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An Edge-preserving Variational Method for Image Decomposition*

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Abstract — Variational methods for image decomposition have gained considerable attention in recent years. In such approaches, an image usually can be decomposed into a geometrical (or structure) component and a textured (or noise) feature. In this paper we propose an edge-preserving variational model which can split an image into four components: a first one containing the structure of the image, a second one the texture of the image, a third one the noise and a forth one the edge. Our decomposition model relies on the use of three different terms: the edge-preserving regularization for the geometrical component and the edge, a negative Sobolev norm for the texture, and a negative Besov norm for the noise. We explicitly give numerical scheme that is the synthesis of a projection algorithm, a redundant wavelet (or curvelet) soft threshold and two coupled Partial differential equations (PDE's). Finally we show image decomposition results on synthetic and real image.

Key words — Image decomposition, Edge-preserving, Structure, Texture, Noise, Edge, Variational approach.

I. Introduction

Image decomposition is an important and challenging problem in image processing. In principle, this problem can be regarded as an inverse problem. Consequently, image decomposition can be done by means of regularization techniques and minimization of related variational functional.

A classical approach consists in considering that an image f can be decomposed into two components $u+v$. The first component u has a simple geometric structure. The second component v contains the oscillating patterns (both textures and noise). A variational approach to this problem is by solving the following energy minimization problem:

$$\inf_{(u,v)}\{E(u,v)=E_{smooth}(u)+\lambda E_{fidelity}(v), f=u+v\} \tag{1}$$

where E_{smooth} is a smoothing term which rewards smooth signals and penalizes oscillatory ones, $E_{fidelity}$ is fidelity term which accounts for the closeness to the input image f.

In the 90's, most of the research was consisted in considering the smoothness term. A main contribution was obtained by the total variation minimization model of Rudin, Osher and Fatemi[1]. The ROF model performs very well for removing noise while preserving edges. However, it fails to separate well oscillatory component from high-frequencies components. For example, edges of an object are high-frequencies components and noise is oscillatory components. Both the u and v components in the ROF model contain high-frequencies components. That is, the v component not only contains oscillatory components, it also contains high-frequencies components. We can see shadows of edges in the noise or textured component v, even when the parameter λ is not so small. Alternatively, to remedy this situation, Meyer[2] proposed that one should focus on the role of the fidelity term. This has inspired many new image decomposition models and algorithms[3−5,9].

In this paper, we are interested in the variational decomposition model introduced by Aujol and colleagues in Ref.[4]. That is,

$$\inf_{(u,v,\omega)\in BV\times\mu B_G\times\delta B_E}\left\{F_{\lambda,\mu,\delta}(u,v,\omega)=|u|_{BV}+\frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2(\Omega)}\right\} \tag{2}$$

where $\mu B_G=\{v\in G|\|v\|_G\le\mu\}$ ($G=W^{-1,\infty}(\Omega)$ is the dual space of the Sobolev space $W_0^{1,1}(\Omega)$) and $\delta B_E=\{\omega|\|\omega\|_E\le\delta\}$ ($E=\dot{B}^{-1}_{\infty,\infty}$ is the dual space of the usual homogeneous Besov space $\dot{B}^1_{1,1}$). But, due to the G and E norm, this problem cannot be solved directly. Therefore, the authors proposed the following regularization method:

$$\inf_{(u,v,\omega)\in BV\times\mu B_G\times\delta B_E}\left\{F(u,v,\omega)=|u|_{BV}+J^*\left(\frac{v}{\mu}\right)+B^*\left(\frac{\omega}{\delta}\right)+\frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2}\right\} \tag{3}$$

where $J^*(v/\mu)=\chi_{\{\|v\|_G\le\mu\}}$ and $B^*(\omega/\delta)=\chi_{\{\|\omega\|_E\le\delta\}}$. J^* and B^* is the indicator functions of the convex sets G and E[3].

S. Teboul and colleagues developed the nonquadratic vari-

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ational approach for image reconstruction[6]:

$$\inf_u \left\{ J(u) = \int_\Omega |u-f|^2 dx + \lambda^2 \int_\Omega \varphi[|\nabla u(x)|] dx \right\} \tag{4}$$

where φ is an edge-preserving regularization function which satisfies the following conditions[7]:

(1) $0 < \lim_{t\to 0}[\varphi'(t)/2t] = 1$ which can keep isotropic smoothing in homogeneous areas.

(2) $\lim_{t\to\infty}[\varphi'(t)/2t] = 0$ which preserves edges.

(3) $\varphi'(t)/2t$ is strictly decreasing, which can avoid instabilities.

Since the properties (1)–(3) imply that the function φ can be represented as the infimum of quadratic functions, the model (4) results in two coupled systems of PDE's: one processes the intensity, the other the edges.

Inspired by Teboul[6], it is a nature idea that we propose to incorporate edge-preserving regularization into the Aujol's image decomposition model. Altogether this leads to a new variational model which would split an image into four components: a first one containing the structure of the image u, a second one the texture of the image v, a third one the noise ω and a forth one the edges b. We give a powerful algorithm to compute the image decomposition we want to get. Then we show some experimental results. We compare our algorithm with the method introduced in Ref.[4].

