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Multiplicity of Nontrivial Solutions of Semilinear Elliptic Equations[1]

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Two nontrivial solutions are obtained by the reduction method for the nonautonomous semilinear elliptic Dirichlet boundary value problem. Some well-known multiplicity results are generalized. © 2000 Academic Press

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1. INTRODUCTION AND MAIN RESULTS

Consider the semilinear elliptic Dirichlet boundary problem

$$-\Delta u = f(x, u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\Omega \subset R^N$ ($N \geq 1$) is a bounded smooth domain and f: $\overline{\Omega} \times R \to R$ is a subcritical Carathéodory function; that is, there are positive constants

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C_1, C_2 such that

$$|f(x,t)| \leq C_1|t|^{p-1} + C_2 \tag{2}$$

for all $t \in R$ and a.e. $x \in \Omega$, where $p \in]2, 2N/(N-2)[$ for $N \geq 3$, $p \in]2, +\infty[$ for $N = 1, 2$. Let

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

be the sequence of the distinct eigenvalues of the eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

and let k be a fixed positive integer.

With the reduction method, two nontrivial solutions are obtained for the nonresonant or resonant elliptic problem (1) (see [2, 5–8]). Recall the following multiple existence results.

THEOREM A. *Suppose that* $f \in C^1(R, R)$, $f(0) = 0$, *and* f' *is bounded. Assume that there exists* $0 < m \leq k$ *such that*

$$\lambda_{m-1} < f'(0) < \lambda_m, \qquad \lambda_k < \lim_{|t| \to \infty} f'(t) < \lambda_{k+1} \tag{3}$$

and such that

$$\sup_{t \in R} f'(t) < \lambda_{k+1}. \tag{4}$$

Then problem (1) has at least two nontrivial solutions.

THEOREM B. *Suppose that* $f(t) = \lambda_k t + g(t)$, $g \in C^2(R, R)$,

$$\lim_{|t| \to \infty} \frac{g(t)}{t} = 0,$$

and

$$\lim_{|t| \to \infty} g(t)t = +\infty$$

and that g is bounded or $\liminf_{|t| \to \infty} |g(t)| > 0$ *holds. Assume that there exists* $0 < m \leq k$ *such that*

$$\lambda_{m-1} \leq \lambda_k + \inf_{t \in R \setminus \{0\}} \frac{g(t)}{t} \leq \lambda_k + g'(0) < \lambda_m$$

and that

$$\lambda_k + \sup_{t \in R} g'(t) < \lambda_{k+1}.$$

Then problem (1) has at least two nontrivial solutions.

Theorem A is due to Castro and Lazer [6]. Their approach is based on the reduction method and finite dimensional critical point theory. That is, under assumptions (3) and (4), there exist a function ψ on the finite dimensional space V spanned by the eigenfunctions corresponding to the eigenvalues $\lambda_1, \dots, \lambda_k$ and a function θ which is from V to the Sobolev space $H_0^1(\Omega)$ such that $v \in V$ is a critical point of ψ if and only if $v + \theta(v)$ is a solution of problem (1). Under assumptions (3) and (4), one of the nontrivial solutions of problem (1) is a critical point of ψ at which ψ attains its maximum. The existence of the second nontrivial solution is deduced from the calculation of the Leray–Schauder index of critical points.

Theorem B is due to Hirano and Nishimura [7]. Their approach is based on the reduction method and an abstract multiplicity result which is based on the minimax method.

In this paper, we obtain some multiplicity results which unify and generalize the results mentioned above. Our approach is based on the reduction method and a three-critical-point theorem due to Brezis and Nirenberg [4]. The main results are the following theorems.

THEOREM 1. *Suppose that* (2) *holds and that there exists* $a < \lambda_{k+1}$ *such that*

$$\frac{f(x,s) - f(x,t)}{s - t} \le a \tag{5}$$

for all $s, t \in R$, $s \neq t$, *and a.e.* $x \in \Omega$. *Assume that*

$$F(x,t) - \tfrac{1}{2}\lambda_k t^2 \to +\infty \tag{6}$$

as $|t| \to \infty$ *uniformly for a.e.* $x \in \Omega$ *and that there exist* $\delta > 0$, $b > 0$, *and* $0 < m \le k$ *such that*

$$\tfrac{1}{2}\lambda_{m-1} t^2 \le F(x,t) \le \tfrac{1}{2}(\lambda_m - b)t^2 \tag{7}$$

for all $|t| \le \delta$ *and a.e.* $x \in \Omega$, *where* $F(x,t) = \int_0^t f(x,s)\,ds$. *Then problem* (1) *has at least two nontrivial solutions in* $H_0^1(\Omega)$.

