Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis: Real World Applications 9 (2008) 1837 – 1850
Nonlinear Analysis
Real World Applications
www.elsevier.com/locate/na

Algebro-geometric solutions for some (2 + 1)-dimensional discrete systems

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Received 15 April 2006; accepted 25 May 2007

Abstract

Starting from a discrete spectral problem, a discrete soliton hierarchy is derived. Some $(2+1)$-dimensional discrete systems related to the hierarchy are proposed. The elliptic coordinates are introduced and the equations in the discrete soliton hierarchy are decomposed into solvable ordinary differential equations. The straightening out of the continuous flow and the discrete flow are exactly given through the Abel–Jacobi coordinates. As an application, explicit algebro-geometric solutions for the $(2+1)$-dimensional discrete systems are obtained.

Keywords: $(2+1)$-dimensional discrete systems; Algebro-geometric solutions; Elliptic coordinates; Straightening out; Riemann–Jacobi inversion technique

1. Introduction

There have been several systematic approaches to obtain explicit solutions of the soliton equations, such as the inverse scattering transformation, the Bäcklund transformation, the algebro-geometric method, the polar expansion method and so on [2,3,11–14]. Some interesting explicit solutions have been found, for example, pure-soliton solutions, quasi-periodic solutions, polar expansion solutions, etc. The algebro-geometric method was first developed by Matveev, Its, Novikov et al. as analog of inverse scattering theory [6,10,5]. This method allows us to find an important class of exact solutions to the soliton equations. As a degenerated case of this solutions, the multisoliton solutions and elliptic functions may be obtained [12]. Recently, based on the nonlinearization technique of Lax pairs, algebro-geometric solutions for $(1+1)$-dimensional and $(2+1)$-dimensional soliton equations have been obtained by Cao and Geng [4,7,8].

In recent years, the study of nonlinear integrable lattice equations has become the focus of common concern in the theory of integrable systems. Many nonlinear integrable lattice equations have been proposed and discussed, for example, the Ablowitz–Ladik lattice [1], the Toda lattice [18], and so on. In this paper, we will consider a discrete spectral problem

$$E\psi(n) = U_n\psi(n) = \begin{pmatrix} \lambda^{-1}(1+p_nq_n) & p_n \\ q_n & \lambda \end{pmatrix}\psi(n), \tag{1.1}$$

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1468-1218/$ - see front matter © 2007 Published by Elsevier Ltd.
doi:10.1016/j.nonrwa.2007.05.012

where E is the shift operator, $Ef(n) = f(n+1)$. In Section 2, we will derive a hierarchy of lattice soliton equations from (1.1). We will also propose Some $(2+1)$-dimensional differential-difference equations related to the discrete soliton hierarchy. In Section 3, based on finite-order expansion of the Lax matrix, we introduce elliptic coordinates. The spectral solutions of the differential-difference equations are reduced to solving ordinary differential equation. In Sections 4 and 5, the Abel–Jacobi coordinates are introduced, by which the straightening out of the continuous flow and the discrete flow are studied in detail. In Section 6, the Riemann–Jacobi inversion is discussed, from which the algebro-geometric solutions for the $(2+1)$-dimensional differential-difference equations are obtained by using the Riemann theta functions.

2. The discrete soliton hierarchy

In order to derive the hierarchy related to (1.1), we first introduce Lenard's gradient sequence S_j, $0 \leqslant j \in Z$, by the recursion equation

$$K_n S_j(n) = J_n S_{j+1}(n), \quad J_n S_0(n) = 0, \quad j \geqslant 0 \tag{2.1}$$

with two operators

$$K_n = \begin{pmatrix} (1+p_nq_n)E & 0 & 0 \\ 0 & 1+p_nq_n & 0 \\ -p_n & q_nE & (1+p_nq_n)(E-1) \end{pmatrix},$$

$$J_n = \begin{pmatrix} 1 & 0 & q_n(E+1) \\ 0 & E & p_n(E+1) \\ -p_n & q_nE & (1+p_nq_n)(E-1) \end{pmatrix}.$$

Equation $J_n S_0(n) = 0$ has a special solution

$$S_0(n) = \left(q_n, p_{n-1}, -\tfrac{1}{2}\right)^{\mathrm{T}} \tag{2.2}$$

and we have

$$ker\, J_n = \{cS_0(n) \mid \forall c\ (constant)\}.$$

Then $S_j(n)$ is uniquely determined by the recursion relation (2.1) up to a term $cS_0(n)$, which is always assumed to be zero. The first few numbers are

$$S_1(n) = \begin{pmatrix} q_{n+1} - p_{n-1}q_n^2 \\ p_{n-2} - p_{n-1}^2 q_n \\ p_{n-1}q_n \end{pmatrix},$$

$$S_2(n) = \begin{pmatrix} q_{n+2} - p_nq_{n+1}^2 - 2p_{n-1}q_nq_{n+1} - p_{n-2}q_n^2 + p_{n-1}^2q_n^3 \\ p_{n-3} - p_{n-2}^2q_{n-1} - 2p_{n-2}p_{n-1}q_n - p_{n-1}^2q_{n+1} + p_{n-1}^3q_n^2 \\ p_{n-1}q_{n+1} + p_{n-2}q_n - p_{n-1}^2q_n^2 \end{pmatrix}.$$

Assume that the time dependence of $\psi(n)$ for the spectral problem (1.1) is

$$\psi(n)_{t_m} = V_n^{(m)}\psi(n), \quad V_n^{(m)} = \begin{pmatrix} A_n^{(m)} & B_n^{(m)} \\ C_n^{(m)} & -A_n^{(m)} \end{pmatrix} \tag{2.3}$$

with

$$A_n^{(m)} = \sum_{j=0}^{m} S_j^{(3)}(n)\lambda^{2(m-j)+2} + S_{m+1}^{(3)}(n),$$
$$B_n^{(m)} = \sum_{j=0}^{m} S_j^{(2)}(n)\lambda^{2(m-j)+1},$$
$$C_n^{(m)} = \sum_{j=0}^{m} S_j^{(1)}(n)\lambda^{2(m-j)+1}.$$

Then the compatibility condition between (1.1) and (2.3) yields a discrete zero-curvature equation $U_{n,t_m} + U_n V_n^{(m)} - V_{n+1}^{(m)} U_n = 0$, which is equivalent to the hierarchy of lattice soliton equations

$$p_{n,t_m} = -S_{m+1}^{(2)}(n+1), \quad q_{n,t_m} = S_{m+1}^{(1)}(n). \tag{2.4}$$

