Hokkaido Mathematical Journal Vol. 38 (2009) p. 635–662

Multilinear Riesz potential operators on Herz-type spaces and generalized Morrey spaces*

Yanlong Shi and Xiangxing Tao†

(Received August 11, 2008)

Abstract. Let m, n be integers with $n \geq 2$, $m \geq 1$, the multilinear Riesz potential operators be defined by

$$I_\alpha^{(m)}(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} d\mathbf{y},$$

where $\mathbf{y} = (y_1,\dots,y_m)$ and $\mathbf{f} = (f_1,\dots,f_m)$. In the first part of this paper, the boundedness for the operator $I_\alpha^{(m)}$ on the homogeneous Herz-Morrey product spaces, $M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mathbb{R}^n) \times \cdots \times M\dot{K}_{p_m,q_m}^{n(1-1/q_m),\lambda_m}(\mathbb{R}^n)$, and on the Herz-type Hardy product spaces, $H\dot{K}_{q_1}^{\sigma_1,p_1}(\mathbb{R}^n) \times \cdots \times H\dot{K}_{q_m}^{\sigma_m,p_m}(\mathbb{R}^n)$ for $\sigma_i > n(1-1/q_i)$, are established respectively. The second goal of the paper is to extend the known L^p-boundedness of $I_\alpha^{(m)}$ to generalized Morrey spaces, $L^{p,\phi}(\mathbb{R}^n)$, where $p \in [1,+\infty)$ and ϕ is the suitable doubling and integral functions.

Key words: multilinear fractional integral, homogeneous Herz-Morrey space, Herz-type hardy space, homogeneous Herz space, generalized Morrey space

1. Introduction

The theory of multilinear operators has received increasing attentions, see [1], [2] and [3] among others. It's certainly valuable to extend the Riesz potential's theory to the multilinear content and the generalized cases. In [3], Kenig and Stein studied the multilinear Riesz potential operators as follows

$$I_\alpha^{(m)}(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} d\mathbf{y}, \quad 0 < \alpha < mn,$$

where we always let $y_i \in \mathbb{R}^n$ for $i = 1,\dots,m$, and also $x \in \mathbb{R}^n$, so $\mathbf{y} =$

2000 Mathematics Subject Classification : 42B20, 42B25.

*This work was supported in part by the NNSF of China under Grant ♯10771110 and ♯10471069, and sponsored by K. C. Wong Magna Fund in Ningbo University.

†Corresponding author: xxtao@hotmail.com

$(y_1,\dots,y_m)\in(\mathbb{R}^n)^m$, we denote by $d\mathbf{y}=dy_1\dots dy_m$, and by $\mathbf{f}$ the m-tuple $(f_1,\dots,f_m)$. Kenig and Stein proved that

Theorem 1.1 ([3]) *Let $0<\alpha<mn$, $f_i\in L^{p_i}(\mathbb{R}^n)$ $(i=1,\dots,m)$ with $1\le p_i\le\infty$ and $1/q=1/p_1+\cdots+1/p_m-\alpha/n>0$. Then*

$$\left\|I_\alpha^{(m)}(\mathbf{f})\right\|_{L^q(\mathbb{R}^n)}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R}^n)}\quad for\quad p_i>1,$$

and

$$\left\|I_\alpha^{(m)}(\mathbf{f})\right\|_{L^{q,\infty}(\mathbb{R}^n)}\le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\mathbb{R}^n)}\quad for\quad p_i\ge1,$$

here $L^{q,\infty}(\mathbb{R}^n)$ denotes the weak $L^q(\mathbb{R}^n)$, the constant $C>0$ independent of $\mathbf{f}$.

Recently in [10] and [13], we extended Kenig and Stein's results above to the homogeneous Herz-Morrey product spaces and obtained

Theorem 1.2 ([13]) *Let $0<\alpha<mn$, $0\le\lambda_i<n-\alpha/m$, $0<p_i\le\infty$, $1<q_i<\infty$ and $\lambda_i+\alpha/m-n/q_i<\sigma_i<n(1-1/q_i)$ for $i=1,\dots,m$. Suppose that $\lambda=\lambda_1+\cdots+\lambda_m$, $\sigma=\sigma_1+\cdots+\sigma_n$, $1/p=1/p_1+\cdots+1/p_m-\alpha/n$, $1/q=1/q_1+\cdots+1/q_m-\alpha/n$. Then*

$$\left\|I_\alpha^{(m)}(\mathbf{f})\right\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)}\le C\prod_{i=1}^m\|f_i\|_{M\dot{K}_{p_i,q_i}^{\sigma_i,\lambda_i}(\mathbb{R}^n)}$$

with the constant $C>0$ independent of $\mathbf{f}$.

The definition of the homogeneous Herz-Morrey space $M\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)$ will be given in Section 2, here we point out that $M\dot{K}_{p,p}^{0,0}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ for $1\le p<\infty$.

We observe that, in Theorem 1.2, the boundedness of $I_\alpha^{(m)}$ on product space $\prod_{i=1}^m M\dot{K}_{p_i,q_i}^{\sigma_i,\lambda_i}(\mathbb{R}^n)$ for the cases $\sigma_i\ge n(1-1/q_i)$ remains open. Therefore, our first main aim of the paper is devoted to deal with this question, as the continuation of our works in [13].

In the case $\sigma_i=n(1-1/q_i)$, we will use the weak homogeneous Herz-Morrey space $WM\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)$ and weak homogeneous Herz space $W\dot{K}_q^{\sigma,p}(\mathbb{R}^n)$

to derive the following boundedness for the operator $I_\alpha^{(m)}$.

Theorem 1.3 *Let $0 < \alpha < mn$, $0 \le \lambda_i < n - \alpha/m$, $0 < p_i \le 1$ and $1 \le q_i < \infty$ for $i = 1, \dots, m$. Suppose that $\lambda = \lambda_1 + \cdots + \lambda_m$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n$. Then*

$$\left\| I_\alpha^{(m)}(\mathbf{f}) \right\|_{WM\dot{K}_{p,q}^{n(m-1/q)-\alpha,\lambda}(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}_{p_i,q_i}^{n(1-1/q_i),\lambda_i}(\mathbb{R}^n)}$$

with the constant $C > 0$ independent of $\mathbf{f}$*.*

Letting $\lambda_i \equiv 0$, we immediately get the following theorem.

Theorem 1.4 *Let $0 < \alpha < mn$, $0 < p_i \le 1$ and $1 \le q_i < \infty$ for $i = 1, \dots, m$. Suppose that $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n$. Then*

$$\left\| I_\alpha^{(m)}(\mathbf{f}) \right\|_{W\dot{K}_{q}^{n(m-1/q)-\alpha,p}(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{\dot{K}_{q_i}^{n(1-1/q_i),p_i}(\mathbb{R}^n)}$$

with the constant $C > 0$ independent of $\mathbf{f}$*.*

Remark 1.5 The restriction $0 < p_i \le 1$ in Theorem 1.4 can not be removed, see [9] for an counterexample when $m = 1$.

When $\sigma_i > n(1 - 1/q_i)$, the appropriate substitute spaces for the Herz spaces are the Herz-type Hardy spaces, we will prove the following theorem.

Theorem 1.6 *Let $0 < \alpha < mn$, $0 < p_i < \infty$, $1 < q_i < \infty$ and $\sigma_i > n(1 - 1/q_i)$ for $i = 1, \dots, m$. Suppose that $\sigma = \sigma_1 + \cdots + \sigma_n$, $1/p = 1/p_1 + \cdots + 1/p_m - \alpha/n$, $1/q = 1/q_1 + \cdots + 1/q_m - \alpha/n$. Then*

$$\left\| I_\alpha^{(m)}(\mathbf{f}) \right\|_{\dot{K}_{q}^{\sigma,p}(\mathbb{R}^n)} \le C \prod_{i=1}^m \|f_i\|_{H\dot{K}_{q_i}^{\sigma_i,p_i}(\mathbb{R}^n)}$$

with the constant $C > 0$ independent of $\mathbf{f}$*.*

On the other hand, in 1994, Nakai [4] studied the Riesz potential operator I_α on generalized Morrey spaces, $L^{p,\phi}(\mathbb{R}^n)$. Recently in [12], we established the boundedness of maximal operators on this generalized Morrey spaces. Under the assumptions

$$r \le t \le 2r \Longrightarrow C_1 \le \frac{\phi(x_0,t)}{\phi(x_0,r)} \le C_2 \tag{1.1}$$

and

$$\int_r^{+\infty} \frac{\phi(x_0,t)}{t^{n-\alpha p+1}}dt \le C\frac{\phi(x_0,r)}{r^{n-\alpha p}} \tag{1.2}$$

for any $x_0 \in \mathbb{R}^n$ and $r > 0$, Nakai proved that

Theorem 1.7 ([4]) *Assume that $0 < \alpha < n$, $1/q = 1/p - \alpha/n > 0$ and $1 \le p < \infty$. If ϕ satisfies the conditions (1.1) and (1.2), and let $\varphi = \phi^{q/p}$. Then there is a constant $C > 0$ independent of f such that*

$$\|I_\alpha f\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\phi}(\mathbb{R}^n)} \quad \textit{for} \quad p > 1,$$

and

$$\frac{|\{x \in Q : |I_\alpha f(x)| > \lambda\}|}{\varphi(Q)} \le \left(\frac{C}{\lambda}\|f\|_{L^{1,\phi}(\mathbb{R}^n)}\right)^q \quad \textit{for} \quad p = 1,$$

where $\varphi(Q) = \varphi(x_0,r)$ for any cube Q centered at x_0 and of side length r.