II. Description of New Model: Structure+Texture+Noise+Edge Decomposition

In this paper, we characterize the smoothing term $|\cdot|_{BV}$ of Eq.(2) by using edge-preserving regularization $\int_\Omega \varphi[|\nabla u(x)|]dx$ introduced in Eq.(4), while keeping $v\times\omega \in \mu B_G \times \delta B_E$. Thus, one has

$$\inf_{u,v\times\omega\in\mu B_G\times\delta B_E} \left\{ F(u,v) = \int_\Omega \varphi[|\nabla u(x)|]dx + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2(\Omega)} \right\} \tag{5}$$

To avoid solving G and E norm, we incorporate J^* and B^* into Eq.(5). Hence, we have

$$\inf_{u,v,\omega} \left\{ F(u,v,\omega) = \int_\Omega \varphi[|\nabla u(x)|]dx + J^*\left(\frac{v}{\mu}\right) + B^*\left(\frac{\omega}{\delta}\right) + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{6}$$

Since the function φ can be represented as the infimum of quadratic functions, the functional Eq.(6) becomes

$$F(u,v,\omega,b) = \inf\left\{ \int_\Omega \{b(x)|\nabla u(x)|^2 + \psi[b(x)]\} + J^*\left(\frac{v}{\mu}\right) + B^*\left(\frac{\omega}{\delta}\right) + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{7}$$

where $\psi(b) = \theta[(\theta')^{-1}(b)] - b(\theta')^{-1}(b)$, $\theta(t) = \varphi(\sqrt{t})$. The minimum on b when u is fixed is unique and given by $b_{\inf}(x) = \dfrac{\varphi'[|\nabla u(x)|]}{2|\nabla u(x)|}$. Since the variable b, which weights the quadratic smoothing on $|\nabla u|$, represents the edges of the image, and the function ψ acts as a penalty for introducing an edge ($b\to 0$). Thus, a model is defined, depending on both the intensity u, the edge b, the texture v and the noise ω, as follows:

$$\inf_{u,v,\omega,b} \left\{ F^*(u,v,\omega,b) = \int_\Omega \{b|\nabla u|^2 + \psi(b)\} + J^*\left(\frac{v}{\mu}\right) + B^*\left(\frac{\omega}{\delta}\right) + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{8}$$

To enhance the quality of edges, we add a segmentation constraint on b and parameter k to the functional Eq.(11). Hence, we have

$$\inf_{u,v,\omega,b} \left\{ F^*(u,v,\omega,b) = \int_\Omega \{b|\nabla u|^2 + k\psi(b)\} + \frac{\beta}{k}\int_\Omega \varphi_b(|\nabla b|) + J^*\left(\frac{v}{\mu}\right) + B^*\left(\frac{\omega}{\delta}\right) + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{9}$$

Now we consider the three following problems to solve Eq.(9):

(1) u, v, ω being fixed, we search for b as a solution of Eq.(10):

$$\inf_b \left\{ \int_\Omega [b|\nabla u|^2 + k\psi(b)] + \frac{\beta}{k}\int_\Omega \varphi_b(|\nabla b|) \right\} \tag{10}$$

(2) v, ω, b being fixed, we search for u as a solution of Eq.(11):

$$\inf_u \left\{ \int_\Omega [b|\nabla u|^2 + k\psi(b)] + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{11}$$

(3) u, ω, b being fixed, we search for v as a solution of Eq.(12):

$$\inf_v \left\{ J^*\left(\frac{v}{\mu}\right) + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{12}$$

(4) u, v, b being fixed, we search for ω as a solution of Eq.(13):

$$\inf_\omega \left\{ B^*\left(\frac{\omega}{\delta}\right) + \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{13}$$

If minima of Eqs.(10) and (11) exist, one can formally verify the following system of Euler-Lagrange equations:

$$\|\nabla u\|^2 + k\psi'(b) - \frac{\beta}{k}\mathrm{div}\left[\frac{\varphi'(|\nabla b|)}{|\nabla b|}\nabla b\right] = 0 \tag{14}$$

$$-(f-u-v-\omega) - 2\lambda\mathrm{div}(b\nabla u) = 0 \tag{15}$$

The minimum of Eq.(12) is equivalent to the problem

$$\inf_{v\in\mu B_G} \left\{ \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{16}$$

Thus the solution of Eq.(16) is given by

$$v = P_{\mu B_G}(f-u-\omega) \tag{17}$$

where P is the orthogonal projector on $\mu B_G = \{v \in G \| \|v\|_G \le \mu\}$[3,4]. Similarly, minimum of Eq.(13) can be denoted by

$$\inf_{\omega\in\delta B_E} \left\{ \frac{1}{2\lambda}\|f-u-v-\omega\|^2_{L^2} \right\} \tag{18}$$