The first and second discrete systems ($m = 0, 1$) in the hierarchy (2.4) are

$$\begin{aligned} p_{n,t_0} &= -p_{n-1} + p_n^2 q_{n+1}, \\ q_{n,t_0} &= q_{n+1} - p_{n-1} q_n^2 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} p_{n,t_1} &= -p_{n-2} + p_{n-1}^2 q_n + 2p_{n-1}p_n q_{n+1} + p_n^2 q_{n+2} - p_n^3 q_{n+1}^2, \\ q_{n,t_1} &= q_{n+2} - p_n q_{n+1}^2 - 2p_{n-1} q_n q_{n+1} - p_{n-2} q_n^2 + p_{n-1}^2 q_n^3. \end{aligned} \tag{2.6}$$

Substituting u_n, v_n for q_n, p_{n-1} in (2.5) and (2.6), we can obtain the differential-difference equations proposed in [15]. If $p(n, t_0, t_1)$ and $q(n, t_0, t_1)$ are the compatible solutions of (2.5) and (2.6), then following the idea of [19], we can get that $p(n, t_0, t_1)$ and $q(n, t_0, t_1)$ are also the solutions of the following $(2+1)$-dimensional differential-difference equations

$$\begin{aligned} p_{n,t_1} &= p_{n-1,t_0} + p_n^2 q_{n+2} + 2p_{n-1}p_n q_{n+1} - p_n^3 q_{n+1}^2, \\ q_{n,t_1} &= q_{n+1,t_0} - p_{n-2} q_n^2 - 2p_{n-1} q_n q_{n+1} + p_{n-1}^2 q_n^3, \end{aligned} \tag{2.7}$$

$$\begin{aligned} p_{n,t_1} &= -p_{n-2} + p_{n-1}^2 q_n + 2p_n^3 q_{n+1}^2 + p_n^2 q_{n+1,t_0} - 2p_n p_{n,t_0} q_{n+1}, \\ q_{n,t_1} &= q_{n+2} - p_n q_{n+1}^2 - 2p_{n-1}^2 q_n^3 - p_{n-1,t_0} q_n^2 + 2p_{n-1} q_n q_{n,t_0}, \end{aligned} \tag{2.8}$$

$$r_{n,t_1} = -r_n r_{n,t_0} + r_{n-1}\Delta^{-1} r_{n,t_0} + r_n(\Delta^*)^{-1} r_{n,t_0} - (\Delta^*)^{-1} r_{n,t_0t_0}, \tag{2.9}$$

where, $r_n = p_{n-1}q_n$ and $\Delta = E - 1$, $\Delta^* = E^{-1} - 1$. We can also found that (p_n, q_{n+2}) satisfies the coupled NLS equations

$$\begin{aligned} &p_{n,t_1} + p_{n,t_0t_0} - p_n^2 q_{n+2} = 0, \\ &q_{n+2,t_1} - q_{n+2,t_0t_0} + p_n q_{n+2}^2 = 0. \end{aligned} \tag{2.10}$$

In the following sections, we will try to construct the algebro-geometric solutions for the $(2+1)$-dimensional differential-difference equations (2.7)–(2.9).

3. Decomposition of the discrete systems

In this section, we will decompose the discrete systems (2.5) and (2.6) into solvable ordinary differential equations. Assume that (1.1) and (2.3) have two basic solutions $\psi(n) = (\psi^{(1)}(n), \psi^{(2)}(n))^{\mathrm{T}}$ and $\phi(n) = (\phi^{(1)}(n), \phi^{(2)}(n))^{\mathrm{T}}$. We define a Lax matrix W_n of three functions $f(n), g(n), h(n)$ by

$$W_n = \frac{1}{2}(\phi(n)\psi(n)^{\mathrm{T}} + \psi(n)\phi(n)^{\mathrm{T}})\sigma = \begin{pmatrix} f(n) & g(n) \\ h(n) & -f(n) \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

It is easy to verify by (1.1) and (2.3) that

$$W_{n+1}U_n - U_nW_n = 0, \quad W_{n,t_m} = [V_n^{(m)}, W_n], \tag{3.2}$$

which imply that the function $\det W_n$ is a constant independent of n and t_m. In fact, we obtain by the first expression of (3.2) that $W_{n+1} = U_nW_nU_n^{-1}$. Then $\det W_{n+1} = \det W_n$, which means that $\det W_n$ is independent of n. In a way similar to the continuous case, a direct calculation shows that $(\det W_n)_{t_m} = 0$. Eq. (3.2) can be written as

$$\begin{aligned}
&\lambda^{-1}(1+p_nq_n)(f(n+1)-f(n)) + q_ng(n+1) - p_nh(n) = 0,\\
&p_n(f(n)+f(n+1)) + \lambda g(n+1) - \lambda^{-1}(1+p_nq_n)g(n) = 0,\\
&q_n(f(n)+f(n+1)) + \lambda h(n) - \lambda^{-1}(1+p_nq_n)h(n+1) = 0,\\
&\lambda(f(n+1)-f(n)) + q_ng(n) - p_nh(n+1) = 0
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
f(n)_{t_m} &= B_n^{(m)}h(n) - C_n^{(m)}g(n),\\
g(n)_{t_m} &= 2A_n^{(m)}g(n) - 2B_n^{(m)}f(n),\\
h(n)_{t_m} &= 2C_n^{(m)}f(n) - 2A_n^{(m)}h(n).
\end{aligned} \tag{3.4}$$

It is easy to see that (3.3) is equivalent to its first three equations. Suppose that the functions $f(n)$, $g(n)$ and $h(n)$ are finite-order polynomials in λ:

$$\begin{aligned}
&f(n) = \sum_{j=0}^{N+1} f_j(n)\lambda^{2(N+1-j)+2}, \quad g(n) = \sum_{j=0}^{N+1} g_j(n)\lambda^{2(N+1-j)+1},\\
&h(n) = \sum_{j=0}^{N+1} h_j(n)\lambda^{2(N+1-j)+1}.
\end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.3) and comparing the coefficients of the same power of λ yields

$$K_nG_j(n) = J_nG_{j+1}(n), \quad J_nG_0(n) = 0, \quad K_nG_{N+1}(n) = 0 \tag{3.6}$$

with $G_j(n) = (h_j(n), g_j(n), f_j(n))^{\mathrm{T}}$. It is easy to see that the equation $J_nG_0(n) = 0$ has the general solution

$$G_0(n) = \alpha_0S_0(n), \tag{3.7}$$

where α_0 is a constant. Therefore, if we take (3.7) as starting point, then $G_j(n)$ can be recursively determined by the first two expressions of (3.6). Acting with $(J_n^{-1}K_n)^k$ upon (3.7), we obtain