THEOREM 2. *Suppose that* $f \in C^1(R,R)$ *is subcritical and*

$$\sup_{t \in R} f'(t) < \lambda_{k+1}. \tag{8}$$

Assume that

$$f(t)t - \lambda_k t^2 \to +\infty \tag{9}$$

at $|t| \to \infty$ *and that there exist* $\delta > 0$ *and* $0 < m \le k$ *such that*

$$\lambda_{m-1} \le \inf_{0 < |t| < \delta} \frac{f(t)}{t}, \qquad f'(0) < \lambda_m. \tag{10}$$

Then the problem

$$-\Delta u = f(u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has at least two nontrivial solutions.

COROLLARY 1. *Suppose that* $g \in C^1(R, R)$ *and*

$$\sup_{t \in R} g'(t) < \lambda_{k+1} - \lambda_k.$$

Assume that

$$g(t)t \to +\infty$$

as $|t| \to \infty$ *and that there exists* $\delta > 0$ *and* $0 < m \le k$ *such that*

$$\lambda_{m-1} - \lambda_k \le \inf_{0<|t|\le\delta} \frac{g(t)}{t}, \qquad g'(0) < \lambda_m - \lambda_k.$$

Then the problem

$$-\Delta u = \lambda_k u + g(u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

has at least two nontrivial solutions.

Remark 1. Theorem A is a consequence of Theorem 2. In fact, $\lim_{|t|\to\infty} f'(t) > \lambda_k$ implies that

$$f(t)t - \lambda_k t^2 \to +\infty$$

as $|t| \to \infty$ and $f'(0) < \lambda_{m-1}$ implies that

$$\lambda_{m-1} \le \inf_{0<|t|<\delta} \frac{f(t)}{t}$$

for some $\delta > 0$. There are functions $f \in C^1(R, R)$ satisfying our Theorem 2 and not satisfying Theorem A. For example,

$$f(t) = \begin{cases} \frac{1}{2}(\lambda_{k+1} + \lambda_k)t - \frac{1}{4}(\lambda_{k+1} + \lambda_k - 2\lambda_{m-1})\dfrac{t}{|t|} \\ \qquad |t| \ge 1, \\ \lambda_{m-1}t + \frac{1}{4}(\lambda_{k+1} + \lambda_k - 2\lambda_{m-1})|t|t, \\ \qquad |t| \le 1 \end{cases}$$

where $f'(0) = \lambda_{m-1}$.

Remark 2. Corollary 1 generalizes Theorem B. In fact, Corollary 1 has no need of the conditions that

$$\lim_{|t|\to\infty} \frac{g(t)}{t} = 0$$

and that g is bounded or $\liminf_{|t|\to\infty} |g(t)| > 0$ holds; besides, the condition that

$$\lambda_{m-1} \leq \lambda_k + \inf_{t \in R \setminus \{0\}} \frac{g(t)}{t}$$

is replaced by the weaker one that

$$\lambda_{m-1} - \lambda_k \leq \inf_{0<|t|\leq \delta} \frac{g(t)}{t}$$

for some $\delta > 0$. There are functions $g \in C^1(R, R)$ satisfying our Corollary 1 and not satisfying Theorem B. For example,

$$g(t) = f(t) - \lambda_k t,$$

where f is the same as Remark 1 and

$$\lim_{|t|\to\infty} \frac{g(t)}{t} \neq 0.$$

Remark 3. From Remarks 1 and 2, we know that Theorem 2 unifies and generalizes Theorems A and B. Furthermore, Theorem 1 generalizes Theorems A and B to the nonautonomous case; it needs only the basic regularity.