Our second goal of the paper is to show the bounded estimates for the operator $I_\alpha^{(m)}$ on the product space of the generalized Morrey spaces $L^{p,\phi}(\mathbb{R}^n)$, which is a extension of Theorem 1.7 in multilinear sense.

Theorem 1.8 *Let $0 < \alpha < mn$, $1 \le p_i < mn/\alpha$, $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$, $\varphi = \left(\prod_{i=1}^m \phi_i^{1/p_i}\right)^q$ with ϕ_i satisfy the condition (1.1) and*

$$\int_r^{+\infty} \frac{\phi_i(x_0,t)}{t^{n-\alpha p_i/m+1}}dt \le C\frac{\phi_i(x_0,r)}{r^{n-\alpha p_i/m}}. \tag{1.3}$$

Then there is a constant $C > 0$ independent of $\mathbf{f}$ such that

$$\left\|I_\alpha^{(m)}(\mathbf{f})\right\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(\mathbb{R}^n)} \quad \textit{for} \quad p_i > 1,$$

and

$$\frac{\left|\left\{x \in Q : \left|I_\alpha^{(m)}(\mathbf{f})(x)\right| > \lambda\right\}\right|}{\varphi(Q)} \leq \frac{C}{\lambda^q}\left(\prod_{i=1}^m \|f_i\|_{L^{p_i,\phi_i}(\mathbb{R}^n)}\right)^q \quad \text{for} \quad p_i \geq 1,$$

where $\varphi(Q) = \varphi(x_0, r)$ *for any cube* Q *centered at* x_0 *and of side length* r*.*

This paper is organized as follows. We will introduce in next section the definitions for the (weak) Herz-Morrey spaces and Herz-Hardy spaces used in our theorems, we will give some remarks on the relations between these spaces. The proof of Theorem 1.3 and 1.4 will be given in Section 3. The proof of Theorem 1.6 will be given in Section 4. In Section 5, we will devote to the estimates of $I_\alpha^{(m)}$ on the generalized Morrey product spaces and show Theorem 1.8.

Throughout this paper, the letter C always remains to denote a positive constant that may varies at each occurrence but is independent of the essential variable.

2. Some notations and definitions

We start with some notations and definitions. Here and in what follows, denote by $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $E_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{E_k}$ for $k \in \mathbb{Z}$ be the characteristic function of the set E_k.

Definition 2.1 ([5]) Let $\sigma \in \mathbb{R}$ and $0 < p,\ q \leq \infty$. The homogeneous Herz space $\dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ is defined to be the following space of functions

$$\dot{K}_q^{\sigma,p}(\mathbb{R}^n) = \left\{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\sigma,p}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\dot{K}_q^{\sigma,p}(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\sigma p}\|f\chi_k\|_{L^q(\mathbb{R}^n)}^p\right\}^{\frac{1}{p}}$$

and the usual modification should be made when $p = \infty$.

Definition 2.2 ([6]) Let $\sigma \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q < \infty$. The weak homogeneous Herz space $W\dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ is defined by

$$W\dot{K}_q^{\sigma,p}(\mathbb{R}^n) = \big\{f: \text{ measurable on } \mathbb{R}^n \text{ and } \|f\|_{W\dot{K}_q^{\sigma,p}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{W\dot{K}_q^{\sigma,p}(\mathbb{R}^n)} = \sup_{\gamma>0} \gamma \Big\{ \sum_{k=-\infty}^{\infty} 2^{k\sigma p} |\{x \in E_k : |f(x)| > \gamma\}|^{\frac{p}{q}} \Big\}^{\frac{1}{p}}$$

and the usual modification should be made when $p = \infty$.

Definition 2.3 ([7]) Let $0 \le \lambda < \infty$, $\sigma \in \mathbb{R}$ and $0 < p,\ q \le \infty$. The homogeneous Herz-Morrey space $M\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)$ is defined by

$$M\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n) = \big\{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \Big\{ \sum_{k=-\infty}^{k_0} 2^{k\sigma p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \Big\}^{\frac{1}{p}}$$

and the usual modifications should be made when $p = \infty$.

Definition 2.4 ([7]) Let $0 \le \lambda < \infty$, $\sigma \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q < \infty$. The weak homogeneous Herz-Morrey space $WM\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)$ is defined by

$$WM\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n) = \big\{f: \text{ measurable on } \mathbb{R}^n \text{ and } \|f\|_{WM\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)} < \infty\big\},$$

where

$$\|f\|_{WM\dot{K}_{p,q}^{\sigma,\lambda}(\mathbb{R}^n)} = \sup_{\gamma>0} \gamma \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \Big\{ \sum_{k=-\infty}^{k=k_0} 2^{k\sigma p} |\{x \in E_k : |f(x)| > \gamma\}|^{\frac{p}{q}} \Big\}^{\frac{1}{p}}$$

and the usual modifications should be made when $p = \infty$.

Definition 2.5 ([8]) Let $\sigma \in \mathbb{R}$ and $0 < p,\ q \le \infty$, then the homogeneous Herz type Hardy spaces $H\dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ are defined by

$$H\dot{K}_q^{\sigma,p}(\mathbb{R}^n) = \big\{f \in S'(\mathbb{R}^n) : G(f) \in \dot{K}_q^{\sigma,p}(\mathbb{R}^n)\big\},$$

and

$$\|f\|_{H\dot{K}_q^{\sigma,p}(\mathbb{R}^n)} = \|G(f)\|_{\dot{K}_q^{\sigma,p}(\mathbb{R}^n)}.$$

Here $S'(\mathbb{R}^n)$ is the space of the temperate distributions on $\mathbb{R}^n$ and $G(f)$ is the grand maximal function of f.

Remark 2.6 We remark that the (weak) homogeneous Herz-Morrey spaces generalizes the (weak) homogeneous Herz spaces. Particularly, we have $M\dot{K}_{p,q}^{\sigma,0}(\mathbb{R}^n) = \dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ and $WM\dot{K}_{p,q}^{\sigma,0}(\mathbb{R}^n) = W\dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ for any $0 < p,\ q < \infty$ and $\sigma \in \mathbb{R}$. Moreover, we have $\dot{K}_p^{\sigma/p,p}(\mathbb{R}^n) = L^p_{|x|^\sigma}(\mathbb{R}^n)$, the weighted L^p spaces, when $1 \le p < \infty$ and $\sigma \in \mathbb{R}$.

Remark 2.7 The appropriate substitute spaces for the Herz spaces are the Herz-type Hardy spaces. Here we figure out that $H\dot{K}_q^{\sigma,p}(\mathbb{R}^n) = \dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ when $-n/q < \sigma < n(1-1/q)$ and $H\dot{K}_q^{\sigma,p}(\mathbb{R}^n) \neq \dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ when $\sigma \ge n(1-1/q)$.

3. The proof of Theorem 1.3 and 1.4

By Remark 2.6, it's clear that Theorem 1.4 is a corollary of Theorem 1.3, so we give the proof of Theorem 1.3 in the section. In order to simplify the proof, we only consider the situation when $m = 2$. Actually, the similar procedure works for all $m \in \mathbb{N}$.

Let f_1, f_2 be functions in $M\dot{K}_{p_1,q_1}^{n(1-1/q_1),\lambda_1}(\mathbb{R}^n)$ and $M\dot{K}_{p_2,q_2}^{n(1-1/q_2),\lambda_2}(\mathbb{R}^n)$, respectively. Obviously, to prove theorem, we only need to find a constant $C > 0$ independent of $\mathbf{f}$ such that

$$\begin{aligned}
&\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big\{ \sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\big|\{x \in E_k : |I_\alpha^{(2)}(\mathbf{f})(x)| > 9\gamma\}\big|^{\frac{p}{q}} \Big\}^{\frac{1}{p}} \\
&\quad \le C\|f_1\|_{M\dot{K}_{p_1,q_1}^{n(1-1/p_1),\lambda_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{n(1-1/p_2),\lambda_2}(\mathbb{R}^n)}
\end{aligned}$$

for all $\gamma > 0$, where and hereinafter we let $1/g = 1/q_1 + 1/q_2$ and $1/l = 1/p_1 + 1/p_2$.