$$G_k(n) = \alpha_0S_k(n) + \alpha_1S_{k-1}(n) + \cdots + \alpha_kS_0(n), \tag{3.8}$$

where $\alpha_0, \ldots, \alpha_k$ are constants. Substituting (3.8) into the third expression of (3.6), we arrive at a discrete stationary equation

$$\alpha_0K_nS_{N+1}(n) + \cdots + \alpha_{N+1}K_nS_0(n) = 0. \tag{3.9}$$

This means that (p_n, q_n) is finite-band solution. In another way, we can get from the equation $K_nG_{N+1}(n) = 0$ that $G_{N+1}(n)$ should possess the following form:

$$G_{N+1}(n) = \beta_0\widehat{S}_0(n) = \beta_0\left(0, 0, \tfrac{1}{2}\right)^{\mathrm{T}}, \tag{3.10}$$

where β_0 is a constant. If we take (3.10) as a starting point and with the help of (3.6), we can get that $G_k(n)$ should also possess the following forms:

$$G_{N+1-k}(n) = \beta_0\widehat{S}_{-k}(n) + \beta_1\widehat{S}_{-k+1}(n) + \cdots + \beta_k\widehat{S}_0(n), \tag{3.11}$$

where $\widehat{S}_j(n)$, $0 \geqslant j \in Z$ is Lenard's gradient sequence determined by the following equation:

$$K_n \widehat{S}_{j-1}(n) = J_n \widehat{S}_j(n), \quad K_n \widehat{S}_0(n) = 0. \tag{3.12}$$

Without any loss of generality we can set $\alpha_0 = 1$. From (3.7), (3.8), (3.10) and (3.11), we have

$$\begin{aligned}
&h_0(n) = q_n, g_0(n) = p_{n-1}, \quad f_0(n) = -\tfrac{1}{2},\\
&h_1(n) = q_{n+1} - p_{n-1}q_n^2 + \alpha_1 q_n, \quad g_1(n) = p_{n-2} - p_{n-1}^2 q_n + \alpha_1 p_{n-1},\\
&f_1(n) = p_{n-1}q_n - \tfrac{1}{2}\alpha_1,\\
&h_2(n) = q_{n+2} - p_n q_{n+1}^2 - 2p_{n-1}q_n q_{n+1} - p_{n-2}q_n^2 + p_{n-1}^2 q_n^3\\
&\qquad + \alpha_1(q_{n+1} - p_{n-1}q_n^2) + \alpha_2 q_n,\\
&g_2(n) = p_{n-3} - p_{n-2}^2 q_{n-1} - 2p_{n-2}p_{n-1}q_n - p_{n-1}^2 q_{n+1} + p_{n-1}^3 q_n^2\\
&\qquad + \alpha_1(p_{n-2} - p_{n-1}^2 q_n) + \alpha_2 p_{n-1},\\
&f_2(n) = p_{n-1}q_{n+1} + p_{n-2}q_n - p_{n-1}^2 q_n^2 + \alpha_1 p_{n-1}q_n - \frac{\alpha_2}{2},\\
&h_{N+1}(n) = 0, \quad g_{N+1}(n) = 0, \quad f_{N+1}(n) = \tfrac{1}{2}\beta_0,\\
&h_N(n) = \frac{\beta_0 q_{n-1}}{1 + p_{n-1}q_{n-1}}, \quad g_N(n) = \frac{\beta_0 p_n}{1 + p_n q_n},\\
&f_N(n) = -\frac{\beta_0 p_n q_{n-1}}{(1 + p_n q_n)(1 + p_{n-1}q_{n-1})} + \frac{\beta_1}{2}.
\end{aligned} \tag{3.13}$$

We use $g(n)$ and $h(n)$ as polynomials of λ to define the elliptic coordinates $\{\mu_j(n)\}$ and $\{v_j(n)\}$:

$$g(n) = \lambda^3 p_{n-1} \prod_{j=1}^{N} (\lambda^2 - \mu_j(n)), \quad h(n) = \lambda^3 q_n \prod_{j=1}^{N} (\lambda^2 - v_j(n)). \tag{3.14}$$

By comparing coefficients of the same power of λ, we get

$$g_1(n) = -p_{n-1} \sum_{j=1}^{N} \mu_j(n), \quad h_1(n) = -q_n \sum_{j=1}^{N} v_j(n), \tag{3.15}$$

$$g_2(n) = p_{n-1} \sum_{i<j} \mu_i(n)\mu_j(n), \quad h_2(n) = q_n \sum_{i<j} v_i(n)v_j(n). \tag{3.16}$$

Eq. (3.15) can be written as

$$\frac{p_{n-2}}{p_{n-1}} - p_{n-1}q_n + \alpha_1 = -\sum_{j=1}^{N} \mu_j(n),$$

$$\frac{q_{n+1}}{q_n} - p_{n-1}q_n + \alpha_1 = -\sum_{j=1}^{N} v_j(n). \tag{3.17}$$

Resorting to (2.5), we arrive at

$$\partial_{t_0} \ln p_n = \sum_{j=1}^{N} \mu_j(n+1) + \alpha_1, \quad \partial_{t_0} \ln q_n = -\sum_{j=1}^{N} v_j(n) - \alpha_1. \tag{3.18}$$

Similarly, we can get

$$\partial_{t_1} \ln p_n = \frac{1}{2}\sum_{j=1}^{N}\mu_j^2(n+1) - \frac{1}{2}\left(\sum_{j=1}^{N}\mu_j(n+1)\right)^2 - \alpha_1\sum_{j=1}^{N}\mu_j(n+1) - \alpha_1^2 + \alpha_2,$$

$$\partial_{t_1} \ln q_n = -\frac{1}{2}\sum_{j=1}^{N}v_j^2(n) + \frac{1}{2}\left(\sum_{j=1}^{N}v_j(n)\right)^2 + \alpha_1\sum_{j=1}^{N}v_j(n) + \alpha_1^2 - \alpha_2. \tag{3.19}$$

Consider the function $\det W_n$, which is a $(2N+4)$th-order polynomial in $\zeta=\lambda^2$ with constant coefficients of the n-flow and t_m-flow:

$$-\det W_n = f(n)^2 + g(n)h(n) = \frac{1}{4}\zeta^2\prod_{j=1}^{2N+2}(\zeta-\zeta_j) = \frac{1}{4}\zeta^2 R(\zeta). \tag{3.20}$$