2. PROOFS OF THEOREMS

Define the functional φ on the Sobolev space $H_0^1(\Omega)$ by

$$\varphi(u) = -\tfrac{1}{2}\|u\|^2 + \int_\Omega F(x,u)\,dx, \qquad u \in H_0^1(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)\,ds$, $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$ is the usual norm in $H_0^1(\Omega)$. Then φ is continuously differentiable and

$$\langle \varphi'(u), v\rangle = -\int_\Omega \nabla u \nabla v\,dx + \int_\Omega f(x,u)v\,dx$$

for $u, v \in H_0^1(\Omega)$. It is well known that $u \in H_0^1(\Omega)$ is a solution of problem (1) if and only if u is a critical point of φ. Let

$$V = E(\lambda_1) + \cdots + E(\lambda_k)$$

and $W = V^{\perp}$, where $E(\lambda_i)$ stands for the eigenspace corresponding to λ_i, i.e., the finite dimensional space spanned by the eigenfunctions corresponding to λ_i. Define the functional ψ

$$\psi(v) = \sup_{w \in W} \varphi(v + w), \qquad v \in V$$

It follows easily from Theorem 2.3 of Amann [1] that

LEMMA 1. *Suppose that* (2) *and* (5) *hold. Then* $\psi\colon V \to R$ *is continuously differentiable and*

$$\psi'(v) = P_V \varphi'(v + \theta(v)), \qquad v \in V,$$

where $P_V\colon H_0^1(\Omega) \to V$ *is the corresponding projection onto* V *along* W, θ: $V \to W$ *and is a continuous mapping satisfying that*

$$\psi(v) = \varphi(v + \theta(v))$$

for every $v \in V$.

It is a simple corollary that $v + \theta(v)$ is a critical point of φ if v is a critical point of ψ.

LEMMA 2. *Suppose that* (2) *and* (6) *hold. Then* φ *is coercive on* V, *i.e.*,

$$\varphi(v) + \infty \qquad \text{as } \|v\| \to \infty \text{ in } V.$$

Thus ψ *is coercive.*

Proof. By (6) there exists $M > 0$ such that

$$F(x,t) - \tfrac{1}{2}\lambda_k t^2 \geq 0$$

for all $|t| \geq M$ and a.e. $x \in \Omega$. It follows from (2) that

$$|F(x,t)| \leq \frac{C_1}{p} M^{p-1} + C_2 M$$

for all $|t| \leq M$ and a.e. $x \in \Omega$. Hence we have

$$F(x,t) \geq \frac{1}{2}\lambda_k t^2 - \frac{1}{2}\lambda_k M^2 - \frac{C_1}{p} M^p - C_2 M$$

for all $t \in R$ and a.e. $x \in \Omega$.

If φ is not coercive, there exist $M_0 > 0$ and a sequence (v_n) in V such that $\|v_n\| \to \infty$ and $\varphi(v_n) \leq M_0$. Let $v_n = a_n + b_n$, $a_n \in E(\lambda_1) + \cdots + E(\lambda_{k-1})$, and $b_n \in E(\lambda_k)$. Then

$$\|v_n\|^2 = \|a_n\|^2 + \|b_n\|^2.$$

In the case that (a_n) has a subsequence (a_{n_i}) such that

$$\|a_{n_i}\| \to \infty \qquad \text{as } i \to \infty$$

one has

$$\begin{aligned}\varphi(v_{n_i}) &\geq -\frac{1}{2}\|a_{n_i}\|^2 + \frac{1}{2}\lambda_k\|a_{n_i}\|_{L^2}^2 - C_3 \\ &\geq \frac{1}{2}\left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right)\|a_{n_i}\|^2 - C_3 \\ &\to +\infty\end{aligned}$$

as $i \to +\infty$, where $C_3 = (\frac{1}{2}\lambda_k M^2 + (C_1/p)M^p + C_2 M)$ meas Ω. This is a contradiction.

In the case that $(\|a_n\|)$ is bounded by C_4, one has $\|b_n\| \to \infty$ as $n \to \infty$. It follows from the first part of the proof of Lemma 3.2 in [3] that for every $\alpha > 0$ there exists $m_\alpha > 0$ such that

$$\operatorname{meas}\{x \in \Omega \mid |v(x)| < m_\alpha\|v\|\} < \alpha$$

for all $v \in E(\lambda_k)$. Let

$$A_n = \{x \in \Omega \mid |b_n(x)| \geq m_\alpha\|b_n\|\}.$$

Then one has $\operatorname{meas}(\Omega \setminus A_n) < \alpha$. By the finite dimensionality of V, there exists $C_5 > 0$ such that