Indeed, we decompose f_i as

$$f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x) =: \sum_{l_i=-\infty}^{\infty} f_{l_i}(x), \quad i = 1, 2, \quad l_i \in \mathbb{Z}.$$

Then

$$\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\big|\big\{x\in E_k : \big|I_\alpha^{(2)}(\mathbf{f})(x)\big| > 9\gamma\big\}\big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}}$$

$$\leq C\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_i=-\infty}^{\infty}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > 9\gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}}$$

$$\leq C\sum_{i=1}^{9}\Big(\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda} G_i(k_0)\Big) := C\sum_{i=1}^{9} H_i,$$

where

$$G_1(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_2(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_3(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=-\infty}^{k-2}\sum_{l_2=k+2}^{\infty}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_4(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=k-1}^{k+1}\sum_{l_2=-\infty}^{k-2}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_5(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=k-1}^{k+1}\sum_{l_2=k-1}^{k+1}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_6(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=k-1}^{k+1}\sum_{l_2=k+2}^{\infty}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_7(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=k+2}^{\infty}\sum_{l_2=-\infty}^{k-2}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_8(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=k+2}^{\infty}\sum_{l_2=k-1}^{k+1}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}},$$

$$G_9(k_0) = \Big\{\sum_{k=-\infty}^{k_0} 2^{kn(2-\frac{1}{g})p}\Big|\Big\{x\in E_k : \sum_{l_1=k+2}^{\infty}\sum_{l_2=k+2}^{\infty}\big|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\big| > \gamma\Big\}\Big|^{\frac{p}{q}}\Big\}^{\frac{1}{p}}.$$

By the symmetry of f_1 and f_2, we will know that the estimate of H_2 is analogous to that of H_4, the estimate of H_3 is similar with that of H_7, the estimate of H_6 is equal with that of H_8. Then we will estimate H_1, H_2, H_3, H_5, H_6 and H_9 respectively.

(i) For H_1, notice that $l_i \le k-2$ for every $i = 1, 2$, and so

$$|x-y_i| \ge |x|-|y_i| > 2^{k-1}-2^{l_i} > 2^{k-1}-2^{k-2} = 2^{k-2}, \quad \text{for } x \in E_k,\ y_i \in E_{l_i}.$$

Thus, for $x \in E_k$, we get

$$\left|I_\alpha^{(2)}(f_{l_1}, f_{l_2})(x)\right| \le C2^{-k(n-\frac{\alpha}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-k(n-\frac{\alpha}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

From the Chebychev inequality, the estimates above and the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\left\{x \in E_k : \sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\left|I_\alpha^{(2)}(f_{l_1}, f_{l_2})(x)\right| > \gamma\right\}\right|^{\frac{1}{q}} \\
&\quad \le C\gamma^{-1}\left\|\left(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}\left|I_\alpha^{(2)}(f_{l_1}, f_{l_2})\right|\chi_k\right)\right\|_{L^q(\mathbb{R}^n)} \\
&\quad \le C\gamma^{-1}\|\chi_k\|_{L^q(\mathbb{R}^n)}\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{k-2}2^{-k(n-\frac{\alpha}{2})}\|f_{l_i}\|_{L^1(\mathbb{R}^n)}\right) \\
&\quad \le C\gamma^{-1}\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{k-2}2^{n(l_i-k)(1-\frac{1}{q_i})}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\right). \qquad (3.1)
\end{aligned}$$

Therefore, recalling the fact that $(\sum|a_i|)^s \le \sum|a_i|^s$ for any $0 < s \le 1$, and using the Cauchy inequality, and since $p > l$ and $0 < p_i \le 1$, we have

$$\begin{aligned}
H_1 &\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\prod_{i=1}^{2}\left(2^{kn(1-\frac{1}{q}_i)}\sum_{l_i=-\infty}^{k-2}2^{n(l_i-k)(1-\frac{1}{q_i})}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\right)^p\right\}^{\frac{1}{p}} \\
&\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{k-2}2^{nl_i(1-\frac{1}{q_i})}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\right)^l\right\}^{\frac{1}{l}}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\prod_{i=1}^{2}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_i=-\infty}^{k-2}2^{nl_i(1-\frac{1}{q_i})}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\right)^{p_i}\right\}^{\frac{1}{p_i}}\\
&\leq C\prod_{i=1}^{2}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}\sum_{l_i=-\infty}^{k-2}2^{l_in(1-\frac{1}{q_i})p_i}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}^{p_i}\right\}^{\frac{1}{p}_i}\\
&\leq C\prod_{i=1}^{2}\|f_i\|_{M\dot{K}^{n(1-1/q_i),\lambda_i}_{p_i,q_i}(\mathbb{R}^n)}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_i}\left\{\sum_{k=-\infty}^{k_0}2^{(k-2)\lambda_ip_i}\right\}^{\frac{1}{p}_i}\\
&\leq C\|f_1\|_{M\dot{K}^{n(1-1/q_1),\lambda_1}_{p_1,q_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}.
\end{aligned}
$$

(ii) Now we consider the estimate of the term H_2. Similar to the the estimates of H_1, we have $l_1 \leq k-2$ and $k-1 < l_2 \leq k+1$, then

$$|(x-y_1,x-y_2)| \geq |x-y_1| \geq |x|-|y_1| > 2^{k-2}, \quad \text{for } x\in E_k,\ y_i\in E_{l_i},\ i=1,2.$$

and so

$$\left|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\right| \leq C2^{-k(n-\frac{\alpha}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-k(n-\frac{\alpha}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

Thus, similar as (3.1), we get by the Chebychev inequality that

$$
\begin{aligned}
&\left|\left\{x\in E_k:\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{k+1}\left|I_\alpha^{(2)}(f_{l_1},f_{l_2})(x)\right|>\gamma\right\}\right|^{\frac{1}{q}}\\
&\quad\leq C\gamma^{-1}\sum_{l_1=-\infty}^{k-2}2^{n(l_1-k)(1-\frac{1}{q_1})}\|f_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\sum_{l_2=k-1}^{k+1}2^{n(l_2-k)(1-\frac{1}{q_2})}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Using analogous arguments as that of H_1, we can deduce that

$$
\begin{aligned}
H_2 \leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda_1}&\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_1=-\infty}^{k-2}2^{nl_1(1-\frac{1}{q_1})}\|f_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\right)^{p_1}\right\}^{\frac{1}{p_1}}\\
&\times 2^{-k_0\lambda_2}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{l_2=k-1}^{k+1}2^{nl_2(1-\frac{1}{q_2})}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_2}\right\}^{\frac{1}{p_2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\Big\{\sum_{k=-\infty}^{k_0}\sum_{l_1=-\infty}^{k-2} 2^{l_1 n(1-\frac{1}{q_1})p_1}\|f_{l_1}\|^{p_1}_{L^{q_1}(\mathbb{R}^n)}\Big\}^{\frac{1}{p_1}}\\
&\quad\times 2^{-k_0\lambda_2}\Big\{\sum_{k=-\infty}^{k_0+1} 2^{kn(1-\frac{1}{q}_2)p_2}\|f_2\chi_k\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Big\}^{\frac{1}{p}_2}\\
&\leq C\|f_1\|_{M\dot{K}^{n(1-1/q_1),\lambda_1}_{p_1,q_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}.
\end{aligned}
$$

(iii) To estimate H_3. Since $l_1 \leq k-2$ and $l_2 \geq k+2$, we first see that

$$
\begin{gathered}
|x-y_1| \geq |x| - |y_1| > 2^{k-2}, \quad \text{for } x\in E_k,\ y_1\in E_{l_1},\\
|x-y_2| \geq |y_2| - |x| > 2^{l_2-1} - 2^k > 2^{l_2-1} - 2^{l_2-2} > 2^{l_2-2},\\
\text{for } x\in E_k,\ y_2\in E_{l_2}.
\end{gathered}
$$

Thus

$$
\big|I^{(2)}_\alpha(f_{l_1}, f_{l_2})(x)\big| \leq C2^{-k(n-\frac{\alpha}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\alpha}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.
$$

Therefore, by the Chebychev inequality and the Cauchy inequality again, and since $0 < p_1,\, p_2 \leq 1$, we can use the same method as for H_1 and H_2 to show that

$$
\begin{aligned}
H_3 &\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\Big\{\sum_{k=-\infty}^{k_0}\Big(2^{kn(1-\frac{1}{q}_1)}\sum_{l_i=-\infty}^{k-2} 2^{n(l_1-k)(1-\frac{1}{q_1})}\|f_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\Big)^{p_1}\Big\}^{\frac{1}{p_1}}\\
&\quad\times 2^{-k_0\lambda_2}\Big\{\sum_{k=-\infty}^{k_0}\Big(2^{kn(1-\frac{1}{q}_2)}\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-\frac{n}{q_2})}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}\Big)^{p_2}\Big\}^{\frac{1}{p_2}}\\
&\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\Big\{\sum_{k=-\infty}^{k_0}\sum_{l_1=-\infty}^{k-2} 2^{l_1 n(1-\frac{1}{q}_1)p_1}\|f_{l_1}\|^{p_1}_{L^{q_1}(\mathbb{R}^n)}\Big\}^{\frac{1}{p}_1}\\
&\quad\times 2^{-k_0\lambda_2}\Big\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2+l_2 n(1-\frac{1}{q}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Big\}^{\frac{1}{p_2}}\\
&=: \sup_{k_0\in\mathbb{Z}} H_{31}(k_0)\times H_{32}(k_0).
\end{aligned}
$$

From the estimates of H_1, we know $H_{31}(k_0) \leq C\|f_1\|_{M\dot{K}^{n(1-1/q_1),\lambda_1}_{p_1,q_1}(\mathbb{R}^n)}$, so we only need to show that $H_{32}(k_0) \leq C\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}$.

