matrix $Q = (q_{ij})_{n\times n}$, we denote that Q^{T} is the transposed matrix of Q and Q^{-1} is the inverse matrix of Q. For any n-dimensional vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$, we denote the norm $|x| = \sqrt{\sum_{i=1}^n x_i^2}$.

In this paper, for system (1) we introduce the following assumptions.

(H_1) Functions $c_i(t)$, $a_{ij}(t)$, $b_{ij}(t)$ and $I_i(t)$ $(i,j = 1,2,\ldots,n)$ are bounded and continuous defined on R_+, functions $\tau_i(t)$ $(i = 1,2,\ldots,n)$ are nonnegative, bounded and continuously differentiable defined on R_+ and $\inf_{t\in R_+}\{1-\dot{\tau}_i(t)\} > 0$, where $\dot{\tau}_i(t)$ expresses the derivative of $\tau_i(t)$ with respect to time t.

(H_2) There are positive constants k_i and h_i $(i = 1,2,\ldots,n)$ such that

$$0 \le \frac{f_i(u) - f_i(u^*)}{u - u^*} \le k_i, \qquad |g_i(u) - g_i(u^*)| \le h_i|u - u^*|$$

for all $u, u^* \in R = (-\infty, +\infty)$ and $i = 1,2,\ldots,n$.

(H_3) There are positive definite matrix S, diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$, $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n) > 0$, $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) > 0$ and a constant $a > 0$ such that

$$\lambda_{\min}(D_1(t,\eta)) \ge a$$

for all $t \in R_+$ and $0 \le \eta \le K$, where

$$D_1(t,\eta) = SC(t) + C(t)S - (SA(t) - C(t)\gamma)\eta - SB(t)\beta^{-1}B^{\mathrm{T}}(t)S - \eta(A^{\mathrm{T}}(t)S - \gamma C(t)) - \eta(\gamma A(t) + A^{\mathrm{T}}(t)\gamma)\eta - H^2(\alpha+\beta)P(t) - \eta\gamma B(t)\alpha^{-1}B^{\mathrm{T}}(t)\gamma\eta,$$

$\eta = \mathrm{diag}(\eta_1, \eta_2, \dots, \eta_n)$, $K = \mathrm{diag}(k_1, k_2, \dots, k_n)$, $H = \mathrm{diag}(h_1, h_2, \dots, h_n)$ and

$$P(t) = \mathrm{diag}\left(\frac{1}{1-\dot{\tau}_1(\psi_1^{-1}(t))}, \frac{1}{1-\dot{\tau}_2(\psi_2^{-1}(t))}, \dots, \frac{1}{1-\dot{\tau}_n(\psi_n^{-1}(t))}\right),$$

here $\psi_i^{-1}(t)$ is inverse function of $\psi_i(t) = t - \tau_i(t)$ $(i = 1, 2, \dots, n)$.

(H_4) There are positive definite matrix S, diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_n) > 0$, $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) \ge 0$ and a constant $a > 0$ such that

$$\gamma A(t) + A^{\mathrm{T}}(t)\gamma < 0 \quad \text{and} \quad \lambda_{\min}(D_2(t,\eta)) \ge a$$

for all $t \in R_+$ and $0 \le \eta \le K$, where

$$D_2(t,\eta) = SC(t) + C(t)S - H^2(\alpha - \beta^*)P(t) - SB(t)\alpha^{-1}B^{\mathrm{T}}(t)S - (SA(t) - C(t)\gamma)\eta - \eta(A^{\mathrm{T}}(t)S - \gamma C(t)),$$

$$\beta^* = \inf_{t\in R_+}\{\lambda_{\min}(B^{\mathrm{T}}(t)\gamma(\gamma A(t) + A^{\mathrm{T}}(t)\gamma)^{-1}\gamma B(t))\}E$$

and E is unit matrix.

Let $\tau = \sup\{\tau_i(t) : t \in R_+, i = 1, 2, \dots, n\}$. We denote by $C[-\tau, 0]$ the Banach space of n-dimensional continuous functions $\phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^{\mathrm{T}} : [-\tau, 0] \to R^n$ with the norm $\|\phi\| = \max_{-\tau\le s\le 0}|\phi(s)|$. In this paper we always assume that all solutions of system (1) satisfy the following initial conditions

$$x_i(\theta) = \phi_i(\theta) \quad \text{for all } \theta \in [-\tau, 0], \quad i = 1, 2, \dots, n, \tag{2}$$

where $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in C[-\tau, 0]$. It is well known that, by the fundamental theory of functional differential equations (see Burton, 1985), system (1) has a unique solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ satisfying the initial condition (2).

Definition 1. System (1) is said to be globally exponentially stable, if there are constants $\epsilon > 0$ and $M \ge 1$ such that for any two solutions $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \dots, y_n(t))$ of systems (1) with the initial functions $\phi, \psi \in C[-\tau, 0]$, respectively, one has

$$|x(t) - y(t)| \le M\|\phi - \psi\|\exp(-\epsilon t) \qquad \text{for all } t \ge 0.$$

3. Main results

We first introduce the following result on the boundedness of solutions of system (1). This result can be found (Jiang & Teng, 2003).

Lemma 1. *Suppose that* (H_1), (H_2) *and at least one of* (H_3) *and* (H_4) *hold, then system* (1) *is uniformly bounded and uniformly ultimately bounded.*

On the global exponential stability of solutions for system (1), we have the following result.

Theorem 1. *Suppose that* (H_1), (H_2) *and at least one of* (H_3) *and* (H_4) *hold, then system* (1) *is globally exponentially stable.*

Proof. Let $x^{(i)}(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^{\mathrm{T}}$ $(i = 1, 2)$ be any two solutions of system (1) satisfying the initial conditions $x^{(i)}(\theta) = \phi_i(\theta)$ for all $\theta \in [-\tau, 0]$, where $\phi_i(\theta) = (\phi_{i1}(\theta), \phi_{i2}(\theta), \dots, \phi_{in}(\theta)) \in C[-\tau, 0]$ for $i = 1, 2$. By Lemma 1, we know that $x^{(i)}(t)$ $(i = 1, 2)$ are defined for all $t \in R_+$ and are bounded. Let $z(t) = x^{(1)}(t) - x^{(2)}(t) = (z_1(t), z_2(t), \dots, z_n(t))$, where $z_i(t) = x_{1i}(t) - x_{2i}(t)$ $(i = 1, 2, \dots, n)$, then system (1) transformed into the following form:

$$\dot{z}(t) = -C(t)z(t) + A(t)\Phi(z(t)) + B(t)\Psi(z(t - \tau(t))), \tag{3}$$

where $\Phi(z(t)) = (\Phi_1(z_1(t)), \Phi_2(z_2(t)), \dots, \Phi_n(z_n(t)))^{\mathrm{T}}$, $\Psi(z(t - \tau(t))) = (\Psi_1(z_1(t - \tau_1(t))), \Psi_2(z_2(t - \tau_2(t))), \dots, \Psi_n(z_n(t - \tau_n(t))))^{\mathrm{T}}$, $\Phi_i(z_i(t)) = f_i(x_{1i}(t)) - f_i(x_{2i}(t))$ and $\Psi_i(z_i(t - \tau_i(t))) = g_i(x_{1i}(t - \tau_i(t))) - g_i(x_{2i}(t - \tau_i(t)))$ $(i = 1, 2, \dots, n)$. By assumption (H_2), we obtain $|\Phi_i(z_i(t))| \le k_i|z_i(t)|$ and $|\Psi_i(z_i(t - \tau_i(t)))| \le h_i|z_i(t - \tau_i(t))|$ for each $i = 1, 2, \dots, n$. Let $\epsilon > 0$ be a constant which will be determined in the following. We construct the Liapunov functional as follows.