Substituting (3.5) into (3.20) and comparing the coefficients of ζ, we obtain

$$\alpha_1 = -\frac{1}{2}\sum_{j=1}^{2N+2}\zeta_j, \quad \alpha_2 = \frac{1}{2}\sum_{i<j}\zeta_i\zeta_j - \frac{1}{8}\left(\sum_{j=1}^{2N+2}\zeta_j\right)^2, \quad \beta_0^2 = \prod_{j=1}^{2N+2}\zeta_j. \tag{3.21}$$

Using (3.20), (3.14) and (4.4), we obtain

$$f(n)|_{\zeta=\mu_k(n)} = -\tfrac{1}{2}\mu_k(n)\sqrt{R(\mu_k(n))}, \quad f(n)|_{\zeta=v_k(n)} = -\tfrac{1}{2}v_k(n)\sqrt{R(v_k(n))}, \tag{3.22}$$

$$\frac{\partial_{t_0}\mu_k(n)}{\sqrt{R(\mu_k(n))}} = \frac{-1}{\prod_{j\neq k}(\mu_k(n)-\mu_j(n))},$$
$$\frac{\partial_{t_0}v_k(n)}{\sqrt{R(v_k(n))}} = \frac{1}{\prod_{j\neq k}(v_k(n)-v_j(n))}. \tag{3.23}$$

Similarly, we have

$$\frac{\partial_{t_1}\mu_k(n)}{\sqrt{R(\mu_k(n))}} = -\frac{\mu_k(n)-\sum_{j=1}^{N}\mu_j(n)-\alpha_1}{\prod_{j\neq k}(\mu_k(n)-\mu_j(n))},$$
$$\frac{\partial_{t_1}v_k(n)}{\sqrt{R(v_k(n))}} = \frac{v_k(n)-\sum_{j=1}^{N}v_j(n)-\alpha_1}{\prod_{j\neq k}(v_k(n)-v_j(n))}. \tag{3.24}$$

Therefore, if $\zeta_1,\ldots,\zeta_{2N+2}$ are $2N+2$ distinct parameters, and $\mu_k(n)$, $v_k(n)$ are compatible solutions of differential equations (3.23) and 3.24. Then (p_n, q_n) determined by (3.18) and (3.19) solves the $(2+1)$-dimensional differential-difference equations (2.7) and (2.8), $r_n = p_{n-1}q_n$ solves the $(2+1)$-dimensional differential-difference equations (2.9).

4. Straightening out of the continuous flow

In order to obtain the algebro-geometric solutions of systems (2.5) and (2.6), we first introduce the Riemann surfaces Γ of the hyperelliptic curve $\xi^2 = R(\zeta) = \prod_{j=1}^{2N+2}(\zeta-\zeta_j)$, of genus N. On Γ there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. Equip Γ with the canonical basis of cycles: $a_1,\ldots,a_N; b_1,\ldots,b_N$, and the holomorphic differentials

$$\widetilde{\omega}_l = \frac{\zeta^{l-1}d\zeta}{\sqrt{R(\zeta)}}, \quad 1\leqslant l\leqslant N.$$

Then the period matrices A and B defined by

$$A_{ij} = \int_{a_j} \widetilde{\omega}_i, \quad B_{ij} = \int_{b_j} \widetilde{\omega}_i$$

are invertible [9,16]. Let $C = A^{-1}$, $\tau = A^{-1}B$. If we normalize $\widetilde{\omega}_l$ into the new basis ω_j

$$\omega_j = \sum_{l=1}^{N} C_{jl}\widetilde{\omega}_l, \quad 1 \leqslant j \leqslant N,$$

then we have

$$\int_{a_i} \omega_j = \delta_{ji}, \quad \int_{b_i} \omega_j = \tau_{ji}. \tag{4.1}$$

Now we introduce the Able map $\mathscr{A}(P)$

$$\mathscr{A}(P) = \int_{P_0}^{P} \omega. \tag{4.2}$$

Then the Able–Jacobi coordinates are defined as

$$\rho^{(1)}(n) = \mathscr{A}\left(\sum_{k=1}^{N} P(\mu_k(n))\right) = \sum_{k=1}^{N} \int_{P_0}^{P(\mu_k(n))} \omega, \tag{4.3}$$

$$\rho^{(2)}(n) = \mathscr{A}\left(\sum_{k=1}^{N} P(v_k(n))\right) = \sum_{k=1}^{N} \int_{P_0}^{P(v_k(n))} \omega, \tag{4.4}$$

where $P(\mu_k(n)) = (\lambda = \mu_k(n), \xi = \sqrt{R(\mu_k(n))})$, $P(v_k(n)) = (\lambda = v_k(n), \xi = \sqrt{R(v_k(n))}) \in \Gamma$, and P_0 is a chosen base point on Γ. The components of the Able–Jacobi coordinates in (4.3) and (4.4) are

$$\rho_j^{(1)}(n, t_0, t_1) = \sum_{k=1}^{N} \int_{P_0}^{P(\mu_k(n,t_0,t_1))} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \int_{\zeta(p_0)}^{\mu_k(n)} \frac{\zeta^{l-1}\,\mathrm{d}\zeta}{\sqrt{R(\zeta)}}, \quad 1 \leqslant j \leqslant N, \tag{4.5}$$

$$\rho_j^{(2)}(n, t_0, t_1) = \sum_{k=1}^{N} \int_{P_0}^{P(v_k(n,t_0,t_1))} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \int_{\zeta(p_0)}^{v_k(n)} \frac{\zeta^{l-1}\,\mathrm{d}\zeta}{\sqrt{R(\zeta)}}, \quad 1 \leqslant j \leqslant N, \tag{4.6}$$

where $\zeta(P_0)$ is the local coordinate of P_0.