For $H_{32}(k_0)$, we write

$$\begin{aligned}H_{32}(k_0) \leq\ & 2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{k_0} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2+l_2n(1-\frac{1}{q}{}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Bigg\}^{\frac{1}{p_2}}\\ &+2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k_0+1}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2+l_2n(1-\frac{1}{q}{}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Bigg\}^{\frac{1}{p_2}}\\ &:= H^1_{32}(k_0)+H^2_{32}(k_0).\end{aligned}$$

Firstly, the fact $0 < \alpha < 2n$ yields

$$\begin{aligned}H^1_{32}(k_0) &\leq 2^{-k_0\lambda_2}\Bigg\{\sum_{l_2=-\infty}^{k_0}\sum_{k=-\infty}^{l_2-2} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2+l_2n(1-\frac{1}{q}{}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Bigg\}^{\frac{1}{p_2}}\\ &\leq 2^{-k_0\lambda_2}\Bigg\{\sum_{l_2=-\infty}^{k_0} 2^{l_2n(1-\frac{1}{q}{}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Bigg(\sum_{s=2}^{\infty}2^{s(\alpha/2-n)p_2}\Bigg)\Bigg\}^{\frac{1}{p}{}_2}\\ &\leq C2^{-k_0\lambda_2}\Bigg\{\sum_{l_2=-\infty}^{k_0} 2^{l_2n(1-\frac{1}{q}{}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Bigg\}^{\frac{1}{p}{}_2}\\ &\leq C\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}.\end{aligned}$$

Secondly, one sees from the definition of Herz-Morrey space that $2^{l_2n(1-1/q_2)}\|f_{l_2}\|_{L^{q_2}} \leq 2^{l_2\lambda_2}\|f\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}}$ and then, by the fact $\lambda_2 < n-\alpha/2$, we get

$$\begin{aligned}H^2_{32}(k_0) &\leq 2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2}2^{l_2n(1-\frac{1}{q}{}_2)p_2}\|f_{l_2}\|^{p_2}_{L^{q_2}(\mathbb{R}^n)}\Bigg\}^{\frac{1}{p_2}}\\ &\leq C\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}\\ &\quad\times 2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2}2^{l_2\lambda_2p_2}\Bigg\}^{\frac{1}{p}{}_2}\end{aligned}$$

$$
\begin{aligned}
&\le C\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)} \\
&\quad\times 2^{-k_0\lambda_2}\Bigg(\sum_{k=-\infty}^{k_0} 2^{k(n-\frac{\alpha}{2})p_2}\sum_{l_2=k+2}^{\infty} 2^{l_2(\lambda_2-n+\frac{\alpha}{2})p_2}\Bigg)^{\frac{1}{p_2}} \\
&\le C\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)},
\end{aligned}
$$

as desired.

(iv) To estimate the term H_5, by Theorem 1.1, the weak L^q-boundedness for $I_\alpha^{(2)}$, we note that

$$
\big\|I_\alpha^{(2)}(f_{l_1}f_{l_2})\chi_k\big\|_{L^{q,\infty}(\mathbb{R}^n)} \le C\|f_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}.
$$

Thus, noting $p > l$, then the similar arguments shows that

$$
\begin{aligned}
H_5 &\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg\{\sum_{k=-\infty}^{k_0}\prod_{i=1}^{2}\Bigg(2^{kn(1-1/q_i)}\sum_{l_i=k-1}^{k+1}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\Bigg)^{p}\Bigg\}^{\frac{1}{p}} \\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Bigg\{\sum_{k=-\infty}^{k_0}\prod_{i=1}^{2}\Bigg(2^{kn(1-1/q_i)}\sum_{l_i=k-1}^{k+1}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\Bigg)^{l}\Bigg\}^{\frac{1}{l}} \\
&\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\prod_{i=1}^{2}\Bigg\{\sum_{k=-\infty}^{k_0}\Bigg(2^{kn(1-1/q_i)}\sum_{l_i=k-1}^{k+1}\|f_{l_i}\|_{L^{q_i}(\mathbb{R}^n)}\Bigg)^{p_i}\Bigg\}^{\frac{1}{p_i}} \\
&\le C\|f_1\|_{M\dot{K}^{n(1-1/q_1),\lambda_1}_{p_1,q_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}.
\end{aligned}
$$

(v) To estimate the term H_6, we note $k-1 < l_1 \le k+1$ and $l_2 \ge k+2$, and then

$$
|(x-y_1, x-y_2)| \ge |x-y_2| > 2^{l_2-2}, \quad \text{for } x\in E_k,\ y_1\in E_{l_1}\ y_2\in E_{l_2}.
$$

Thus, for $x \in E_k$, there is

$$
\big|I_\alpha^{(2)}(f_{l_1}, f_{l_2})(x)\big| \le C2^{-k(n-\frac{\alpha}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\alpha}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.
$$

Hence, we obtain

$$
\begin{aligned}
H_6 \le\; & C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\Bigg\{\sum_{k=-\infty}^{k_0} 2^{kn(1-\frac{1}{q}_1)p_1}\Bigg(\sum_{l_1=k-1}^{k+1} 2^{n(l_1-k)(1-\frac{1}{q_1})}\|f_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\Bigg)^{p_1}\Bigg\}^{\frac{1}{p}_1}\\
& \times 2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0} 2^{kn(1-\frac{1}{q}_2)p_2}\Bigg(\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-\frac{n}{q_2})}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}\Bigg)^{p_2}\Bigg\}^{\frac{1}{p}_2}\\
\le\; & C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\Bigg\{\sum_{k=-\infty}^{k_0+1} 2^{kn(1-\frac{1}{q}_1)p_1}\|f_1\chi_k\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\Bigg\}^{\frac{1}{p}_1}\\
& \times 2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0}\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-n)p_2+l_2n(1-\frac{1}{q}_2)p_2}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}^{p_2}\Bigg\}^{\frac{1}{p}_2}\\
\le\; & C\|f_1\|_{M\dot{K}^{n(1-1/q_1),\lambda_1}_{p_1,q_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{n(1-1/q_2),\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}.
\end{aligned}
$$

(vi) Finally, we have to estimate the term H_9, we note $l_i \ge k+2$ and

$$|x-y_i| > 2^{l_i-2}, \quad \text{for } x\in E_k,\ y_i\in E_{l_i}\ i=1,2.$$

Similarly, for $x\in E_k$, there is

$$\big|I^{(2)}_\alpha(f_{l_1},f_{l_2})(x)\big| \le C2^{-l_1(n-\frac{\alpha}{2})}\|f_{l_1}\|_{L^1(\mathbb{R}^n)}2^{-l_2(n-\frac{\alpha}{2})}\|f_{l_2}\|_{L^1(\mathbb{R}^n)}.$$

Moreover, by Hölder inequality, it can be deduced that

$$
\begin{aligned}
H_9 \le\; & \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda_1}\Bigg\{\sum_{k=-\infty}^{k_0} 2^{kn(1-\frac{1}{q}_1)p_1}\Bigg(\sum_{l_1=k+2}^{\infty} 2^{(l_1-k)(\frac{\alpha}{2}-\frac{n}{q_1})}\|f_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\Bigg)^{p_1}\Bigg\}^{\frac{1}{p}_1}\\
& \times 2^{-k_0\lambda_2}\Bigg\{\sum_{k=-\infty}^{k_0} 2^{kn(1-\frac{1}{q}_2)p_2}\Bigg(\sum_{l_2=k+2}^{\infty} 2^{(l_2-k)(\frac{\alpha}{2}-\frac{n}{q_2})}\|f_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}\Bigg)^{p_2}\Bigg\}^{\frac{1}{p}_2}\\
=\; & \sup_{k_0\in\mathbb{Z}} H_{91}(k_0)\times H_{92}(k_0) \le C\|f_1\|_{M\dot{K}^{\sigma_1,\lambda_1}_{p_1,q_1}(\mathbb{R}^n)}\|f_2\|_{M\dot{K}^{\sigma_2,\lambda_2}_{p_2,q_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Here the estimate of $H_{91}(k_0)$ and $H_{92}(k_0)$ is similar as that of $H_{32}(k_0)$.

Finally, a combination for the estimates of H_i $(i=1,2,\dots,9)$ finishes the proof of Theorem 1.3.

4. The proof of Theorem 1.6

It is well known that the Herz type Hardy spaces have the central atomic decomposition characterization, which make it convenient to study the boudedness of operators on these spaces.

Definition 4.1 Let $\sigma \in \mathbb{R}$ and $0 < q \leq \infty$ and $s \geq [\sigma - n(1 - 1/q)]$. A function a on $\mathbb{R}^n$ is called a central (σ, q) atom if

(1) $supp(a) \subset B(0, r)$ for some $r > 0$;
(2) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(0, r)|^{-\sigma/n}$;
(3) $\int_{\mathbb{R}^n} a(x)x^\nu dx = 0$ for $\nu \leq s$.

where $B(0, r)$ denotes a ball in $\mathbb{R}^n$ with center origin and radius r.

Lemma 4.2 ([8]) *Let $0 < p < \infty$, $1 < q < \infty$ and $\sigma \geq n(1 - 1/q)$. A function f on $\mathbb{R}^n$ belongs to $H\dot{K}_q^{\sigma,p}(\mathbb{R}^n)$ if and only if it can be written as $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in distributional sense with each a_k a central (σ, q) atom supported on $B_k = B(0, 2^k)$ and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Moreover*

$$\|f\|_{H\dot{K}_q^{\sigma,p}(\mathbb{R}^n)} \sim \inf \left\{ \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p} \right\}$$

with the infimum taken over all decomposition of f.