$$V(t,z_t) = z^{\mathrm{T}}(t)Sz(t)\mathrm{e}^{\epsilon t} + 2\sum_{i=1}^{n}\int_0^{z_i}\gamma_i\Phi_i(s)\mathrm{d}s\,\mathrm{e}^{\epsilon t} + \sum_{i=1}^{n}\int_{t-\tau_i(t)}^{t}\frac{\alpha_i+\beta_i}{1-\dot{\tau}_i(\psi_i^{-1}(s))}\Psi_i^2(z_i(s))\mathrm{e}^{\epsilon(s+\tau_i(\psi_i^{-1}(s)))}\mathrm{d}s, \tag{4}$$

where S, γ_i, α_i and β_i are decided by (H_3). From the boundedness of $z(t)$ on R_+, we obtain that $V(t, z_t)$ also is bounded on R_+. Calculating the derivative of $V(t, z_t)$ along system (3), we have

$$\frac{dV(t,z_t)}{dt} = e^{\epsilon t}\Big[-2z^T(t)SC(t)z(t) + 2z^T(t)SA(t)\Phi(z(t)) + 2z^T(t)SB(t)\Psi(z(t-\tau(t))) - 2\Phi^T(z(t))\gamma C(t)z(t) + 2\Phi^T(z(t))\gamma A(t)\Phi(z(t)) + 2\Phi^T(z(t))\gamma B(t) \times \Psi(z(t-\tau(t))) + \Psi^T(z(t))(\alpha+\beta)\bar{P}(t,\epsilon)\Psi(z(t)) - \Psi^T(z(t-\tau(t)))(\alpha+\beta)\Psi(z(t-\tau(t))) + \epsilon z^T(t)Sz(t) + 2\epsilon\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)ds\Big], \quad (5)$$

where

$$\bar{P}(t,\epsilon) = \mathrm{diag}\left(\frac{e^{\epsilon\tau_1(\psi_1^{-1}(t))}}{1-\dot{\tau}_1(\psi_1^{-1}(t))}, \frac{e^{\epsilon\tau_2(\psi_2^{-1}(t))}}{1-\dot{\tau}_2(\psi_2^{-1}(t))}, \dots, \frac{e^{\epsilon\tau_n(\psi_n^{-1}(t))}}{1-\dot{\tau}_n(\psi_n^{-1}(t))}\right).$$

Firstly, we assume that (H_3) holds. Since

$$2z^T(t)SB(t)\Psi(z(t-\tau(t)) - \Psi^T(z(t-\tau(t)))\beta\Psi(z(t-\tau(t))) \le z^T(t)SB(t)\beta^{-1}B^T(t)Sz(t) \quad (6)$$

and

$$2\Phi^T(z(t))\gamma B(t)\Psi(z(t-\tau(t))) - \Psi^T(z(t-\tau(t)))\alpha\Psi(z(t-\tau(t))) \le \Phi^T(z(t))\gamma B(t)\alpha^{-1}B^T(t)\gamma\Phi(z(t)). \quad (7)$$

From (5)–(7), we further obtain

$$\frac{dV(t,z_t)}{dt} \le e^{\epsilon t}\Big\{ -z^T(t)\Big[SC(t)+C(t)S-\epsilon S - 2\epsilon E\frac{1}{z^2(t)}\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)ds\Big]z(t) + z^T(t)(SA(t)-C(t)\gamma)\Phi(z(t)) + \Phi^T(z(t))(A^T(t)S-\gamma C(t))z(t) + \Phi^T(z(t))(\gamma A(t) + A^T(t)\gamma)\Phi(z(t)) + z^T(t)H^2(\alpha+\beta)\bar{P}(t,\epsilon)z(t) + z^T(t)SB(t)\beta^{-1}B^T(t)Sz(t) + \Phi^T(z(t))\gamma B(t)\alpha^{-1}B^T(t)\gamma\Phi(z(t)).\Big\} \quad (8)$$

We let, for each $i=1,2,\dots,n$ and $t\in R_+$, $\omega_i(z_i(t)) = z_i^{-1}(t)\Phi_i(z_i(t))$ if $z_i(t)\ne 0$ and $\omega_i(z_i(t))=0$ if $z_i(t)=0$. Then by assumption (H_2) we have $0\le\omega_i(z_i(t))\le k_i$ for all $t\in R_+$ and $i=1,2,\dots,n$. Let $\eta=\mathrm{diag}(\omega_1(z_1(t)),\omega_2(z_2(t)),\dots,\omega_n(z_n(t)))$, then we have

$$\Phi(z(t)) = \eta z(t) \quad \text{for all } t\in R_+. \quad (9)$$

Therefore, from (8) and (9) we further obtain

$$\frac{dV(t,z_t)}{dt} \le e^{\epsilon t}\Big\{ -z^T(t)\Big[SC(t)+C(t)S-\epsilon S - 2\epsilon E\frac{1}{z^2(t)}\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)sf3s\Big]z(t) + z^T(t)(SA(t)-C(t)\gamma)\eta z(t) + z^T(t)\eta(A^T(t)S-\gamma C(t))z(t) + z^T(t)\eta(\gamma A(t)+A^T(t)\gamma)\eta z(t) + z^T(t)H^2(\alpha+\beta)\bar{P}(t,\epsilon)z(t) + z^T(t)SB(t)\beta^{-1}B^T(t)Sz(t) + z^T(t)\eta\gamma B(t)\alpha^{-1}B^T(t)\gamma\eta z(t)\Big\}$$
$$= -e^{\epsilon t}z^T(t)\Big[SC(t)+C(t)S-\epsilon S - H^2(\alpha+\beta)\bar{P}(t,\epsilon) - SB(t)\beta^{-1}B^T(t)S - (SA(t)-C(t)\gamma)\eta - \eta(A^T(t)S-\gamma C(t)) - \eta(\gamma A(t)+A^T(t)\gamma)\eta - \eta\gamma B(t)\beta^{-1}B^T(t)\gamma\eta - 2\epsilon E\frac{1}{z^2(t)} \times \sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)ds\Big]z(t). \quad (10)$$

Let

$$\bar{D}_1(t,\eta,\epsilon) = SC(t)+C(t)S-H^2(\alpha+\beta)\bar{P}(t,\epsilon) - SB(t)\beta^{-1}B^T(t)S - \epsilon S - (SA(t)-C(t)\gamma)\eta - \eta(A^T(t)S-\gamma C(t)) - \eta(\gamma A(t)+A^T(t)\gamma)\eta - \eta\gamma B(t)\beta^{-1}B^T(t)\gamma\eta - 2\epsilon E\frac{1}{z^2(t)}\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)ds.$$