$$\sup\{|a_n(x)| \mid x \in \Omega\} \leq C_5$$

for all n. For every $\beta > 0$, there exists $M > 0$ such that

$$F(x,t) - \tfrac{1}{2}\lambda_k t^2 \geq \beta$$

for all $|t| \geq M$ and a.e. $x \in \Omega$ by (6). Let

$$B_n = \{x \in \Omega \mid |v_n(x)| \geq M\}.$$

For $x \in A_n$, one has

$$|v_n(x)| \geq |b_n(x)| - |a_n(x)| \geq m_\alpha\|b_n\| - C_5,$$

which implies $A_n \subset B_n$ for large n. Now we have

$$\begin{aligned}
&\int_\Omega \left(F(x, v_n) - \frac{1}{2}\lambda_k |v_n|^2 \right) dx \\
&\quad \geq \beta \text{ meas } B_n - \left(\frac{1}{2}\lambda_k M^2 + \frac{C_1}{p} M^p + C_2 M \right) \text{meas}(\Omega \setminus B_n) \\
&\quad \geq \beta \text{ meas } A_n - \left(\frac{1}{2}\lambda_k M^2 + \frac{C_1}{p} M^p + C_2 M \right) \text{meas}(\Omega \setminus A_n) \\
&\quad \geq \beta(\text{meas } \Omega - \alpha) - \left(\frac{1}{2}\lambda_k M^2 + \frac{C_1}{p} M^p + C_2 M \right) \alpha
\end{aligned}$$

for large n. Hence one has

$$\begin{aligned}
&\liminf_{n \to \infty} \int_\Omega \left(F(x, v_n) - \frac{1}{2}\lambda_k |v_n|^2 \right) dx \\
&\quad \geq \beta(\text{meas } \Omega - \alpha) - \left(\frac{1}{2}\lambda_k M^2 + \frac{C_1}{p} M^p + C_2 M \right) \alpha.
\end{aligned}$$

Letting $\alpha \to 0$, we obtain

$$\liminf_{n \to \infty} \int_\Omega \left(F(x, v_n) - \tfrac{1}{2}\lambda_k |v_n|^2 \right) dx \geq \beta \text{ meas } \Omega.$$

By the arbitrariness of β we have

$$\liminf_{n \to \infty} \int_\Omega \left(F(x, v_n) - \tfrac{1}{2}\lambda_k |v_n|^2 \right) dx = +\infty.$$

Hence one has

$$\varphi(v_n) \geq \int_\Omega \left(F(x, v_n) - \tfrac{1}{2}\lambda_k |v_n|^2 \right) dx \to +\infty$$

as $n \to \infty$, which is a contradiction. Therefore φ is coercive on V. Thus ψ is coercive.

LEMMA 3. *Let $V_1 = E(\lambda_1) + \cdots + E(\lambda_{m-1})$, $V_2 = E(\lambda_m) + \cdots + E(\lambda_k)$. Suppose that* (7) *holds. Then there exists $\delta_0 > 0$ such that*

$$\psi(v) \geq 0 \qquad \textit{for } v \in V_1 \qquad \textit{and} \qquad \|v\| \leq \delta_0$$

and

$$\psi(v) \leq 0 \qquad \textit{for } v \in V_2 \qquad \textit{and} \qquad \|v\| \leq \delta_0.$$

Proof. By the finite dimensionality of V_1, there exists $C_6 > 0$ such that

$$\sup\{|v(x)|\,|\,x \in \Omega\} \leq C_6\|v\|$$

for all $v \in V_1$. Let $\delta_1 = (1/C_6)\delta$. Then for $v \in V_1$, $\|v\| \leq \delta_1$, one has

$$\psi(v) \geq \varphi(v) \geq -\tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}\lambda_{m-1}\|v\|_{L^2}^2 \geq 0.$$

On the other hand, it follows from (7) that $f(x,0) = 0$ for a.e. $x \in \Omega$. Associating with (5) we have $f(x,t)t \leq at^2$ for all $t \in R$ and a.e. $x \in \Omega$, which implies that

$$F(x,t) \leq \tfrac{1}{2}at^2$$

for all $t \in R$ and a.e. $x \in \Omega$ by the fact that $F(x,t) = \int_0^1 f(x,ts)t\,ds$. Hence one has