Theorem 4.1 (*Straightening out of the continuous flow*).

$$\partial_{t_0}\rho_j^{(1)}(n) = \Omega_j^{(1)}, \quad \partial_{t_0}\rho_j^{(2)}(n) = -\Omega_j^{(1)}, \tag{4.7}$$

$$\partial_{t_1}\rho_j^{(1)}(n) = \Omega_j^{(2)}, \quad \partial_{t_1}\rho_j^{(2)}(n) = -\Omega_j^{(2)}{}', \tag{4.8}$$

where

$$\Omega_j^{(1)} = -C_{jN}, \quad \Omega_j^{(2)} = -C_{j,N_1} + \alpha_1 C_{jN}, \quad 1 \leqslant j \leqslant N.$$

Proof. Using (4.5) and (3.23), we have

$$\partial_{t_0}\rho_j^{(1)}(n) = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl}\frac{\mu_k(n)^{l-1}\partial_{t_0}\mu_k(n)}{\sqrt{R(\mu_k(n))}} = -\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{C_{jl}\mu_k(n)^{l-1}}{\prod_{j\neq k}(\mu_k(n) - \mu_j(n))} = -C_{jN} = \Omega_j^{(1)},$$

where we use the equalities

$$\sum_{k=1}^{N} \frac{\mu_k^{l-1}(n)}{\prod_{j\neq k}^{N}(\mu_k(n)-\mu_i(n))} = \delta_{lN}, \quad 1\leqslant l \leqslant N. \tag{4.9}$$

In a similar way, we can prove the second expression of (4.7) and (4.8). The proof is completed. □

5. Straightening out of the discrete flow

Let us denote the fundamental solution matrix of (1.1) by

$$Q_n = (\phi(n), \widehat{\phi}(n)) = \begin{pmatrix} \phi^{(1)}(n) & \widehat{\phi}^{(1)}(n) \\ \phi^{(2)}(n) & \widehat{\phi}^{(2)}(n) \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

which implies

$$Q_{n+1} = U_n U_{n-1} \cdots U_0. \tag{5.2}$$

It is easy to prove by mathematical induction that

$$\begin{aligned}
\phi^{(1)}(n) &= \prod_{j=0}^{n-1}(1+p_j q_j)\lambda^{-n} + \cdots + p_{n-1}q_0\lambda^{n-2},\\
\phi^{(2)}(n) &= \prod_{j=0}^{n-2}(1+p_j q_j)q_{n-1}\lambda^{-n+1} + \cdots + q_0\lambda^{n-1},\\
\widehat{\phi}^{(1)}(n) &= \prod_{j=1}^{n-1}(1+p_j q_j)p_0\lambda^{-n+1} + \cdots + P_{n-1}\lambda^{n-1},\\
\widehat{\phi}^{(2)}(n) &= \prod_{j=1}^{n-2}(1+p_j q_j)p_0 q_{n-1}\lambda^{-n+2} + \cdots + \lambda^{n}.
\end{aligned} \tag{5.3}$$

The Lax matrix W_n satisfies the discrete stationary Lax equation $W_{n+1}U_n - U_n W_n = 0$, which implies that the solution space of the linear equation $\psi(n+1) = U_n\psi(n)$ is invariant under the action of W_n. Let ϱ be the eigenvalue of W_n in the solution space, and $\psi(n)$ be the associated eigenfunction, which is called the Baker function:

$$\psi(n+1) = U_n\psi(n), \quad W_n\psi(n) = \varrho\psi(n). \tag{5.4}$$

It is easy to see that $\det|\varrho - W_n| = \varrho^2 - f(n)^2 - g(n)h(n) = 0$. Thus there are two eigenvalues $\varrho^{\pm} = \pm\varrho$, where

$$\varrho = \sqrt{f(n)^2 + g(n)h(n)} = \tfrac{1}{2}\zeta\sqrt{R(\zeta)}. \tag{5.5}$$

An elementary discussion shows that the corresponding Baker functions can be taken as

$$\phi^{\pm}(n) = \phi(n) + b^{\pm}\widehat{\phi}(n), \quad \widehat{\phi}^{\pm}(n) = c^{\pm}\phi(n) + \widehat{\phi}(n) \tag{5.6}$$

with

$$b^{\pm} = \frac{\pm\varrho - f(0)}{g(0)}, \quad c^{\pm} = \frac{f(0) \pm \varrho}{h(0)}. \tag{5.7}$$

Theorem 5.1. *Let $p^{\pm}(n,\lambda)$ and $q^{\pm}(n,\lambda)$ be the first component and the second one, respectively, of the Baker functions $\phi^{\pm}(n,\lambda)$ and $\widehat{\phi}^{\pm}(n,\lambda)$. Then*

$$p^{+}(n,\lambda)p^{-}(n,\lambda) = \frac{g(n)}{g(0)}, \quad q^{+}(n,\lambda)q^{-}(n,\lambda) = \frac{h(n)}{h(0)}. \tag{5.8}$$

Proof. Resorting to (5.2) and the first expression of (3.2), we have

$$W_n Q_n = Q_n W_0, \tag{5.9}$$

from which a direct calculation derives (5.8). The proof is completed. □

Proposition 5.2. *For $\lambda \to \infty$, we have*

$$\begin{aligned} p^+(n,\lambda) &= \frac{p_{n-1}}{p_{-1}}\lambda^n(1+O(\lambda^{-2})),\\ p^-(n,\lambda) &= \lambda^{-n}(1+O(\lambda)^{-2}), \end{aligned} \tag{5.10}$$

$$\begin{aligned} q^+(n,\lambda) &= \lambda^n(1+O(\lambda^{-2})),\\ q^-(n,\lambda) &= \frac{q_n}{q_0}\lambda^{-n}(1+O(\lambda^{-2})). \end{aligned} \tag{5.11}$$

Proof. When $\lambda \to \infty$, by using (5.5) and (3.5), we obtain

$$\varrho = \frac{\lambda^{2N+4}}{2}(1+\alpha_1\lambda^{-2}+O(\lambda^{-4})), \tag{5.12}$$

which implies

$$b^+ = \frac{\lambda}{p_{-1}}(1+O(\lambda^{-2})), \quad b^- = -q_0\lambda^{-1}(1+O(\lambda^{-2})), \tag{5.13}$$

$$c^+ = p_{-1}\lambda^{-1}(1+O(\lambda^{-2})), \quad c^- = -\frac{\lambda}{q_0}(1+O(\lambda^{-2})). \tag{5.14}$$

Substitute (5.3) and the first expression of (5.13) into $p^+(n,\lambda)=\phi^{(1)}(n,\lambda)+b^+\widehat{\phi}^{(1)}(n,\lambda)$, we have the first expression of (5.10). The second expression of (5.10) is obtained from the first one and

$$p^+(n,\lambda)p^-(n,\lambda) = \frac{g(n)}{g(0)} = \frac{p_{n-1}}{p_{-1}}\prod_{j=1}^{N}\frac{\lambda^2-\mu_j(n)}{\lambda^2-\mu_j(0)}, \quad \lambda\to\infty. \tag{5.15}$$