Obviously, $\lim_{\epsilon\to 0}\bar{P}(t,\epsilon) = P(t)$ uniformly for all $t\in R_+$. From assumption (H_2) we obtain

$$\frac{1}{z^2(t)}\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)ds \le \frac{1}{2}\max_{1\le i\le n}\{\gamma_i k_i\} \quad \text{for all } t\in R_+.$$

Hence, we further have $\lim_{\epsilon\to 0}\bar{D}_1(t,\eta,\epsilon) = D_1(t,\eta)$ uniformly for all $t\in R_+$. and $0\le\eta\le K$. Thus, by assumption (H_3) there exists a constant $\epsilon>0$ such that

$$\lambda_{\min}(\bar{D}_1(t,\eta,\epsilon)) \ge \frac{1}{2}a$$

for all $t\in R_+$ and $0\le\eta\le K$. Therefore, by (10), we finally obtain

$$\frac{dV(t,z_t)}{dt} \le -\frac{1}{2}a\,e^{\epsilon t}z^T(t)z(t) < 0 \quad \text{for all } t\in R_+. \quad (11)$$

Secondly, we assume that (H_4) holds. Since

$$\begin{aligned}&2\Phi^{\mathrm{T}}(z(t))\gamma B(t)\Psi(z(t-\tau(t)))+\Phi^{\mathrm{T}}(z(t))(\gamma A(t)+A^{\mathrm{T}}(t)\gamma)\Phi(z(t))\\&\quad=2\Phi^{\mathrm{T}}(z(t))\gamma B(t)\Psi(z(t-\tau(t)))-\Phi^{\mathrm{T}}(z(t))\\&\qquad\times[-(\gamma A(t)+A^{\mathrm{T}}(t)\gamma)]\Phi(z(t))\le\Psi^{\mathrm{T}}(z(t-\tau(t)))B^{\mathrm{T}}(t)\\&\qquad\times\gamma[-(\gamma A(t)+A^{\mathrm{T}}(t)\gamma)]^{-1}\gamma B(t)\Psi(z(t-\tau(t)))\\&\quad=-\Psi^{\mathrm{T}}(z(t-\tau(t)))B^{\mathrm{T}}(t)\gamma(\gamma A(t)+A^{\mathrm{T}}(t)\gamma)^{-1}\\&\qquad\times\gamma B(t)\Psi(z(t-\tau(t)))\end{aligned}\tag{12}$$

and

$$\begin{aligned}&2z^{\mathrm{T}}(t)SB(t)\Psi(z(t-\tau(t)))-\Psi^{\mathrm{T}}(z(t-\tau(t)))\alpha\Psi(z(t-\tau(t)))\\&\quad\le z^{\mathrm{T}}(t)SB(t)\alpha^{-1}B^{\mathrm{T}}(t)Sz(t),\end{aligned}\tag{13}$$

then, from (5) and (9), when $\beta=-\beta^*$ we obtain

$$\begin{aligned}\frac{\mathrm{d}V(t,z_t)}{\mathrm{d}t}&\le\mathrm{e}^{\epsilon t}\Bigg[-2z^{\mathrm{T}}(t)SC(t)z(t)+2z^{\mathrm{T}}(t)SA(t)\Phi(z(t))\\&\quad+z^{\mathrm{T}}(t)SB(t)\alpha^{-1}B^{\mathrm{T}}(t)Sz(t)-2\Phi^{\mathrm{T}}(z(t))\gamma C(t)z(t)\\&\quad-\Psi^{\mathrm{T}}(z(t-\tau(t)))B^{\mathrm{T}}(t)\gamma(\gamma A(t)+A^{\mathrm{T}}(t)\gamma)^{-1}\gamma B(t)\\&\quad\times\Psi(z(t-\tau(t)))+\Psi^{\mathrm{T}}(z(t))(\alpha+\beta)\bar{P}(t,\epsilon)\Psi(z(t))\\&\quad-\Psi^{\mathrm{T}}(z(t-\tau(t)))\beta\Psi(z(t-\tau(t)))+\epsilon z^{\mathrm{T}}(t)Sz(t)\\&\quad+2\epsilon\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)\mathrm{d}s\Bigg]\\&\le\mathrm{e}^{\epsilon t}\Bigg\{-z^{\mathrm{T}}(t)\Bigg[SC(t)+C(t)S-\epsilon S\\&\quad-2\epsilon E\frac{1}{z^2(t)}\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)\mathrm{d}s)\Bigg]z(t)\\&\quad+z^{\mathrm{T}}(t)(SA(t)-C(t)\gamma)\eta z(t)+z^{\mathrm{T}}(t)\eta(A^{\mathrm{T}}(t)S\\&\quad-\gamma C(t))z(t)+z^{\mathrm{T}}(t)SB(t)\alpha^{-1}B^{\mathrm{T}}(t)Sz(t)\\&\quad+z^{\mathrm{T}}(t)H^2(\alpha-\beta^*)\bar{P}(t,\epsilon)z(t)\Bigg\}.\end{aligned}\tag{14}$$

Let

$$\begin{aligned}\bar{D}_2(t,\eta,\epsilon)&=SC(t)+C(t)S-\epsilon S-2\epsilon E\frac{1}{z^2(t)}\sum_{i=1}^{n}\int_0^{z_i(t)}\gamma_i\Phi_i(s)\mathrm{d}s\\&\quad-H^2(\alpha-\beta^*)\bar{P}(t,\epsilon)-SB(t)\alpha^{-1}B^{\mathrm{T}}(t)S\\&\quad-(SA(t)-C(t)\gamma)\eta-\eta(A^{\mathrm{T}}(t)S-\gamma C(t)).\end{aligned}$$

A similar argument as about $\bar{D}_1(t,\eta,\epsilon)$, we also have $\lim_{\epsilon\to0}\bar{D}_2(t,\eta,\epsilon)=D_2(t,\eta)$ uniformly for all $t\in R_+$ and $0\le\eta\le K$. Thus, by assumption (H_4) there exists a constant $\epsilon>0$ such that

$$\lambda_{\min}(\bar{D}_2(t,\eta,\epsilon))\ge\frac{1}{2}a \text{ for all } t\in R_+, 0\le\eta\le K.$$

Therefore, by (14) we finally have

$$\frac{\mathrm{d}V(t,z_t)}{\mathrm{d}t}\le-\frac{1}{2}az^{\mathrm{T}}(t)z(t)\mathrm{e}^{\epsilon t}<0 \text{ for all } t\in R_+.\tag{15}$$