The proof of Theorem 1.6. In order to simplify the proof, we also only consider the situation when $m = 2$. Without loss of generality, by Lemma 4.2, we can write

$$f_i = \sum_{l_i=-\infty}^{\infty} \lambda_{l_i} a_{l_i}, \quad i = 1, 2,$$

where a_{l_i} are central (σ_i, q_i)-atoms supported on $B_{l_i} = B(0, 2^{l_i})$. Also, for fixed constant N such that $N > \sigma - n(2 - 1/g)$, we assume that a_{l_i} satisfy the cancellation conditions up to order N. We can find s_1 and s_2 such that $s_1 > \sigma_1 - n(1 - 1/q_1)$, $s_2 > \sigma_2 - n(1 - 1/q_2)$ and $s_1 + s_2 = N$. Then we have following decomposition,

$$
\begin{aligned}
&\big\|I_\alpha^{(2)}(\mathbf{f})\big\|_{\dot K_q^{\sigma,p}(\mathbb{R}^n)} \le C(J_1+J_2+J_3+J_4)\\
&\quad =: C\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\sigma p}\bigg(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=-\infty}^{k-2}|\lambda_{l_1}||\lambda_{l_2}|\big\|I_\alpha^{(2)}(a_{l_1},a_{l_2})\chi_k\big\|_{L^q(\mathbb{R}^n)}\bigg)^p\bigg\}^{\frac1p}\\
&\qquad + C\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\sigma p}\bigg(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{\infty}|\lambda_{l_1}||\lambda_{l_2}|\big\|I_\alpha^{(2)}(a_{l_1},a_{l_2})\chi_k\big\|_{L^q(\mathbb{R}^n)}\bigg)^p\bigg\}^{\frac1p}\\
&\qquad + C\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\sigma p}\bigg(\sum_{l_1=k-1}^{\infty}\sum_{l_2=-\infty}^{k-2}|\lambda_{l_1}||\lambda_{l_2}|\big\|I_\alpha^{(2)}(a_{l_1},a_{l_2})\chi_k\big\|_{L^q(\mathbb{R}^n)}\bigg)^p\bigg\}^{\frac1p}\\
&\qquad + C\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\sigma p}\bigg(\sum_{l_1=k-1}^{\infty}\sum_{l_2=k-1}^{\infty}|\lambda_{l_1}||\lambda_{l_2}|\big\|I_\alpha^{(2)}(a_{l_1},a_{l_2})\chi_k\big\|_{L^q(\mathbb{R}^n)}\bigg)^p\bigg\}^{\frac1p}.
\end{aligned}
$$

For J_1, we use the fact that $l_1 \le k-2$, $l_2 \le k-2$ and $x \in E_k$. Without loss of generality, for fixed l_1 and l_2, we assume $l_1 \le l_2$. Since a_{l_1} has zero vanishing moments up to order N, we can subtract the Taylor polynomial $P_0^N(x,\cdot,y_2)$ of the function $K(x,y_1,y_2) = |(x-y_1,x-y_2)|^{-2n+\alpha}$ at the origin to obtain that

$$
\begin{aligned}
&\big\|I_\alpha^{(2)}(a_{l_1},a_{l_2})\chi_k\big\|_{L^q(\mathbb{R}^n)}\\
&\quad\le \bigg\{\int_{E_k}\bigg|\int_{B_{l_2}} a_{l_2}(y_2)\int_{B_{l_1}} a_{l_1}(y_1)\big(K(x,y_1,y_2)-P_0^N(x,\cdot,y_2)\big)dy_1dy_2\bigg|^q dx\bigg\}^{\frac1q}\\
&\quad\le \bigg\{\int_{E_k}\bigg(\int_{B_{l_2}}|a_{l_2}(y_2)|\int_{B_{l_1}}|a_{l_1}(y_1)|\\
&\qquad\qquad \times\sum_{|\beta|=N+1}\big|\big(\partial_{y_1}^\beta K\big)(x,\xi,y_2)\big|\frac{|y_1|^{|\beta|}}{\beta!}dy_1dy_2\bigg)^q dx\bigg\}^{\frac1q}
\end{aligned}
$$

for some ξ on the line segment joining y_1 to the origin by Taylor's theorem. We remark that $|\xi| \le |y_1| \le 2^{l_1} \le 2^{k-2} < \frac12|x|$ and so $|x-\xi| \ge |x|-|\xi| \ge \frac12|x|$. Similarly we have $|x-y_2| \ge \frac12|x|$. Hence

$$
\big|\big(\partial_{y_1}^\beta K\big)(x,y_1,y_2)\big| \le \frac{C_\beta}{|(x-\xi,x-y_2)|^{2n-\alpha+|\beta|}} \le \frac{C_\beta}{|x|^{2n-\alpha+|\beta|}}. \tag{4.1}
$$

From this and the size of atom a_{l_i}, by the Minkowski inequality and the Hölder inequality, we give the following estimates for the expression above:

$$\begin{aligned}
&\left\{ \int_{E_k} \left[\int_{B_{l_2}} |a_{l_2}(y_2)| \int_{B_{l_1}} |a_{l_1}(y_1)| \frac{|y_1|^{N+1}}{|x|^{2n-\alpha+N+1}} dy_1 dy_2 \right]^q dx \right\}^{\frac{1}{q}} \\
&\quad \le C \int_{B_{l_2}} |a_{l_2}(y_2)| \int_{B_{l_1}} |a_{l_1}(y_1)||y_1|^{N+1} \left\{ \int_{E_k} \frac{1}{|x|^{(2n-\alpha+N+1)q}} dx \right\}^{\frac{1}{q}} dy_1 dy_2 \\
&\quad \le C 2^{-k(2n-\alpha+N-\frac{n}{q})+l_1 N} \int_{B_{l_2}} |a_{l_2}(y_2)| \int_{B_{l_1}} |a_{l_1}(y_1)| dy_1 dy_2 \\
&\quad \le C 2^{-k(2n-\alpha+s_1+s_2-\frac{n}{q})+l_1 s_1 + l_2 s_2} 2^{l_1 n(1-\frac{1}{q}_1)} 2^{l_2 n(1-\frac{1}{q}_2)} \\
&\qquad\qquad \times \|a_{l_1}\|_{L^{q_1}(\mathbb{R}^n)} \|a_{l_2}\|_{L^{q_2}(\mathbb{R}^n)} \\
&\quad \le C 2^{-k(2n-\alpha+s_1+s_2-\frac{n}{q})} 2^{l_1[n(1-\frac{1}{q}_1)-\sigma_1+s_1]} 2^{l_2[n(1-\frac{1}{q}_2)-\sigma_2+s_2]} \\
&\quad \le C 2^{-k\sigma} 2^{(l_1-k)[n(1-\frac{1}{q}_1)-\sigma_1+s_1]} 2^{(l_2-k)[n(1-\frac{1}{q}_2)-\sigma_2+s_2]}.
\end{aligned}$$

Thus, since $l \le p$ and $1/l = 1/p_1 + 1/p_2$, by the Cauchy inequality we have

$$\begin{aligned}
J_1 &\le C \left\{ \sum_{k=-\infty}^{\infty} 2^{k\sigma l} \left(\sum_{l_1=-\infty}^{k-2} \sum_{l_2=-\infty}^{k-2} |\lambda_{l_1}||\lambda_{l_2}| \left\| I_\alpha^{(2)}(a_{l_1}, a_{l_2}) \chi_k \right\|_{L^q(\mathbb{R}^n)} \right)^l \right\}^{\frac{1}{l}} \\
&\le C \left\{ \sum_{k=-\infty}^{\infty} \prod_{i=1}^{2} \left(\sum_{l_i=-\infty}^{k-2} |\lambda_{l_i}| 2^{(l_i-k)\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right)^l \right\}^{\frac{1}{l}} \\
&\le C \prod_{i=1}^{2} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{l_i=-\infty}^{k-2} |\lambda_{l_i}| 2^{(l_i-k)\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right)^{p_i} \right\}^{\frac{1}{p_i}} \\
&:= C(J_{11} \times J_{12}).
\end{aligned}$$

Notice that $n(1-1/q_i) - \sigma_i + s_i > 0$ for $i = 1, 2$. If $0 < p_i \le 1$, then

$$J_{1i} \le \left\{ \sum_{k=-\infty}^{\infty} \sum_{l_i=-\infty}^{k-2} |\lambda_{l_i}|^{p_i} 2^{(l_i-k)p_i\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right\}^{\frac{1}{p}_i}$$

$$\leq \left\{ \sum_{l_i=-\infty}^{\infty} |\lambda_{l_i}|^{p_i} \sum_{k=l_i+2}^{\infty} 2^{(l_i-k)p_i\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right\}^{\frac{1}{p}_i}$$

$$\leq C\left(\sum_{l_i=-\infty}^{\infty} |\lambda_{l_i}|^{p_i} \right)^{\frac{1}{p}_i} \leq C\|f_i\|_{H\dot{K}_{q_i}^{\sigma_i,p_i}(\mathbb{R}^n)}.$$