Similarly, we can prove (5.11). The proof is completed. □

Proposition 5.3. *For $\lambda \to 0$, we have*

$$\begin{aligned} p^+(n,\lambda) &= \prod_{j=0}^{n-1}(1+p_jq_j)\lambda^{-n}(1+O(\lambda^2)),\\ p^-(n,\lambda) &= \frac{p_n}{p_0\prod_{j=1}^{n}(1+p_jq_j)}\lambda^n(1+O(\lambda)^2), \end{aligned} \tag{5.16}$$

$$\begin{aligned} q^+(n,\lambda) &= \frac{q_{n-1}\prod_{j=-1}^{n-2}(1+p_jq_j)}{q_{-1}}\lambda^{-n}(1+O(\lambda^2)),\\ q^-(n,\lambda) &= \frac{1}{\prod_{j=0}^{n-1}(1+p_jq_j)}\lambda^n(1+O(\lambda^2)). \end{aligned} \tag{5.17}$$

Proof. When $\lambda \to 0$, by using (5.5) and (3.5), we obtain

$$\varrho = \frac{\lambda^2}{2}(\beta_0+\beta_1\lambda^2+O(\lambda^4)), \tag{5.18}$$

which implies

$$b^+ = \frac{q_{-1}}{1+p_{-1}q_{-1}}\lambda(1+O(\lambda^2)), \quad b^- = -\frac{1+p_0q_0}{p_0}\lambda^{-1}(1+O(\lambda^2)), \tag{5.19}$$

$$c^+ = \frac{1+p_{-1}q_{-1}}{q_{-1}}\lambda^{-1}(1+O(\lambda^2)), \quad c^- = -\frac{p_0}{1+p_0q_0}\lambda(1+O(\lambda^2)). \tag{5.20}$$

Substitute (5.3) and the first expression of (5.19) into $p^+(n,\lambda)=\phi^{(1)}(n,\lambda)+b^+\widehat{\phi}^{(1)}(n,\lambda)$, we have the first expression of (5.16). The second expression of (5.16) is obtained from the first one and

$$p^+(n,\lambda)p^-(n,\lambda) = \frac{g(n)}{g(0)} = \frac{p_n(1+p_0q_0)(1+O(\lambda^2))}{p_0(1+p_nq_n)(1+O(\lambda^2))}, \quad \lambda \to 0. \tag{5.21}$$

Similarly, we can prove (5.17). The proof is completed. □

According to (5.7) it is easy to see that λb^+ and λb^-, λc^+ and λc^- are functions of ζ, which can be regarded as the values of the single-valued functions $[\lambda b](P)$ and $[\lambda c](P)$ on the upper and lower sheets of Γ, respectively. Therefore, with the help of (5.3), we know that

$$\begin{aligned} p^\pm(2k,\lambda) &= \phi^{(1)}(2k,\lambda) + \lambda b^\pm\{\lambda^{-1}\widehat{\phi}^{(1)}(2k,\lambda)\}, \\ \lambda p^\pm(2k+1,\lambda) &= \lambda\phi^{(1)}(2k+1,\lambda) + \lambda b^\pm\{\widehat{\phi}^{(1)}(2k+1,\lambda)\}, \end{aligned} \tag{5.22}$$

$$\begin{aligned} q^\pm(2k,\lambda) &= \lambda c^\pm\{\lambda^{-1}\phi^{(2)}(2k,\lambda)\} + \widehat{\phi}^{(2)}(2k,\lambda), \\ \lambda q^\pm(2k+1,\lambda) &= \lambda c^\pm\{\phi^{(2)}(2k+1,\lambda)\} + \lambda\widehat{\phi}^{(2)}(2k+1,\lambda), \end{aligned} \tag{5.23}$$

determine four meromorphic functions of ζ on Γ: $p(2k,P)$, $[\lambda p](2k+1,P)$, $q(2k,P)$ and $[\lambda q](2k+1,P)$.

In the local coordinates $z=\zeta^{-1}$, $\widehat{\xi}=\zeta^{-(N+1)}\xi$, the equation of Γ near infinity is written as

$$\widehat{\xi}^2 = \widehat{R}(z), \quad \widehat{R}(z) = \prod_{j=1}^{2N+2}(1-\zeta_j z). \tag{5.24}$$

On Γ there are two infinities and two zeros

$$\infty_s = (z=0, \widehat{\xi}=(-1)^s), \quad 0_s = (\zeta=0, \xi=(-1)^s\beta_0), \quad s=1,2 \tag{5.25}$$

which are located on the upper ($s=2$) and lower ($s=1$) sheets, respectively. By Proposition 5.2, the principal asymptotic terms of the four meromorphic functions near ∞_2 are

$$\begin{aligned} &p^+(2k,P) \sim \frac{p_{2k-1}}{p_{-1}}\zeta^k, \quad \lambda p^+(2k+1,P) \sim \frac{p_{2k}}{p_{-1}}\zeta^{k+1}, \\ &q^+(2k,P) \sim \zeta^k, \quad \lambda q^+(2k+1,P) \sim \zeta^{k+1}, \end{aligned} \tag{5.26}$$

and their principal asymptotic terms near ∞_1 are

$$\begin{aligned} &p^-(2k,P) \sim \zeta^{-k}, \quad \lambda p^-(2k+1,P) \sim \zeta^{-k}, \\ &q^-(2k,P) \sim \frac{q_{2k}}{q_0}\zeta^{-k}, \quad \lambda q^+(2k+1,P) \sim \frac{q_{2k+1}}{q_0}\zeta^{-k}. \end{aligned} \tag{5.27}$$

Resorting to Proposition 5.3, the principal asymptotic terms of the four meromorphic functions near 0_2 are

$$\begin{aligned} &p^+(2k,P) \sim \prod_{j=0}^{2k-1}(1+p_jq_j)\zeta^{-k}, \quad \lambda p^+(2k+1,P) \sim \prod_{j=0}^{2k}(1+p_jq_j)\zeta^{-k}, \\ &q^+(2k,P) \sim \frac{q_{2k-1}}{q_{-1}}\prod_{j=-1}^{2k-2}(1+p_jq_j)\zeta^{-k}, \quad \lambda q^+(2k+1,P) \sim \frac{q_{2k}}{q_{-1}}\prod_{j=-1}^{2k-1}(1+p_jq_j)\zeta^{-k}, \end{aligned} \tag{5.28}$$

and their principal asymptotic terms near 0_1 are

$$p^-(2k,P)\sim\frac{p_{2k}\zeta^k}{p_0\prod_{j=1}^{2k}(1+p_jq_j)},\quad \lambda p^-(2k+1,P)\sim\frac{p_{2k+1}\zeta^{k+1}}{p_0\prod_{j=1}^{2k+1}(1+p_jq_j)},$$
$$q^-(2k,P)\sim\frac{\zeta^k}{\prod_{j=0}^{2k-1}(1+p_jq_j)},\quad \lambda q^-(2k+1,P)\sim\frac{\zeta^{k+1}}{\prod_{j=0}^{2k}(1+p_jq_j)}. \tag{5.29}$$

Based on Dubrovin–Novikov's formulas (5.8) and through an elementary analysis, it is easy to see the following assertions.