From (11) and (15), we further obtain

$$V(t)\le V(0) \text{ for all } t\ge0.\tag{16}$$

Directly from (4) and assumption (H_2) we have

$$V(t)\ge z^{\mathrm{T}}(t)Sz(t)\mathrm{e}^{\epsilon t}\ge\lambda_{\min}(S)\mathrm{e}^{\epsilon t}\sum_{i=1}^{n}z_i^2(t)\tag{17}$$

for all $t\ge0$ and

$$\begin{aligned}V(0)&=z^{\mathrm{T}}(0)Sz(0)+2\sum_{i=1}^{n}\int_0^{z_i(0)}\gamma_i\Phi_i(s)\mathrm{d}s\\&\quad+\sum_{i=1}^{n}\int_{-\tau_i(0)}^{0}\frac{\alpha_i+\beta_i}{1-\dot{\tau}_i(\psi_i^{-1}(s))}\Psi_i^2(z_i(s))\mathrm{e}^{\epsilon(s+\tau_i(\psi_i^{-1}(s)))}\mathrm{d}s\\&\le\lambda_{\max}(S)\sum_{i=1}^{n}\sup_{s\in[-\tau,0]}(\phi_{1i}(s)-\phi_{2i}(s))^2\\&\quad+\sum_{i=1}^{n}\gamma_ik_i\sup_{s\in[-\tau,0]}(\phi_{1i}(s)-\phi_{2i}(s))^2+\sum_{i=1}^{n}L_i\sup_{s\in[-\tau,0]}(\phi_{1i}(s)\\&\quad-\phi_{2i}(s))^2\le M\|\phi_1-\phi_2\|^2,\end{aligned}\tag{18}$$

where

$$L_i=\sup_{s\in[-\tau,0]}\left\{\frac{\alpha_i+\beta_i}{1-\dot{\tau}_i(\psi_i^{-1}(s))}\mathrm{e}^{\epsilon(s+\tau_i(\psi_i^{-1}(s)))}\right\}h_i^2\tau_i(0)$$

and $M=\lambda_{\max}(S)+\max_{1\le i\le n}\{\gamma_ik_i+L_i\}$. Hence, by (16)–(18) we finally obtain

$$\sum_{i=1}^{n}z_i^2(t)\le M_0\|\phi_1-\phi_2\|^2\mathrm{e}^{-\epsilon t} \text{ for all } t\ge0,\tag{19}$$

where $M_0\ge1$ is a constant and M_0 is independent of any solution of system (1). From (19) we obtain that system (1) is globally exponentially stable. This completes the proof of Theorem 1.

4. Corollaries and remarks

In this section, we will give a series of corollaries as the special cases of Theorem 1. From these corollaries we will see that many important results (Arik, 2000; Arik & Tavsanoglu, 2000; Cao, 2001; Lu, 2001) are improved and extended in this paper to some more general cases, particularly, to the CNNs with time-varying coefficients and time-varying delays.

In assumption (H_3), if we choose $S=\beta=\sigma E$ and $\gamma=0$, where $\sigma>0$ is a constant, then we have

$$\begin{aligned}D_1(t,\eta)&=\sigma[2C(t)-A(t)\eta-\eta A^{\mathrm{T}}(t)-B(t)B^{\mathrm{T}}(t)\\&\quad-H^2P(t)]-H^2\alpha P(t).\end{aligned}$$

Obviously, if there is a constant $a>0$ such that

$$\lambda_{\min}(2C(t)-A(t)\eta-\eta A^{\mathrm{T}}(t)-B(t)B^{\mathrm{T}}(t)-H^2P(t))\ge a$$

for all $t \in R_+$ and $0 \le \eta \le K$, then there must exist a constant $\sigma > 0$ such that $\lambda_{\min}(D_1(t,\eta)) \ge a$ for all $t \in R_+$ and $0 \le \eta \le K$. Thus, we obtain the following corollary as a special case of Theorem 1.

Corollary 1. *Suppose that* (H_1) *and* (H_2) *hold. If there is a constant* $a > 0$ *such that*

$$\lambda_{\min}(2C(t) - A(t)\eta - \eta A^{\mathrm{T}}(t) - B(t)B^{\mathrm{T}}(t) - H^2P(t)) \ge a$$

for all $t \in R_+$ *and* $0 \le \eta \le K$, *then system* (1) *is globally exponentially stable.*

In assumption (H_4), if we choose $S = \alpha = \beta = \gamma = E$, then we have

$$D_2(t,\eta) = 2C(t) - (A(t) - C(t))\eta - \eta(A^{\mathrm{T}}(t) - C(t)) - B(t)B^{\mathrm{T}}(t) - H^2(1 - \bar{\beta}^*)P(t),$$

where $\bar{\beta}^* = \inf_{t\in R_+}\{\lambda_{\min}(B^{\mathrm{T}}(t)(A(t) + A^{\mathrm{T}}(t))^{-1}B(t))\}$. Therefore, as a special case of Theorem 1 we obtain the following corollary.

Corollary 2. *Suppose that* (H_1) *and* (H_2) *hold. If* $A(t) + A^{\mathrm{T}}(t) < 0$ *and there is a constant* $a > 0$ *such that*

$$\lambda_{\min}(2C(t) - (A(t) - C(t))\eta - \eta(A^{\mathrm{T}}(t) - C(t)) - B(t)B^{\mathrm{T}}(t) - H^2(1 - \bar{\beta}^*)P(t)) \ge a$$

for all $t \in R_+$ *and* $0 \le \eta \le K$, *then system* (1) *is globally exponentially stable.*

When $C(t) \equiv C$, $A(t) \equiv A$, $B(t) \equiv B$, $\tau(t) \equiv \tau$ and $I(t) \equiv I$ for all $t \in R_+$ are constants, then system (1) degenerates into the following autonomous CNNs with delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + Af(x(t)) + Bg(x(t-\tau)) + I. \tag{20}$$

In this case, the matrices $D_1(t,\eta)$ and $D_2(t,\eta)$ given in assumptions (H_3) and (H_4), respectively, become into

$$D_1(\eta) = SC + CS - (SA - C\gamma)\eta - \eta(A^{\mathrm{T}}S - \gamma C) - SB\beta^{-1}B^{\mathrm{T}}S - \eta(\gamma A + A^{\mathrm{T}}\gamma)\eta - H^2(\alpha + \beta) - \eta\gamma B\beta^{-1}B^{\mathrm{T}}\gamma\eta$$

and

$$D_2(\eta) = SC + CS - H^2(\alpha - \beta^*) - SB\alpha^{-1}B^{\mathrm{T}}S - (SA - C\gamma)\eta - \eta(A^{\mathrm{T}}S - \gamma C),$$

where $\beta^* = \lambda_{\min}(B^{\mathrm{T}}\gamma(\gamma A + A^{\mathrm{T}}\gamma)^{-1}\gamma B)E$. Further, assumptions (H_3) and (H_4), respectively, become into the following forms.

(H_3^*) There are positive definite matrix S, diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$, $\beta = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n) > 0$ and $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \ge 0$ such that

$$\lambda_{\min}(D_1(\eta)) > 0 \qquad \text{for all } 0 \le \eta \le K.$$

(H_4^*) There are positive definite matrix S, diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n > 0$ and $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \ge 0$ such that $\gamma A + A^{\mathrm{T}}\gamma < 0$ and

$$\lambda_{\min}(D_2(\eta)) > 0 \qquad \text{for all } 0 \le \eta \le K.$$

Therefore, as consequence of Theorem 1 we have the following result.

Corollary 3. *Suppose that* (H_2) *and at least one of* (H_3^*) *and* (H_4^*) *hold, then system* (18) *has a unique equilibrium* x^* *which is globally exponentially stable.*

The proof of the existence of unique equilibrium x^* in Corollary 3 is similar to Corollary 6 given in Jiang and Teng (2003).

Further, as consequence of Corollaries 1 and 2 we have the following results.

Corollary 4. *Suppose that* (H_2) *holds and*

$$\lambda_{\min}(2C - A\eta - \eta A^{\mathrm{T}} - BB^{\mathrm{T}} - H^2) > 0$$

for all $0 \le \eta \le K$.