If $1 < p_i < \infty$, by the Hölder inequality with exponents p_i, we obtain

$$J_{1i} \leq \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{l_i=-\infty}^{k-2} |\lambda_{l_i}|^{p_i} 2^{(l_i-k)\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right) \right.$$
$$\left. \times \left(\sum_{l_i=-\infty}^{k-2} 2^{(l_i-k)\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right)^{p_i-1} \right\}^{\frac{1}{p}_i}$$

$$\leq \left\{ \sum_{k=-\infty}^{\infty} |\lambda_{l_i}|^{p_i} \left(\sum_{l_i=-\infty}^{k-2} 2^{(l_i-k)\left[n(1-\frac{1}{q}_i)-\sigma_i+s_i\right]} \right) \right\}^{\frac{1}{p}_i}$$

$$\leq C\left(\sum_{l_i=-\infty}^{\infty} |\lambda_{l_i}|^{p_i} \right)^{\frac{1}{p}_i} \leq C\|f_i\|_{H\dot{K}_{q_i}^{\sigma_i,p_i}(\mathbb{R}^n)}.$$

A combination of the estimates above for J_{1i} finish the estimates of J_1. Thus,

$$J_1 \leq C\|f_1\|_{H\dot{K}_{q_1}^{\sigma_1,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot{K}_{q_2}^{\sigma_2,p_2}(\mathbb{R}^n)}.$$

Now we treat term J_2. Observing that $l_1 \leq k-2$, $l_2 \geq k-1$ and $x \in E_k$. Similar to the estimates for J_1

$$\left\| I_\alpha^{(2)}(a_{l_1}, a_{l_2})\chi_k \right\|_{L^q(\mathbb{R}^n)}$$
$$\leq C2^{-k(2n-\alpha+N-\frac{n}{q})+l_1 N} \int_{B_{l_2}} |a_{l_2}(y_2)| \int_{B_{l_1}} |a_{l_1}(y_1)| dy_1 dy_2$$
$$\leq C2^{-k(2n-\alpha+s_1+s_2-\frac{n}{q})+l_1 s_1+l_1 s_2} 2^{l_1 n(1-\frac{1}{q}_1)} 2^{l_2 n(1-\frac{1}{q}_2)}$$
$$\times \|a_{l_1}\|_{L^{q_1}(\mathbb{R}^n)} \|a_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}$$
$$\leq C2^{-k(2n-\alpha+s_1-\frac{n}{q})} 2^{l_1[n(1-\frac{1}{q}_1)-\sigma_1+s_1]} 2^{l_2[n(1-\frac{1}{q}_2)-\sigma_2]}$$
$$\leq C2^{-k\sigma} 2^{(l_1-k)[n(1-\frac{1}{q}_1)-\sigma_1+s_1]} 2^{(l_2-k)[n(1-\frac{1}{q}_2)-\sigma_2]}.$$

By the fact $l/p < 1$ and the Cauchy inequality with exponents l,

$$
\begin{aligned}
J_2 &\le C\bigg\{\sum_{k=-\infty}^{\infty} 2^{k\sigma l}\bigg(\sum_{l_1=-\infty}^{k-2}\sum_{l_2=k-1}^{\infty}|\lambda_{l_1}||\lambda_{l_2}|\big\|I_\alpha^{(2)}(a_{l_1},a_{l_2})\chi_k\big\|_{L^q(\mathbb{R}^n)}\bigg)^l\bigg\}^{\frac{1}{l}}\\
&\le C\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{l_1=-\infty}^{k-2}|\lambda_{l_1}|2^{(l_1-k)[n(1-\frac{1}{q}_1)-\sigma_1+s_1]}\bigg)^l\\
&\qquad\qquad \times\bigg(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|2^{(l_2-k)[n(1-\frac{1}{q}_2)-\sigma_2]}\bigg)^l\bigg\}^{\frac{1}{l}}\\
&\le C\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{l_1=-\infty}^{k-2}|\lambda_{l_1}|2^{(l_1-k)[n(1-\frac{1}{q}_1)-\sigma_1+s_1]}\bigg)^{p_1}\bigg\}^{\frac{1}{p_1}}\\
&\qquad\qquad \times\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|2^{(l_2-k)[n(1-\frac{1}{q}_2)-\sigma_2]}\bigg)^{p_2}\bigg\}^{\frac{1}{p_2}}\\
&=: J_{21}\times J_{22}.
\end{aligned}
$$

We observe that J_{21} is equal to J_{11}, and so we have $J_{21} \le C\|f_1\|_{H\dot{K}_{q_1}^{\sigma_1,p_1}(\mathbb{R}^n)}$ with constant C independent of f_1.

Noting $n(1-1/q_2)-\sigma_2 < 0$. Then, similar with the estimates of J_{1i}, we obtain

$$
\begin{aligned}
J_{22} &\le C\bigg\{\sum_{k=-\infty}^{\infty}\bigg(\sum_{l_2=k-1}^{\infty}|\lambda_{l_2}|2^{(l_2-k)[n(1-\frac{1}{q}_2)-\sigma_2]}\bigg)^{p_2}\bigg\}^{\frac{1}{p_2}}\\
&\le C\begin{cases}\bigg\{\displaystyle\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\sum_{k=-\infty}^{l_2+1}2^{(l_2-k)p_2\left[n(1-\frac{1}{q}_2)-\sigma_2\right]}\bigg\}^{\frac{1}{p}_2} & \text{for } 0<p_2\le 1,\\ \bigg\{\displaystyle\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\bigg(\sum_{k=-\infty}^{l_2+1}2^{(l_2-k)\left[n(1-\frac{1}{q}_2)-\sigma_2\right]}\bigg)\bigg\}^{\frac{1}{p}_2} & \text{for } 1<p_2<\infty,\end{cases}\\
&\le C\bigg(\sum_{l_2=-\infty}^{\infty}|\lambda_{l_2}|^{p_2}\bigg)^{\frac{1}{p}_2} \le C\|f_2\|_{H\dot{K}_{q_2}^{\sigma_2,p_2}(\mathbb{R}^n)}.
\end{aligned}
$$

Thus,

$$J_2 \le C\|f_1\|_{H\dot{K}_{q_1}^{\sigma_1,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot{K}_{q_2}^{\sigma_2,p_2}(\mathbb{R}^n)}.$$

From the analogous argumentation of J_2, we will get the estimate of J_3, at the same time, it is should be pointed that we should use the cancellation condition of a_{l_2} subtracting the Taylor polynomial $P_0^N(x, y_1, \cdot)$ of the function $K(x, y_1, y_2) = |x - y_1, x - y_2|^{-2n+\alpha}$ at the origin.

As for J_4, by Theorem 1.1, we have

$$\begin{aligned}\left\|I_\alpha^{(2)}(a_{l_1}, a_{l_2})\chi_k\right\|_{L^q(\mathbb{R}^n)} &\le C\|a_{l_1}\|_{L^{q_1}(\mathbb{R}^n)}\|a_{l_2}\|_{L^{q_2}(\mathbb{R}^n)}\\ &\le C2^{-k\sigma}2^{(k-l_1)\sigma_1}2^{(k-l_2)\sigma_2}.\end{aligned}$$

Then by the Hölder inequality, the fact $l/p < 1$ and $\sigma_i > 0$, we get

$$\begin{aligned}J_4 &\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\sigma l}\left(\sum_{l_1=k-1}^{\infty}\sum_{l_2=k-1}^{\infty}|\lambda_{l_1}||\lambda_{l_2}|\left\|I_\alpha^{(2)}(a_{l_1}, a_{l_2})\chi_k\right\|_{L^q(\mathbb{R}^n)}\right)^l\right\}^{\frac{1}{l}}\\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l_1=k-1}^{\infty}\sum_{l_2=k-1}^{\infty}|\lambda_{l_1}||\lambda_{l_2}|2^{(k-l_1)\sigma_1}2^{(k-l_2)\sigma_2}\right)^l\right\}^{\frac{1}{l}}\\
&\le C\prod_{i=1}^{2}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l_i=k-1}^{\infty}|\lambda_{l_i}|2^{(k-l_i)\sigma_i}\right)^{p_i}\right\}^{\frac{1}{p_i}}\\
&\le C\begin{cases}\displaystyle\prod_{i=1}^{2}\left\{\sum_{k=-\infty}^{\infty}|\lambda_{l_i}|\left(\sum_{l_i=k-1}^{\infty}2^{(k-l_i)\sigma_i p_i}\right)\right\}^{\frac{1}{p_i}} & \text{for } 0 < p_i \le 1,\\ \displaystyle\prod_{i=1}^{2}\left\{\sum_{k=-\infty}^{\infty}|\lambda_{l_i}|^{p_i}\left(\sum_{l_i=k-1}^{\infty}2^{(k-l_i)\sigma_i}\right)\right\}^{\frac{1}{p_i}} & \text{for } 1 < p_i < \infty,\end{cases}\\
&\le C\prod_{i=1}^{2}\left(\sum_{l_i=-\infty}^{\infty}|\lambda_{l_i}|^{p_i}\right)^{\frac{1}{p_i}} \le C\|f_1\|_{H\dot{K}_{q_1}^{\sigma_1,p_1}(\mathbb{R}^n)}\|f_2\|_{H\dot{K}_{q_2}^{\sigma_2,p_2}(\mathbb{R}^n)}.\end{aligned}$$

This finishes the proof of the Theorem 1.6.

5. The proof of Theorem 1.8

Let us begin with the definition of the generalized Morrey space.