Proposition 5.4. *The Baker function $p(2k,P)$ is of the properties*:

(i) *N simple poles at $\mu_1(0),\dots,\mu_N(0)$ and two poles of kth order at ∞_2 and 0_2;*
(ii) *N simple zeros at $\mu_1(2k),\dots,\mu_N(2k)$ and two zeros of kth order at ∞_1 and 0_1.*

The Baker function $[\lambda p](2k+1,P)$ has

(i) *N simple poles at $\mu_1(0),\dots,\mu_N(0)$, a pole of $(k+1)$th order at ∞_2 and a pole of kth at 0_2;*
(ii) *N simple zeros at $\mu_1(2k+1),\dots,\mu_N(2k+1)$, a zero of kth order at ∞_1 and a zero of $(k+1)$th at 0_1.*

Proposition 5.5. *The Baker function $q(2k,P)$ is of the properties*:

(i) *N simple poles at $v_1(0),\dots,v_N(0)$ and two poles of kth order at ∞_2 and 0_2;*
(ii) *N simple zeros at $v_1(2k),\dots,v_N(2k)$ and two zeros of kth order at ∞_1 and 0_1.*

The Baker function $[\lambda q](2k+1,P)$ has

(i) *N simple poles at $v_1(0),\dots,v_N(0)$, a pole of $(k+1)$th order at ∞_2 and a pole of kth at 0_2;*
(ii) *N simple zeros at $v_1(2k+1),\dots,v_N(2k+1)$, a zero of kth order at ∞_1 and a zero of $(k+1)$th at 0_1.*

Theorem 5.6 (*Straightening out of the discrete flow*).

$$\rho^{(s)}(2k)-\rho^{(s)}(0)=2\Omega^{(0)}k \pmod{\mathscr{T}}, \tag{5.30}$$
$$\rho^{(s)}(2k+1)-\rho^{(s)}(0)=2\Omega^{(0)}k+\eta_2 \pmod{\mathscr{T}}, \tag{5.31}$$

or

$$\rho^{(s)}(n)-\rho^{(s)}(0)=\Omega^{(0)}n+[1-(-1)^n]\eta_0 \pmod{\mathscr{T}}, \tag{5.32}$$

where $\mathscr{T}$ is the lattice spanned by the periodic vectors, and

$$\Omega^{(0)}=\frac{1}{2}(\eta_2-\eta_1),\quad \eta_0=\frac{1}{4}(\eta_1+\eta_2),\quad \eta_s=\int_{0_{3-s}}^{\infty_s}\omega,\quad s=1,2.$$

Proof. For $n=2k$, we introduce the meromorphic differential on Γ:

$$\omega(2k)=\left\{\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\, p(2k,P)\right\}\mathrm{d}\zeta, \tag{5.33}$$

which has poles at $\mu_j(0)$ and $\mu_j(2k)$ with the residues -1, 1, respectively, and poles at ∞_1, ∞_2, 0_1, 0_2 with the residues $k,-k,k,-k$, respectively. Let Ω be the Abel differential of the second kind, and $\omega(P,Q)$ be the normal Abel differential of the third kind with the residue 1, -1 at P, Q, respectively, and the properties

$$\int_{a_j}\omega(P,Q)=0,\quad \int_{b_j}\omega(P,Q)=2\pi\sqrt{-1}\int_Q^P\omega_j.$$

Then (5.33) can be written as [17]

$$\omega(2k)=\Omega+k\omega(\infty_1,\infty_2)+k\omega(0_1,0_2)+\sum_{j=1}^{N}\omega(\mu_j(2k),\mu_j(0))+\sum_{j=1}^{N}e_j\omega_j, \tag{5.34}$$

where e_j are some complex numbers. Integrating (5.34) along a_l and b_l, we obtain that $e_l=2\pi n_l\sqrt{-1}$ and

$$\sum_{j=1}^{N}\int_{\mu_j(0)}^{\mu_j(2k)}\omega_l=k\left(\int_{0_1}^{\infty_2}-\int_{0_2}^{\infty_1}\right)\omega_l+m_l-\sum_{j=1}^{N}n_j\tau_{jl}, \tag{5.35}$$

where n_l and m_l are certain integers. This completes the proof of (5.30) for $s=1$.

For $n=2k+1$, consider the meromorphic differential

$$\begin{aligned}\omega(2k+1)&=\left\{\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln[\lambda p](2k+1,P)\right\}\mathrm{d}\zeta\\&=\Omega+k\omega(\infty_1,\infty_2)+k\omega(0_1,0_2)+\omega(0_1,\infty_2)\\&\quad+\sum_{j=1}^{N}\omega[\mu_j(2k+1),\mu_j(0)]+\sum_{j=1}^{N}\widehat{e}_j\omega_j,\end{aligned} \tag{5.36}$$

which implies (5.31) for $s=1$. Similarly, we can prove (5.30) and (5.31) for $s=2$. The proof is completed. □

Based on Theorems (4.1) and (5.6), the compatible solutions of various flows under the Abel–Jacobi coordinates are obtained simply by a linear superposition. Specifically, we have for the discrete systems (2.5) and (2.6), respectively, that

$$\begin{aligned}\rho^{(1)}(n,t_0)&=\Omega^{(0)}n+\Omega^{(1)}t_0+[1-(-1)^n]\eta_0+\rho_0^{(1)},\\\rho^{(2)}(n,t_0)&=\Omega^{(0)}n-\Omega^{(1)}t_0+[1-(-1)^n]\eta_0+\rho_0^{(2)},\end{aligned} \tag{5.37}$$

and

$$\begin{aligned}\rho^{(1)}(n,t_1)&=\Omega^{(0)}n+\Omega^{(2)}t_1+[1-(-1)^n]\eta_0+\rho_0^{(1)},\\\rho^{(2)}(n,t_1)&=\Omega^{(0)}n-\Omega^{(2)}t_1+[1-(-1)^n]\eta_0+\rho_0^{(2)}.\end{aligned} \tag{5.38}$$