Then system (18) *has a unique equilibrium* x^* *which is globally exponentially stable.*

Corollary 5. *Suppose that* (H_2) *holds. If* $A + A^{\mathrm{T}} < 0$ *and*

$$\lambda_{\min}(2C - A\eta - \eta A^{\mathrm{T}} + 2C\eta - BB^{\mathrm{T}} - H^2(1 - \bar{\beta}^*)) > 0$$

for all $0 \le \eta \le K$, *where* $\beta^* = \lambda_{\min}(B^{\mathrm{T}}(A + A^{\mathrm{T}})^{-1}B)$, *then system* (18) *has a unique equilibrium* x^* *which is globally exponentially stable.*

When system (1) is ω-periodic, that is, $C(t)$, $A(t)$, $B(t)$, $\tau(t)$ and $I(t)$ are ω-periodic functions. Applying the existence theorems of periodic solutions for general functional differential equations (see Burton (1985)), from Lemma 1 and Theorem 1 we have the following result.

Corollary 6. *Suppose that system* (1) *is* ω*-periodic,* (H_1), (H_2) *and at least one of* (H_3) *and* (H_4) *hold. Then system* (1) *has a unique* ω*-periodic solution which is globally exponentially stable.*

As two special cases of system (1), we have the following systems

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -C(t)x(t) + B(t)g(x(t - \tau(t))) + I(t) \tag{21}$$

and

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -C(t)x(t) + A(t)f(x(t)) + I(t). \tag{22}$$

Therefore, as consequences of Theorem 1 we have the following corollaries.

Corollary 7. *Suppose that* (H_1) *and* (H_2) *hold and there are the positive definite matrix* S, *diagonal matrix* $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$ *and a constant* $a > 0$ *such that*

$$\lambda_{\min}(SC(t) + C(t)S - H^2\alpha P(t) - SB(t)\alpha^{-1}B^{\mathrm{T}}(t)S) \geq a$$

for all $t \in R_+$. *Then system* (21) *is globally exponentially stable.*

Corollary 8. *Suppose that* (H_1) *and* (H_2) *hold and there exist the positive definite matrix* S, $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \geq 0$ *and a constant* $a > 0$ *such that* $\lambda_{\min}(D_1(t, \eta)) \geq a$ *for all* $t \in R_+$ *and* $0 \leq \eta \leq K$, *where*

$$D_1(t, \eta) = SC(t) + C(t)S - (SA(t) - C(t)\gamma)\eta - \eta(A^{\mathrm{T}}(t)S - \gamma C(t)) - \eta(\gamma A(t) + A^{\mathrm{T}}(t)\gamma)\eta.$$

Then system (22) *is globally exponentially stable.*

Particularly, when systems (21) and (22) degenerate into the autonomous cases we have the following results which are similar to Corollaries 7 and 8.

Corollary 9. *Suppose that* (H_2) *holds and there are the positive definite matrix* S *and diagonal matrix* $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) > 0$ *such that*

$$\lambda_{\min}(SC + CS - H^2\alpha - SB\alpha^{-1}B^{\mathrm{T}}S) > 0.$$

Then autonomous system (21) *has a unique equilibrium* x^* *which is globally exponentially stable.*

Corollary 10. *Suppose that* (H_2) *holds and there exist the positive definite matrix* S *and* $\gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \geq 0$ *such that*

$$\lambda_{\min}(SC + CS - \eta(\gamma A + A^{\mathrm{T}}\gamma)\eta + (C\gamma - SA)\eta + \eta(C\gamma - A^{\mathrm{T}}S)) > 0$$

for all $0 \leq \eta \leq K$. *Then autonomous system* (22) *has a unique equilibrium* x^* *which is globally exponentially stable.*

Remark 1. In Arik and Tavsanoglu (2000), the following autonomous CNNs

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -Cx(t) + TS(x(t - \tau)) + b \tag{23}$$

are studied, where $x(t) \in R^n, b \in R^n, C = \mathrm{diag}(c_1, c_2, \ldots, c_n) > 0$, $\tau \geq 0$, $T = (T_{ij})_{n \times n}$ and $S(x) = (s_1(x_1), s_2(x_2), \ldots, s_n(x_n))$. Under the assumptions that the response functions $s_i(x_i)$ are monotonically increasing, $s_i(0) = 0$ and each $s_i(x_i)$ satisfies the following condition:

$$0 \leq \frac{s_i(x_i)}{x_i} \leq \sigma_i^M \qquad \text{for all } x_i \in R,\ x_i \neq 0,$$

the author obtained the following result. That is, if there is a positive matrix P and positive diagonal matrix D such that

$$-(PC + CP) + PTD^2T^{\mathrm{T}}P + \Sigma^M D^{-2}\Sigma^M < 0, \tag{24}$$

here $\Sigma^M = \mathrm{diag}(\sigma_1^M, \sigma_2^M, \ldots, \sigma_n^M)$, then the equilibrium $x = x^*$ of system (23) is globally asymptotically stable. Clearly, the condition (24) is equivalent to the condition of Corollary 9. In addition, under this condition we directly obtain the global exponential stability of equilibrium.

Remark 2. In Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001), we see that the authors obtained a series of criteria of the matrix forms on the global asymptotic stability of equilibrium point for autonomous CNNs with constant delay. Comparing with those results given in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001), we find that the results obtained in this paper improve and extend those results in many aspects. Firstly, we see that the systems discussed in this paper are time-varying coefficients and time-varying delay. Secondly, we see that the global exponential stability of solutions is obtained in this paper. Thirdly, we see that in the literature (Arik, 2000; Arik & Tavsanoglu, 2000; Cao, 2001; Lu, 2001) authors assumed the existence of equilibrium point, however, in this paper we have not given this assumption. Fourthly, in our results the response functions may be unbounded, however, in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001), the response functions are assumed to be either bounded or the special case $f(x) = (1/2)(|x + 1| - |x - 1|)$. In addition, we also see that the main results given in this paper are more general than those given in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001) and Lu (2001), because in assumptions (H_3) and (H_4) we can choose many parameters, for example, matrices S, α, β and γ such that (H_3) and (H_4) hold.

Remark 3. The results obtained in this paper are also completely different from the results given in the literature (Cao, 1999; Cao & Wang, 2002; Chu, 2001; Huang et al., 2002; Liang & Cao, 2003; Mohamad, 2001; Mohamad & Gopalsamy, 2000; Peng, Qiao, & Xu, 2002; Zhou & Cao, 2002). In this paper the method of matrix analysis is used and the criteria of matrix forms on the global exponential stability are obtained. However, in the literature (Cao, 1999; Cao & Wang, 2002; Chu, 2001; Huang et al., 2002; Liang & Cao, 2003; Mohamad, 2001; Mohamad & Gopalsamy, 2000; Peng et al., 2002; Zhou & Cao, 2002), the technique of inequality analysis and Young inequality is used and the diagonal domination criteria of the global exponential stability are given. In particular, in Jiang et al. (2003), the diagonal domination criteria on the boundedness, global exponential stability and the existence of periodic solutions are obtained for CNNs with time-varying coefficients and time-varying delay.