Definition 5.1 ([4]) Let $1 \leq p < \infty$ and ϕ be a function from $\mathbb{R}^n \times \mathbb{R}^+$ to $\mathbb{R}^+$, then the generalized Morrey space, $L^{p,\phi} = L^{p,\phi}(\mathbb{R}^n)$, is defined by

$$L^{p,\phi}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \sup_Q \left(\frac{1}{\phi(Q)} \int_Q |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

with norm $\|f\|_{L^{p,\phi}(\mathbb{R}^n)}$ is given by

$$\|f\|_{L^{p,\phi}(\mathbb{R}^n)} = \sup_Q \left(\frac{1}{\phi(Q)} \int_Q |f(x)|^p dx \right)^{\frac{1}{p}},$$

where $\phi(Q) = \phi(x, r)$ if the cube Q centers at x and has side length r.

It is well known that if $\phi(Q) = 1$, $L^{p,\phi}(\mathbb{R}^n)$ is the space $L^p(\mathbb{R}^n)$. If $\phi(Q) = |Q|$, then $L^{p,\phi}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. And if $\phi(Q) = |Q|^{\mu/n}$ with $\mu \geq 0$, then $L^{p,\phi}(\mathbb{R}^n)$ is the classical Morrey space $L^{p,\mu}(\mathbb{R}^n)$.

Let $M(f)(x)$ denote the Hardy-Littlewood maximal function, we will use the following lemmas.

Lemma 5.2 ([4]) *Let $0 < \delta \leq 1$, assume that there are constants C_1, C_2 and C_3 such that ϕ satisfies the condition* (1.1) *and the following condition*

$$\int_r^{+\infty} \frac{\phi(x_0, t)}{t^{n\delta+1}} dt \leq C_3 \frac{\phi(x_0, r)}{r^{n\delta}}. \tag{5.1}$$

Then, for $1 \leq p < \infty$ and any cube Q,

$$\int_{\mathbb{R}^n} |f(x)|^p (M\chi_Q(x))^\delta dx \leq C\phi(Q) \|f\|^p_{L^{p,\phi}},$$

where the constant C depends only on C_1, C_2 and C_3.

Using integration by part, see Lemma 2 in [4], we have that

Lemma 5.3 *Let $\beta \geq 0$, if there is a constant $C_0 > 0$ such that*

$$\int_r^{+\infty} \frac{\phi(x_0, t)}{t^{\beta+1}} dt \leq C \frac{\phi(x_0, r)}{r^\beta},$$

then we can take $C' = C/(1 - C\varepsilon)$ *for small real* $\varepsilon > 0$ *such that*

$$\int_r^{+\infty} \frac{\phi(x_0, t)}{t^{\beta+1-\varepsilon}} dt \le C' \frac{\phi(x_0, r)}{r^{\beta-\varepsilon}}.$$

Now, we are ready to

The proof of Theorem 1.8. We will also only do, for simplicity, the $m = 2$ case, the similar procedure will work for all $m \in \mathbb{N}$.

Recall that, when $m = 2$, we have

$$\left|I_\alpha^{(2)}(\mathbf{f})(x)\right| \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_1(y_1)||f_2(y_2)|}{|(x - y_1, x - y_2)|^{2n-\alpha}} dy_1 dy_2.$$

Fix any cube Q in $\mathbb{R}^n$, we write

$$f_i(x) = f_i(x)\chi_{2Q}(x) + f_i(x)\chi_{(2Q)^c}(x), \quad i = 1, 2.$$

Thus we have

$$\begin{aligned}
\left|I_\alpha^{(2)}(\mathbf{f})(x)\right| &\le \left|I_\alpha^{(2)}(f_1\chi_{2Q}, f_2\chi_{2Q})(x)\right| + \left|I_\alpha^{(2)}(f_1\chi_{(2Q)^c}, f_2\chi_{(2Q)^c})(x)\right| \\
&\quad + \left|I_\alpha^{(2)}(f_1\chi_{(2Q)^c}, f_2\chi_{2Q})(x)\right| + \left|I_\alpha^{(2)}(f_1\chi_{2Q}, f_2\chi_{(2Q)^c})(x)\right| \\
&:= V_1(x) + V_2(x) + V_3(x) + V_4(x).
\end{aligned}$$

We first limit the assumption $p_i > 1$, $i = 1, 2$, and estimate the terms of V_1, V_2 and V_3. The estimation of V_4 is analogous to that of V_3.

(i) As for V_1, by the L^q-boundedness for $I_\alpha^{(2)}$ and the condition (1.1), we obtain

$$\begin{aligned}
&\left(\int_Q \left|I_\alpha^{(2)}(f_1\chi_{2Q}, f_2\chi_{2Q})(x)\right|^q\right)^{\frac{1}{q}} \\
&\le C\|f_1\chi_{2Q}\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\chi_{2Q}\|_{L^{p_2}(\mathbb{R}^n)} \\
&\le C\phi_1^{1/p_1}(2Q)\phi_2^{1/p_2}(2Q)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)} \\
&\le C\varphi^{1/q}(Q)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)},
\end{aligned}$$

and so

$$\|V_1\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}. \tag{5.2}$$

(ii) As for V_2, noting $(1/|x-y_i|)^{n-\alpha/2} \le C(M\chi_Q(y_i)/|Q|)^{1-\alpha/2n}$ for $x \in Q$ and $y_i \in (2Q)^c$. Hence, we get

$$\begin{aligned} \left|I_\alpha^{(2)}(f_1\chi_{(2Q)^c}, f_2\chi_{(2Q)^c})(x)\right| &\le C\prod_{i=1}^{2}\int_{\mathbb{R}^n}\frac{|f_i(y_i)|\chi_{(2Q)^c}(y_i)}{|x-y_i|^{n-\alpha/2}}dy_i \\ &\le C\prod_{i=1}^{2}|Q|^{-1+\alpha/2n}\int_{(2Q)^c}|f_i(y_i)|(M\chi_Q(y_i))^{1-\alpha/2n}dy_i. \end{aligned} \tag{5.3}$$

By the condition (1.3) and Lemma 5.3, we take $0<\varepsilon<\min\{n-\alpha p_i/2\}$ such that

$$\int_r^{+\infty}\frac{\phi_i(x_0,t)}{t^{n-\alpha p_i/2+1-\varepsilon}}dt \le C\frac{\phi_i(x_0,r)}{r^{n-\alpha p_i/2-\varepsilon}},$$

and let $\delta_i = \frac{n-\alpha p_i/2-\varepsilon}{n}$, obviously, $0<\delta_i\le 1$. The Hölder inequality implies

$$\begin{aligned} &\int_{(2Q)^c}|f_i(y_i)|(M\chi_Q(y_i))^{\delta_i/p_i}(M\chi_Q(y_i))^{1-\frac{\alpha}{2n}-\frac{\delta_i}{p_i}}dy_i \\ &\le \left(\int_{(2Q)^c}|f_i(y_i)|^{p_i}(M\chi_Q(y_i))^{\delta_i}dy_i\right)^{\frac{1}{p_i}}\left(\int_{(2Q)^c}(M\chi_Q(y_i))^{(1-\frac{\alpha}{2n}-\frac{\delta_i}{p_i})p_i'}dy_i\right)^{\frac{1}{p_i'}}. \end{aligned}$$

Since $M\chi_Q(y_i)\sim 2^{-kn}$ for $y_i\in 2^{k+1}Q\backslash 2^kQ$, then we get

$$\begin{aligned} &\int_{(2Q)^c}(M\chi_Q(y_i))^{(1-\frac{\alpha}{2n}-\frac{\delta_i}{p_i})p_i'}dy_i \\ &\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\backslash 2^kQ}(M\chi_Q(y_i))^{(1-\frac{\alpha}{2n}-\frac{\delta_i}{p_i})p_i'}dy_i \\ &\le C\sum_{k=1}^{\infty}2^{-knp_i'(1-\frac{\alpha}{2n}-\frac{\delta_i}{p_i})}|2^{k+1}Q| \\ &\le C|Q|\sum_{k=0}^{\infty}2^{\frac{-k\varepsilon}{p_i-1}} \le C|Q|. \end{aligned}$$

By these estimates and the inequality (5.3), using Lemma 5.2 and the condition (1.1), we have

$$\begin{aligned}
&\left|I_\alpha^{(2)}(f_1\chi_{(2Q)^c}, f_2\chi_{(2Q)^c})(x)\right| \\
&\le C\prod_{i=1}^{2}|Q|^{-1/p_i+\alpha/2n}\left(\int_{(2Q)^c}|f_i(y_i)|^{p_i}(M\chi_Q(y_i))^{\delta_i}dy_i\right)^{\frac{1}{p_i}} \\
&\le C|Q|^{-1/q}\prod_{i=1}^{2}\left(\int_{(2Q)^c}|f_i(y_i)|^{p_i}(M\chi_Q(y_i))^{\delta_i}dy_i\right)^{\frac{1}{p_i}} \\
&\le C|Q|^{-1/q}\phi_1^{1/p_1}(Q)\phi_2^{1/p_2}(Q)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)} \\
&= C|Q|^{-1/q}\varphi^{1/q}(Q)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}.
\end{aligned}$$