$$\begin{aligned} F(x,t) &\leq \tfrac{1}{2}(\lambda_m - b)t^2 + \tfrac{1}{2}(a + b - \lambda_m)t^2 \\ &\leq \tfrac{1}{2}(\lambda_m - b)t^2 + \tfrac{1}{2}|a + b - \lambda_m|\delta^{2-p}|t|^p \end{aligned}$$

for all $|t| \geq \delta$ and a.e. $x \in \Omega$. It follows from (7) that

$$F(x,t) \leq \tfrac{1}{2}(\lambda_m - b)t^2 + C_7|t|^p \tag{11}$$

for all $t \in T$, a.e. $x \in \Omega$, and some $C_7 > \frac{1}{2}|a + b - \lambda_m|\delta^{2-p}$. By Sobolev's embedding theorem, there exists $C_8 > 0$ such that

$$\|u\|_{L^p} \leq C_8\|u\|$$

for all $u \in H_0^1(\Omega)$. It follows from the continuity of θ that there exists $\delta_0 \in]0, \delta_1[$ such that

$$\|v + \theta(v)\| \leq \left(\frac{b}{2\lambda_m C_7 C_8^p}\right)^{1/(p-2)}$$

for all $v \in V_2$ with $\|v\| \leq \delta_0$. Thus (11) implies that

$$\begin{aligned} \psi(v) &= \varphi(v + \theta(v)) \\ &\leq -\frac{1}{2}\|v + \theta(v)\|^2 + \frac{1}{2}(\lambda_m - b)\|v + \theta(v)\|_{L^2}^2 \\ &\qquad + C_7\|v + \theta(v)\|_{L^p}^p \\ &\leq -\frac{b}{2\lambda_m}\|v + \theta(v)\|^2 + C_7C_8^p\|v + \theta(v)\|^p \\ &\leq 0 \end{aligned}$$

for all $v \in V_2$ with $\|v\| \leq \delta_0$.

Proof of Theorem 1. By the finite dimensionality of V and Lemma 1 we know that ψ satisfies the (PS) condition.

In the case that $\inf_{v \in V} \psi(v) = 0$, all $v \in V_2$ with $\|v\| \leq \delta_0$ are minima of ψ by Lemma 3, which implies that ψ has infinite critical points.

In the case that $\inf_{v \in V} \psi(v) < 0$, from the proof of Lemma 2 one obtains

$$\inf_{v \in V} \psi(v) > -\infty.$$

It follows from Theorem 4 in [4] that ψ has at least two nonzero critical points. Hence φ has at least two nonzero critical points. Thus problem (1) has at least two nontrivial solutions in $H_0^1(\Omega)$.

Proof of Theorem 2. Let $a = \sup_{t \in R} f'(t)$. It follows from the mean value theorem and (8) that

$$\frac{f(s) - f(t)}{s - t} = f'(\xi) \leq a$$

for all $s, t \in R$, $s \neq t$. Hence (5) holds. For every $\beta > 0$, there exists $M > 0$ such that

$$f(t)t - \lambda_k t^2 \geq \beta$$

for every $|t| \geq M$ by (9). Hence one has

$$f(t) - \lambda_k t \geq \frac{\beta}{t}$$

for all $t \geq M$, which implies that

$$F(t) - F(M) - \tfrac{1}{2}\lambda_k t^2 + \tfrac{1}{2}\lambda_k M^2 \geq \beta \ln t - \beta \ln M.$$

Thus we have

$$F(t) - \tfrac{1}{2}\lambda_k t^2 \to +\infty$$

as $t \to +\infty$. In a similar way we obtain

$$F(t) - \tfrac{1}{2}\lambda_k t^2 \to +\infty$$

as $t \to -\infty$. Hence (6) holds.

Let $b = \frac{1}{2}(\lambda_m - f'(0))$. From the first part of (10) and the continuity of f, one obtains that $f(0) = 0$. By (10), there exists $\delta_0 \in]0, \delta[$ such that

$$\frac{f(t)}{t} \leq \lambda_m - b$$

for all $0 < |t| < \delta_0$. Hence we have

$$\lambda_{m-1} t^2 \leq f(t)t \leq (\lambda_m - b)t^2$$

for all $|t| < \delta_0$. Thus one has

$$\tfrac{1}{2}\lambda_{m-1} t^2 \leq F(t) \leq \tfrac{1}{2}(\lambda_m - b)t^2.$$

for all $|t| < \delta_0$. Therefore (7) is proved. Now Theorem 2 follows from Theorem 1.

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