Remark 4. The results given in this paper also can be improved to the following CNNs with time-varying coefficients and distributed delay

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -C(t)x(t) + A(t)f(x(t)) + B(t)g\left(\int_{-\tau}^{0} k(s)x(t+s)\mathrm{d}s\right) + I(t), \quad (25)$$

where $g(\int_{-\tau}^{0} k(s)x(t+s)\mathrm{d}s) = (g_1(\int_{-\tau_1}^{0} k_1(s)x_1(t+s)\mathrm{d}s), g_2(\int_{-\tau_2}^{0} k_2(s)x_2(t+s)\mathrm{d}s), \ldots, g_n(\int_{-\tau_n}^{0} k_n(s)x_n(t+s)\mathrm{d}s))$ with $\int_{-\tau_i}^{0} k_i(s)\mathrm{d}s = 1$ $(i = 1, 2, \ldots, n)$, and the following recurrent neural networks with time-varying coefficients and delays

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -C(t)h(x(t)) + A(t)f(x(t)) + B(t)g(x(t - \tau(t))) + I(t), \quad (26)$$

where $h(x) = (h_1(x_1), h_2(x_2), \ldots, h_n(x_n))$, as long as each $h_i(u)$ satisfies the condition $\inf_{u\in R}\{\mathrm{d}h_i(u)/\mathrm{d}u\} > 0$.

5. Two examples

We consider the following two-dimensional CNNs with time-varying coefficients and delays

$$\dot{x}(t) = -C(t)x(t) + A(t)f(x(t)) + B(t)g(x(t - \tau(t))) + I(t), \quad (27)$$

where $t \in R_+$, $x = (x_1, x_2)^{\mathrm{T}}$, $f(x) = (f_1(x_1), f_2(x_2))^{\mathrm{T}}$, $g(x(t-\tau(t))) = (g_1(x_1(t - \tau_1(t))),\ g_2(x_2(t - \tau_2(t))))$ and $I(t) = (I_1(t), I_2(t))^{\mathrm{T}}$.

Example 1. In system (27), we take

$$C(t) = \begin{pmatrix} 7 + \sin t & 0 \\ 0 & 9 - \cos t \end{pmatrix},$$

$$A(t) = \begin{pmatrix} -4 + \frac{1}{2}\sin t & 1 \\ 2 & 2 - \frac{1}{2}\cos t \end{pmatrix},$$

and

$$B(t) = \begin{pmatrix} 1 + \frac{1}{5}\sin t & 2 \\ 1 & 1 - \frac{1}{5}\cos t \end{pmatrix}.$$

Further, we let $f_i(u) = g_i(u) = h(u) = u - \arctan(u/2)$ $(i = 1, 2)$ and $\tau_1(t) = \tau_2(t) = 1 + (1/2)\sin t$. We see directly that $h(u)$ is unbounded on $u \in R$ and satisfies

$$0 \le \frac{h(u) - h(u^*)}{u - u^*} \le 1 \qquad \text{for all } u, u^* \in R.$$

Hence, $K = H = \mathrm{diag}(1, 1)$. Further, we see that $\tau_i(t)$ satisfies $\inf_{t\in R_+}\{1 - \dot{\tau}_i(t) \ge (1/2)$ $(i = 1, 2)$.

Choosing $S = \alpha = \beta = E$ and $\gamma = 0$, then we have

$$D_1(t, \eta) = 2C(t) - A(t)\eta - \eta A^{\mathrm{T}}(t) - B(t)B^{\mathrm{T}}(t) - H^2P(t),$$

where $t \in R_+$ and $0 \le \eta \le K$. By directly calculating, we have

$$A(t)\eta + \eta A^{\mathrm{T}}(t) = \begin{pmatrix} 2\eta_1\left(-4 + \frac{1}{2}\sin t\right) & 2\eta_2 + \eta_1 \\ 2\eta_2 + \eta_1 & 2\eta_2\left(2 - \frac{1}{2}\cos t\right) \end{pmatrix},$$

where $\eta = \mathrm{diag}(\eta_1, \eta_2)$

$$B(t)B^{\mathrm{T}}(t) = \begin{pmatrix} \left(1+\frac{1}{5}\sin t\right)^2 + 4 & 1+\frac{1}{5}\sin t + 2\left(1-\frac{1}{5}\cos t\right) \\ 1+\frac{1}{5}\sin t + 2\left(1-\frac{1}{5}\cos t\right) & \left(1-\frac{1}{5}\cos t\right)^2 + 1 \end{pmatrix}$$

and

$$H^2P(t) \le 2E.$$

Hence, we obtain

$$D_1(t, \eta) = \begin{pmatrix} d_{11}(t, \eta) & d_{12}(t, \eta) \\ d_{12}(t, \eta) & d_{22}(t, \eta) \end{pmatrix},$$

where

$$d_{11}(t, \eta) = 14 + 2\sin t - 2\eta_1\left(-4 + \frac{1}{2}\sin t\right) - 4 - \left(1 + \frac{1}{5}\sin t\right)^2 - 2,$$

$$d_{22}(t, \eta) = 18 - 2\cos t - 2\eta_2\left(2 - \frac{1}{2}\cos t\right) - 1 - \left(1 - \frac{1}{5}\cos t\right)^2 - 2$$

and

$$d_{12}(t, \eta) = 2\eta_2 + \eta_1 - \left(3 + \frac{1}{5}\sin t - \frac{2}{5}\cos t\right).$$

Since $d_{11}(t,\eta) > 3$, $d_{22}(t,\eta) > 6$ and $d_{12}^2(t,\eta) < 13$ for all $t \in R_+$ and $0 \le \eta \le K$, we obtain

$$\det\begin{pmatrix} d_{11}(t,\eta) & d_{12}(t,\eta) \\ d_{12}(t,\eta) & d_{22}(t,\eta) \end{pmatrix} = d_{11}(t,\eta)d_{22}(t,\eta) - d_{12}^2(t,\eta) > 5$$

for all $t \in R_+$ and $0 \le \eta \le K$. Hence, we can obtain that there exists a constant $a > 0$ such that

$$\lambda_{\min}(D_1(t,\eta)) \ge a \quad \text{for all } t \in R_+,\ 0 \le \eta \le K.$$

This shows that assumption (H_3) holds with $S=\alpha=\beta=E$ and $\gamma=0$. Therefore, from Corollary 1, we obtain that system (27) is globally exponentially stable.
However, for any $\gamma = \mathrm{diag}(\gamma_1,\gamma_2) \ge 0$ we have

$$\gamma A(t) + A^{\mathrm{T}}(t)\gamma = \begin{pmatrix} 2\gamma_1\left(-4 + \frac{1}{2}\sin t\right) & 2\gamma_2 + \gamma_1 \\ 2\gamma_2 + \gamma_1 & 2\gamma_2\left(2 - \frac{1}{2}\cos t\right) \end{pmatrix}.$$

Obviously, we see that $\gamma A(t) + A^{\mathrm{T}}(t)\gamma$ is not negative definite for all $t \in R_+$. This shows that assumption (H_4) is not true.