For the $(2+1)$-dimensional differential-difference equations (2.7)–(2.9), we have

$$\begin{aligned}\rho^{(1)}(n,t_0,t_1)&=\Omega^{(0)}n+\Omega^{(1)}t_0+\Omega^{(2)}t_1+[1-(-1)^n]\eta_0+\rho_0^{(1)},\\\rho^{(2)}(n,t_0,t_1)&=\Omega^{(0)}n-\Omega^{(1)}t_0-\Omega^{(2)}t_1+[1-(-1)^n]\eta_0+\rho_0^{(2)}.\end{aligned} \tag{5.39}$$

6. Algebro-geometric solutions

In this section, we shall give explicit solutions of the $(2+1)$-dimensional differential-difference Eqs. (2.7)–(2.9). To this end, we take into account the Riemann theorem [9,16], which asserts that there exist constant vectors $M^{(1)}$ and $M^{(2)}$ such that $\theta(\mathscr{A}(P(\zeta))-\rho^{(l)}(n)-M^{(l)})$ has exactly N zeros at $\mu_1(n),\ldots,\mu_N(n)$ for $l=1$ or $v_1(n),\ldots,v_N(n)$ for $l=2$. Here $M^{(m)}$ is the Riemann constant and θ is the Riemann theta function defined by

$$\theta(\sigma\mid\tau)=\sum_{\eta\in Z^N}exp\left(\pi\sqrt{-1}\langle\tau\eta,\eta\rangle+2\pi\sqrt{-1}\langle\sigma,\eta\rangle\right),$$

in which $\sigma = (\sigma_1, \ldots, \sigma_N)^{\mathrm{T}} \in C^N$, $\langle\cdot,\cdot\rangle$ represents the standard inner-product. And we have the inversion formula

$$\sum_{j=1}^{N} \mu_j(n)^k = I_k(\Gamma) - \sum_{s=1}^{2} Res_{\zeta=\infty_s} \zeta^k \mathrm{d}\ \ln\ \theta(\mathscr{A}(P(\zeta)) - \rho^{(1)}(n) - M^{(1)}),$$
$$\sum_{j=1}^{N} \nu_j(n)^k = I_k(\Gamma) - \sum_{s=1}^{2} Res_{\zeta=\infty_s} \zeta^k \mathrm{d}\ \ln\ \theta(\mathscr{A}(P(\zeta)) - \rho^{(2)}(n) - M^{(2)}), \tag{6.1}$$

with the constant

$$I_k(\Gamma) = \sum_{j=1}^{N} \int_{\alpha_j} \zeta^k \omega_j.$$

Through a standard treatment, we arrive at

$$\sum_{j=1}^{N} \mu_j(n) = I_1(\Gamma) + \partial_{t_0}\ \ln \frac{\theta(\rho^{(1)}(n) + M^{(1)} + \pi^{(2)})}{\theta(\rho^{(1)}(n) + M^{(1)} + \pi^{(1)})}, \tag{6.2}$$

$$\sum_{j=1}^{N} \nu_j(n) = I_1(\Gamma) + \partial_{t_0}\ \ln \frac{\theta(\rho^{(2)}(n) + M^{(2)} + \pi^{(1)})}{\theta(\rho^{(2)}(n) + M^{(2)} + \pi^{(2)})}, \tag{6.3}$$

where

$$\pi^{(s)} = \int_{\infty_s}^{P_0} \omega.$$

By using (3.18), (5.18), (6.2) and (6.3), we obtain the algebro-geometric solutions of the (2+1)-dimensional differential-difference equations (2.7) and (2.8), ($p_n = p(n, t_0, t_1)$, $q_n = q(n, t_0, t_1)$):

$$\begin{aligned} p(n, t_0, t_1) &= \frac{\theta(\Omega^{(0)}(n+1) + \Omega^{(1)} t_0 + \Omega^{(2)} t_1 + [1 - (-1)^{n+1}]\eta_0 + \Upsilon^{(2)})}{\theta(\Omega^{(0)}(n+1) + \Omega^{(1)} t_0 + \Omega^{(2)} t_1 + [1 - (-1)^{n+1}]\eta_0 + \Upsilon^{(1)})} \\ &\quad \times \frac{\theta(\Omega^{(0)}(n+1) + \Omega^{(2)} t_1 + [1 - (-1)^{n+1}]\eta_0 + \Upsilon^{(1)})}{\theta(\Omega^{(0)}(n+1) + \Omega^{(2)} t_1 + [1 - (-1)^{n+1}]\eta_0 + \Upsilon^{(2)})} \mathrm{e}^{(I_1(\Gamma)+\alpha_1)t_0} p(n, 0, t_1), \\ q(n, t_0, t_1) &= \frac{\theta(\Omega^{(0)} n - \Omega^{(1)} t_0 - \Omega^{(2)} t_1 + [1 - (-1)^{n}]\eta_0 + \widehat{\Upsilon}^{(2)})}{\theta(\Omega^{(0)} n - \Omega^{(1)} t_0 - \Omega^{(2)} t_1 + [1 - (-1)^{n}]\eta_0 + \widehat{\Upsilon}^{(1)})} \\ &\quad \times \frac{\theta(\Omega^{(0)} n - \Omega^{(2)} t_1 + [1 - (-1)^{n}]\eta_0 + \widehat{\Upsilon}^{(1)})}{\theta(\Omega^{(0)} n - \Omega^{(2)} t_1 + [1 - (-1)^{n}]\eta_0 + \widehat{\Upsilon}^{(2)})} \mathrm{e}^{-(I_1(\Gamma)+\alpha_1)t_0} q(n, 0, t_1), \end{aligned} \tag{6.4}$$

with the constants

$$\Upsilon^{(1)} = \rho_0^{(1)} + M^{(1)} + \pi^{(1)}, \quad \Upsilon^{(2)} = \rho_0^{(1)} + M^{(1)} + \pi^{(2)},$$
$$\widehat{\Upsilon}^{(1)} = \rho_0^{(2)} + M^{(2)} + \pi^{(1)}, \quad \widehat{\Upsilon}^{(2)} = \rho_0^{(2)} + M^{(2)} + \pi^{(2)}.$$

And we can obtain that $r(n, t_0, t_1) = p(n-1, t_0, t_1) \times q(n, t_0, t_1)$ is the algebro-geometric for the (2+1)-dimensional differential-difference equation (2.9). ($p(n, t_0, t_1)$, $q(n+2, t_0, t_1)$) determined by (6.4) is also the algebro-geometric solutions of the coupled NLS (2.10), where n is viewed as a fixed integer.

Acknowledgments

The author expresses his gratitude to Prof. E.G. Fan for his enthusiastic guidance. The work described in this paper was supported by a National Key Basic Research Project of China (2004CB318000), the National Science Foundation of China (10371023) and Shanghai Shuguang Project of China (02SG02).

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