This implies that

$$\|V_2\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}. \tag{5.4}$$

(iii) As for V_3, applying partly the technique used in the estimate of V_2, we get

$$\begin{aligned}
&\left|I_\alpha^{(2)}(f_1\chi_{2Q}, f_2\chi_{(2Q)^c})(x)\right| \\
&\le C\int_{\mathbb{R}^n}|f_1(y_1)|\chi_{2Q}(y_1)dy_1\int_{\mathbb{R}^n}\frac{|f_2(y_2)|\chi_{(2Q)^c}(y_2)}{|x-y_2|^{2n-\alpha}}dy_2 \\
&\le C\int_{2Q}|f_1(y_1)|dy_1\int_{\mathbb{R}^n}\frac{|f_2(y_2)|\chi_{(2Q)^c}(y_2)}{|x-y_2|^{n-\alpha/2}|x-y_2|^{n-\alpha/2}}dy_2 \\
&\le C|Q|^{-1/p_1+\alpha/2n}\left(\int_{2Q}|f_1(y_1)|^{p_1}dy_1\right)^{1/p_1}\int_{\mathbb{R}^n}\frac{|f_2(y_2)|\chi_{(2Q)^c}(y_2)}{|x-y_2|^{n-\alpha/2}}dy_2 \\
&\le C|Q|^{-1/p_1+\alpha/2n}\phi_1^{1/p_1}(x_0,r)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|f_2(y_2)|\chi_{(2Q)^c}(y_2)}{|x-y_2|^{n-\alpha/2}}dy_2 \\
&\le C|Q|^{-1/q}\phi_1^{1/p_1}(x_0,r)\phi_2^{1/p_2}(x_0,r)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)} \\
&= C|Q|^{-1/q}\varphi^{1/q}(x_0,r)\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}.
\end{aligned}$$

Hence, we have

$$\|V_3\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}, \tag{5.5}$$

and similarly we have

$$\|V_4\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}. \tag{5.6}$$

Combing all the estimates for V_i together, $i = 1, 2, 3, 4$, we get

$$\big\|I_\alpha^{(2)}(f_1, f_2)\big\|_{L^{q,\varphi}(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}.$$

This is the desired inequality when $p_i > 1$ of Theorem 1.8.

We now turn our attention to the case $p_1 \ge 1$. Without loss of the generality, we assume that $p_1 = 1$ and $p_2 > 1$. It's enough to show for any $\lambda > 0$ that

$$\frac{|\{x \in Q : |V_i(x)| > \lambda\}|}{\varphi(Q)} \le \frac{C}{\lambda^q}\left(\prod_{i=1}^{2}\|f_i\|_{L^{p_i,\phi_i}(\mathbb{R}^n)}\right)^q, \quad \text{for } i = 1, 2, 3, 4. \tag{5.7}$$

(a) By the weak L^q-boundedness for $I_\alpha^{(2)}$ and the condition (1.1), we obtain

$$\begin{aligned}
&\big|\big\{x \in Q : \big|I_\alpha^{(2)}(f_1\chi_{2Q}, f_2\chi_{2Q})(x)\big| > \lambda\big\}\big| \\
&\quad\le \frac{C}{\lambda^q}\big(\|f_1\chi_{2Q}\|_{L^{p_1}(\mathbb{R}^n)}\|f_2\chi_{2Q}\|_{L^{p_2}(\mathbb{R}^n)}\big)^q \\
&\quad= \frac{C}{\lambda^q}\left(\int_{2Q}|f_1(x)|^{p_1}\right)^{q/p_1}\left(\int_{2Q}|f_2(x)|^{p_2}\right)^{q/p_2} \\
&\quad\le \frac{C}{\lambda^q}\phi_1^{q/p_1}(2Q)\phi_2^{q/p_2}(2Q)\big(\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}\big)^q \\
&\quad\le \frac{C}{\lambda^q}\varphi(Q)\big(\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}\|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}\big)^q,
\end{aligned}$$

which implies (5.7) for $i = 1$.

(b) Recall the inequality (5.3),

$$|V_2(x)| \le C \prod_{i=1}^{2} |Q|^{-1+\alpha/2n} \int_{\mathbb{R}^n} |f_i(y_i)|(M\chi_Q(y_i))^{1-\alpha/2n} dy_i.$$

For f_1, we choose $\delta_1 = 1 - \alpha/2n$, then Lemma 5.2 show that

$$\begin{aligned} |Q|^{-1+\alpha/2n} \int_{\mathbb{R}^n} |f_1(y_1)|(M\chi_Q(y_1))^{1-\alpha/2n} dy_1 \\ \le C|Q|^{-1+\alpha/2n} \phi_1(Q) \|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)}. \end{aligned}$$

For f_2, we take $\delta_2 = \frac{n-\alpha p_2/2-\varepsilon}{n}$, by similar argument as that in (ii), we have

$$\begin{aligned} |Q|^{-1+\alpha/2n} \int_{\mathbb{R}^n} |f_2(y_2)|(M\chi_Q(y_2))^{1-\alpha/2n} dy_2 \\ \le C|Q|^{-1/p_2+\alpha/2n} \phi_2^{1/p_2}(Q) \|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we derive that

$$|V_2(x)| \le C|Q|^{-1/q} \phi_1(Q)\phi_2^{1/p_2}(Q) \|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}.$$

The Chebychev's inequality yields that

$$\begin{aligned} |\{x \in Q : |V_2(x)| > \lambda\}| &\le C \int_Q \left|\frac{V_2(x)}{\lambda}\right|^q dx \\ &\le \frac{C}{\lambda^q} \phi_1^q(Q)\phi_2^{q/p_2}(Q) \big(\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}\big)^q \\ &= \frac{C}{\lambda^q} \varphi(Q) \big(\|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}\big)^q, \end{aligned}$$

because of $p_1 = 1$. We then obtain (5.7) for $i = 2$.

(c) For V_3 and V_4, similar arguments show get

$$|V_3(x)| + |V_4(x)| \le C|Q|^{-1/q} \varphi^{1/q}(Q) \|f_1\|_{L^{p_1,\phi_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2,\phi_2}(\mathbb{R}^n)}.$$

Then we immediately obtain the desired inequality (5.7) for $i = 3, 4$.

Finally, when $p_1 = p_2 = 1$. We only need take $\delta_1 = \delta_2 = 1 - \alpha/2n$ and use Lemma 5.2, the similar argument as that above will imply the desired inequality (5.7). This completes the proof of the Theorem 1.8.

References

[1] Coifman R. R. and Meyer Y., *On commutators of singular integrals and bilinear singular integrals.* Trans. Amer. Math. Soc. **212** (1975), 315–331.

[2] Grafakos L. and Torres R., *Multilinear Calderón-Zygmund theory.* Adv. Math. **165** (2002), 124–164.

[3] Kenig C. E. and Stein E. M., *Multilinear estimates and fractional integration.* Math. Res. Lett. **6** (1999), 1–15.

[4] Nakai E., *Hardy-Littlewood Maximal Operator, Singular Integral Operators and the Riesz Potentials on Generalized Morrey Spaces.* Math. Nachr. **166** (1994), 95–103.

[5] Herz C., *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transform.* J. Math. Mech. **18** (1968), 283–324.

[6] Hu G. E., Lu S. Z. and Yang D. C., *The application of weak Herz spaces.* Adv. Math. (China). **26** (1997), 417–428.

[7] Lu S. Z. and Xu L., *Boundedness of rough singular integral operators on homogeneous Morrey-Herz spaces.* Hokkaido Math. J. **34** (2005), 299–314.

[8] Lu S. Z. and Yang D. C., *The weighted Herz-type Hardy space and its application.* Sci. in China (Ser. A). **38** (1995), 662–673.

[9] Komori Y., *Weak type estimates for Calderón-Zygmund operators on Herz spaces at critical indexes.* Math. Nachr. **259** (2003), 42–50.

[10] Shi Y. L. and Tao X. X., *Boundedness for multilinear fractional integral operators on Herz type spaces, Applied Mathematics*, A J. Chinese Univ. Ser. B. **23** (2008)(4), 437–446.

[11] Stein E. M., *Harmonic Analysis: Real-Variable methods, Orthogonality, and Oscillatory Integrals.* Princeton N J. Princeton Univ Press, 1993.

[12] Tao X. X. and Chen Q. Q., *The boundedness of maximal function in Orlicz-Campanato spaces of homogeneous type.* Georgian Mathematical Journal **15** (2008)(2), 377–388.

[13] Tao X. X., Shi Y. L. and Zhang S. Y., *Boundedness of multilinear Riesz potential operators on product of Morrey spaces and Herz-Morrey spaces.* Acta Math. Sinica, Chinese Series **52** (2009)(3), 535–548.

Y.-L. Shi
Department of Mathematics Faculty of Science
Ningbo University
Ningbo, 315211, P.R. China
E-mail: shiyan-long@hotmail.com

Current address:
Zhejiang Pharmaceutical College
Ningbo, 315100, P.R. China
E-mail: shiyan-long@hotmail.com

X.-X. Tao
Department of Mathematics Faculty of Science
Ningbo University
Ningbo, 315211, P.R. China
E-mail: xxtao@hotmail.com

Current address:
Department of Mathematics
Zhejiang University of Science and Technology
Hangzhou, 310023, P.R. China
E-mail: xxtao@hotmail.com