Example 2. In system (27), we take

$$C(t) = \begin{pmatrix} 21 - 4\sin t & 0 \\ 0 & 36.5 - 4\cos t \end{pmatrix},$$

$$A(t) = \begin{pmatrix} -2 + \frac{1}{2}\sin t & 0 \\ 1 & -2 + \frac{1}{2}\cos t \end{pmatrix},$$

and

$$B(t) = \begin{pmatrix} -4 + \sin t & 1 \\ 1 & -4 + \cos t \end{pmatrix}.$$

Further, we let $f_1(u) = f_2(u) = h(u) = 29u + \sin u$, $g_1(u) = g_2(u) = p(u) = u - \arctan(1/2)u$ and $\tau_1(t) = \tau_2(t) = 1 + (1/2)\sin t$. We see that $h(u)$ and $p(u)$ are unbounded on $u \in R_+$ and satisfy

$$0 \le \frac{h(u) - h(u^*)}{u - u^*} \le 30 \quad \text{and} \quad 0 \le \frac{p(u) - p(u^*)}{u - u^*} \le 1$$

for all $u, u^* \in R$. Hence, $K = \mathrm{diag}(30,30)$ and $H = \mathrm{diag}(1,1)$. In addition, we have $\inf_{t \in R_+}\{1 - \dot{\tau}_i(t)\} \ge (1/2)$ $(i = 1,2)$. Choosing $S = \alpha = \beta = \gamma = E$, we have

$$\begin{aligned} D_1(t,\eta) = {} & 2C(t) - (A(t) - C(t))\eta - \eta(A^{\mathrm{T}}(t) - C(t)) \\ & - B(t)B^{\mathrm{T}}(t) - \eta(A(t) + A^{\mathrm{T}}(t))\eta \\ & - \eta B(t)B^{\mathrm{T}}(t)\eta - 2H^2P(t) \end{aligned}$$

and

$$\begin{aligned} D_2(t,\eta) = {} & 2C(t) - (A(t) - C(t))\eta - \eta(A^{\mathrm{T}}(t) - C(t)) \\ & - B(t)B^{\mathrm{T}}(t) - H^2(1 - \bar{\beta}^*)P(t), \end{aligned}$$

where $\eta = \mathrm{diag}(\eta_1,\eta_2)$ and $0 \le \eta \le K$. Since

$$(A(t) - C(t))\eta + \eta(A^{\mathrm{T}}(t) - C(t)) = \begin{pmatrix} 2\eta_1(-23 + 4.5\sin t) & \eta_1 \\ \eta_1 & 2\eta_2(-38.5 + 4.5\cos t) \end{pmatrix},$$

we obtain

$$\begin{aligned} (A(t)-C(t))\eta+\eta(A^{\mathrm{T}}(t)-C(t)) & \le \begin{pmatrix} 2\eta_1(-23+4.5\sin t)+\eta_1 & 0 \\ 0 & 2\eta_2(-38.5+4.5\cos t)+\eta_1 \end{pmatrix} \\ & \le \begin{pmatrix} 0 & 0 \\ 0 & \eta_1 \end{pmatrix} \le \begin{pmatrix} 0 & 0 \\ 0 & 30 \end{pmatrix} \end{aligned} \tag{28}$$

and

$$\begin{aligned} (A(t)-C(t))\eta+\eta(A^{\mathrm{T}}(t)-C(t)) & \ge \begin{pmatrix} 2\eta_1(-23+4.5\sin t)-\eta_1 & 0 \\ 0 & 2\eta_2(-38.5+4.5\cos t)-\eta_1 \end{pmatrix} \\ & \ge \begin{pmatrix} -54\times 30 & 0 \\ 0 & -87\times 30 \end{pmatrix}. \end{aligned} \tag{29}$$

Further, we obtain

$$B(t)B^{\mathrm{T}}(t) = \begin{pmatrix} 15-8\sin t+\sin^2 t & -8+\sin t+\cos t \\ -8+\sin t+\cos t & 15-8\cos t+\cos^2 t \end{pmatrix}, \tag{30}$$

$$A(t)+A^{\mathrm{T}}(t) = \begin{pmatrix} -4+\sin t & 1 \\ 1 & -4+\cos t \end{pmatrix}, \tag{31}$$

$$\det(A(t)+A^{\mathrm{T}}(t)) = 15-4\sin t-4\cos t+\sin t\cos t,$$

$$(A(t)+A^{\mathrm{T}}(t))^{-1} = \frac{1}{\det(A(t)+A^{\mathrm{T}}(t))}\begin{pmatrix} -4+\cos t & 1 \\ 1 & -4+\sin t \end{pmatrix}$$

and

$$B^{\mathrm{T}}(t)(A(t)+A^{\mathrm{T}}(t))^{-1}B(t)=\begin{pmatrix}-4+\sin t & 1\\ 1 & -4+\cos t\end{pmatrix}.$$

By directly calculating, we can obtain

$$\begin{aligned}\bar{\beta}^{*}&=\inf_{t\in R}\{\lambda_{\min}(B^{\mathrm{T}}(t)(A(t)+A^{\mathrm{T}}(t))^{-1}B(t))\}\\ &\geq\inf_{t\in R_{+}}\{-8+\sin t+\cos t\}\\ &=(-8+\cos t+\sin t)|_{t=\pi+(\pi/4)}\\ &\geq-9.5.\end{aligned}$$

Hence

$$H^{2}(1-\bar{\beta}^{*})P(t)\geq 21E \text{ for all } t\in R_{+}. \quad (32)$$

From (28), (30) and (32) we have

$$\begin{aligned}D_{2}(t,\eta)&\geq\begin{pmatrix}42-8\sin t & 0\\ 0 & 73-8\cos t\end{pmatrix}-\begin{pmatrix}0 & 0\\ 0 & 30\end{pmatrix}-\begin{pmatrix}21 & 0\\ 0 & 21\end{pmatrix}\\ &\quad-\begin{pmatrix}15-8\sin t+\sin^{2}t & -8+\sin t+\cos t\\ -8+\cos t+\sin t & 15-8\cos t+\cos^{2}t\end{pmatrix}\\ &\geq\begin{pmatrix}6-\sin^{2}t & 4-\sin t-\cos t\\ 4-\sin t-\cos t & 7-\cos^{2}t\end{pmatrix}.\end{aligned}$$

Since $6-\sin^{2}t\geq 5$, $7-\cos^{2}t\geq 6$ and

$$\begin{aligned}\det(D_{2}(t,\eta))&\geq(6-\sin^{2}t)(7-\cos^{2}t)-(4-\sin t-\cos^{2}t)^{2}\\ &>30-30=0,\end{aligned}$$

we can obtain that there exists a constant $a>0$ such that

$$\lambda_{\min}(D_{2}(t,\eta))\geq a \text{ for all } t\in R_{+} \text{ and } 0\leq\eta\leq K.$$

This shows that assumption (H_4) holds with $S=\alpha=\beta=\gamma=E$. Therefore, from Corollary 2, we obtain that system (27) is globally exponentially stable.

However, from (29)–(31), we further obtain

$$\begin{aligned}D_{1}(t,\eta)&=2C(t)-(A(t)-C(t))\eta-\eta(A^{\mathrm{T}}(t)-C(t))\\ &\quad-B(t)B^{\mathrm{T}}(t)-\eta(A(t)+A^{\mathrm{T}}(t))\eta-2H^{2}P(t)\\ &\quad-\eta B(t)B^{\mathrm{T}}(t)\eta\\ &\leq\begin{pmatrix}42-8\sin t & 0\\ 0 & 73-8\cos t\end{pmatrix}\\ &\quad-\begin{pmatrix}-54\times 30 & 0\\ 0 & -87\times 30\end{pmatrix}\\ &\quad-\eta\begin{pmatrix}11-7\sin t+\sin^{2}t & -7+\sin t+\cos t\\ -7+\sin t+\cos t & 11-7\cos t+\cos^{2}t\end{pmatrix}\eta.\end{aligned}$$

In particular, when we choose $\eta=K$, then we have

$$D_{1}(t,\eta)\leq\begin{pmatrix}-8238+6292\sin t-30^{2}\sin^{2}t & -30^{2}(-7+\sin t+\cos t)\\ -30^{2}(-7+\sin t+\cos t) & -7817-8\sin t-30^{2}(-7\cos t+\cos^{2}t)\end{pmatrix}.$$

Since $-8238+6292\sin t-30^{2}\sin^{2}t<0$, we obtain that $D_{1}(t,\eta)$ is not positive definite. From this, we see that assumption (H_3) is not true with $S=\alpha=\beta=\gamma=E$.

6. Conclusions

In this paper, we study a class of CNNs with time-varying coefficients and delays and obtain new sufficient conditions of the global exponential stability of solutions by utilizing the Liapunov function method and the technique of matrix analysis. We introduce two new important assumptions (H_3) and (H_4) to ensure the global exponential stability of the systems. From Examples 1 and 2, we see that these two assumptions are completely different from each other. From Corollaries 1–10 of main Theorem 1, we see that the results obtained in this paper conclude many special cases. In particular, some special case of system (1), like autonomous cellular networks with delay, periodic cellular networks with delay, systems (21) and (22), etc. are concluded. Comparing with the results given in Arik (2000), Arik and Tavsanoglu (2000), Cao (2001), Lu (2001) and Jiang and Teng (2003), we see the results obtained in this paper improve and extend those results in many aspects. Comparing with the results given in Cao (1999, 2001), Cao and Wang (2002), Chu (2001), Huang et al. (2002), Jiang et al. (2003), Liang and Cao (2003), Lu (2001), Mohamad (2001), Mohamad and Gopalsamy (2000), Peng et al., (2002) and Zhou and Cao (2002), we see that the results obtained in this paper are also completely different from those results. In addition, the results given in this paper can also be improved to the CNNs with time-varying coefficients and distributed delay and the recurrent neural networks with time-varying coefficients and delays, like systems (25) and (26). Therefore, we see that the results given in this paper are new, more general and useful in the theory and applications of the stability of CNNs with delay.

Acknowledgements

The authors would like to thank the referees for their helpful suggestions and comments. This paper was supported by The National Natural Science Foundation of P.R. China (10361004) and The Natural Science Foundation of Xinjiang University.

References

Arik, S. (2000). Stability analysis of delayed neural networks. *IEEE Transactions on Circuits Systems I*, *47*, 1089–1092.

Arik, S., & Tavsanoglu, V. (2000). On the global asymptotic stability of delay cellular neural networks. *IEEE Transactions on Circuits Systems I*, *47*, 571–574.

Burton, T. A. (1985). *Stability and periodic solutions of ordinary and functional differential equations*. New York: Academic Press.

Cao, J. (1999). Periodic solutions and exponential stability in delayed cellular neural networks. *Physical Review E*, *60*(3), 3244–3248.

Cao, J. (2000a). Periodic oscillation and exponential stability of delayed CNNs. *Physics Letters A*, *270*, 157–163.

Cao, J. (2000b). Global exponential stability and periodic solutions of delayed cellular neural networks. *Journal Comp. Syst. Sci.*, *60*, 38–46.

Cao, J. (2001). Global stability conditions for delayed CNNs. *IEEE Transactions on Circuits Systems I*, *48*, 1330–1333.

Cao, J., & Wang, L. (2002). Exponential stability and periodic oscillator solution in BAM networks with delays. *IEEE Transactions on Neural Networks*, *13*, 457–463.

Cao, J., & Wang, J. (2003). Global asymptotic stability of a general class of recurrent neural networks with time-varying delays. *IEEE Transactions on Circuits Systems I*, *50*, 34–44.

Chen, A., & Cao, J. (2003). Existence and attractitivity of almost periodic solutions for cellular neural networks with distributed delays and variable coefficients. *Applied Mathematics and Computation*, *134*, 125–140.

Chen, A., Huang, L., & Cao, J. (2003). Existence and stability of almost periodic solution for BAM neural networks with delays. *Applied Mathematics and Computation*, *137*, 177–193.

Chu, T. (2001). An exponential convergence estimate for analog neural networks with delay. *Physics Letters A*, *283*, 113–118.

Dong, Q., Matsui, K., & Huang, X. (2002). Existence and stability of periodic solutions for Hopfield neural network equations with periodic input. *Nonlinear Analysis*, *49*, 471–479.

Huang, H., Cao, J., & Wang, J. (2002). Global exponential stability and periodic solutions of recurrent neural networks with delays. *Physics Letters A*, *298*, 393–404.

Jiang, H., Li, Z., & Teng, Z. (2003). Boundedness and stability for nonautonomous cellular neural networks with delay. *Physics Letters A*, *306*, 313–325.

Jiang, H., & Teng, Z. (2003). Boundedness and stability for nonautonomous cellular neural networks with delay. *Applied Mathematical and Mechanics*, in press.

Liang, J., & Cao, J. (2003). Boundedness and stability for recurrent neural networks with variable coefficients and time-varying delays. *Physics Letters A*, *318*, 53–64.

Lu, H. (2001). Stability criteria for delayed neural networks. *Physical Review E*, *64*, 1–13. 051901.

Mohamad, S. (2001). Global exponential stability in continuous-time and discrete-time delayed bidirectional neural networks. *Physica D*, *159*, 233–251.

Mohamad, S., & Gopalsamy, K. (2000). Dynamics of a class of discrete-time neural networks and their continuous-time counterparts. *Mathematics and Computers in Simulation*, *53*, 1–39.

Peng, J., Qiao, H., & Xu, Z. (2002). A new approach to stability of neural networks with time-varying delays. *Neural Networks*, *15*, 95–103.

Shayer, L. P., & Campbell, S. A. (2000). Stability, bifurcation, and multistability in a system of two coupled neural with multiple time delays. *SIAM Journal of Applied Mathematics*, *61*, 673–700.

Takahashi, N. (2000). A new sufficient condition for complete stability of cellular neural networks with delay. *IEEE Transactions on Circuits Systems I*, *47*(6), 791–799.

Van Den Driessche, P., & Zou, X. (1998). Global attractivity in delayed Hopfield neural network models. *SIAM Journal of Applied Mathematics*, *58*, 1878–1890.

Wei, J., & Ruan, S. (1999). Stability and bifurcation in a neural network model with two delays. *Physica D*, *130*, 255–272.

Zhang, J., & Jin, X. (2000). Global stability analysis in delay Hopfield neural network models. *Neural Networks*, *13*, 745–753.

Zhang, Y., Pheng, A. H., & Kwong, S. L. (2001). Convergence analysis of cellular neural networks with unbounded delay. *IEEE Transactions on Circuits Systems I*, *48*(6), 680–687.

Zhou, D., & Cao, J. (2002). Global exponential stability conditions for cellular neural networks with time-varying delays. *Applied Mathematics and Computation*, *131*, 487–496.