In fact, Eq.(18) can be solved by the dual problem:

$$\inf_{\omega}\left\{\delta\|\omega\|_{\dot{B}^1_{1,1}}+\frac{1}{2\lambda}\|f-u-c-\omega\|^2_{L^2}\right\}$$

Therefore, the solution ω of Eq.(21) can be obtained by

$$\omega=P_{\delta B_E}(f-u-v)=f-u-v-WST(f-u-v,2\delta) \quad (19)$$

where $WST(f-u-v,2\delta)$ stands for the wavelet soft-thresholding of $f-u-v$ with threshold 2δ. Aujol chose discrete wavelet shrinkage in Ref.[4], but one can find that there are a lot of block structure due to artifacts effect in numerical results. For example, the structure u is given in Figs.2 and 5. To overcome this problem, we incorporate redundant wavelet threshold and curvelet threshold respectively into the new model.

To solve Eq.(14), one has to give the choice for φ and ψ. However, the function ψ is defined from the function φ[8]. So we only consider the choices of φ, which satisfies the edge-preserving properties (1)–(3). To recover and preserve image edges, we should impose the growth condition of the type $\lim_{t\to\infty}\varphi(t)=c>0$ on φ. Since we want to obtain the best image decomposition results, one have to choose the proper φ. Thus we consider the function $\varphi=\dfrac{t^2}{1+t^2}$ which is introduced by Geman and McClure for nonlinear image filtering with edge and corner enhancement[10]. Using the half-quadratic theorem[8], the associated ψ is $b-2\sqrt{b}+1$. Due to the infinite limit of $\psi'(b)$ when b tends to 0, the Eq.(14) is difficult to solve. To remedy this, we propose to use b^2 rather than b[6]. Thus, ψ becomes $\psi(b)=(b-1)^2$. Eqs.(14) and (15) become

$$b\left(\frac{|\nabla u|^2}{k}+1\right)-\frac{\beta}{k^2}\mathrm{div}\left[\frac{\nabla b}{(1+|\nabla b|^2)^2}\right]=1 \quad (20)$$

$$-(f-u-v-\omega)-2\lambda\mathrm{div}(b^2\nabla u)=0 \quad (21)$$

III. Algorithm of New Model

1. Discretization of the model

To discretize the Eqs. (23) and (24), we use a semi-implicit finiter differences scheme and an iterative algorithm based on a fixed point iteration. The details of new numerical algorithm are as follows. We use the classical notations $u_{i,j}\approx u(ih,jh)$, $f_{i,j}\approx f(ih,jh)$, $v_{i,j}\approx v(ih,jh)$ and $b_{i,j}\approx b(ih,jh)$ denotes a discrete point for $0\le i,\ j\le M$, where $h>0$ is the step. To simplify the presentation, let us introduce the notation $H(u,k)=\left(1+\dfrac{|\nabla u|^2}{k}\right)^{-1}$ and

$$c_1=\frac{1}{\left(1+\left(\dfrac{b_{i+1,j}-b_{i,j}}{h}\right)^2+\left(\dfrac{b_{i,j+1}-b_{i,j-1}}{2h}\right)^2\right)^2}$$

$$c_2=\frac{1}{\left(1+\left(\dfrac{b_{i,j}-b_{i-1,j}}{h}\right)^2+\left(\dfrac{b_{i-1,j+1}-b_{i-1,j-1}}{2h}\right)^2\right)^2}$$

$$c_3=\frac{1}{\left(1+\left(\dfrac{b_{i+1,j}-b_{i-1,j}}{2h}\right)^2+\left(\dfrac{b_{i,j+1}-b_{i,j}}{h}\right)^2\right)^2}$$

$$c_4=\frac{1}{\left(1+\left(\dfrac{b_{i+1,j-1}-b_{i-1,j-1}}{2h}\right)^2+\left(\dfrac{b_{i,j}-b_{i,j-1}}{h}\right)^2\right)^2}$$

Therefore, the discrete fixed point Gauss-Seidel iteration method for Eqs.(20) and (21) are

$$b^{n+1}_{i,j}=\frac{1+\dfrac{\beta}{k^2h^2}(c_1b^n_{i+1,j}+c_2b^n_{i-1,j}+c_3b^n_{i,j+1}+c_4b^n_{i,j-1})}{H(u^n_{i,j},k)+c_1+c_2+c_3+c_4} \quad (22)$$

and

$$\begin{aligned}u^{n+1}_{i,j}=&\frac{1}{\left(1+\dfrac{2\lambda}{h^2}(b_1+b_2+b_3+b_4)\right)}\\&\cdot\Big[f_{i,j}-v^n_{i,j}-\omega^n_{i,j}+\frac{2\lambda}{h^2}(b_1u^n_{i+1,j}+b_2u^n_{i-1,j}\\&+b_3u^n_{i,j+1}+b_4u^n_{i,j-1})\Big]\end{aligned} \quad (23)$$

where $b_1=(b^n_{i+1,j})^2$, $b_2=(b^n_{i-1,j})^2$, $b_3=(b^n_{i,j+1})^2$, $b_4=(b^n_{i,j-1})^2$.

For Eq.(17), the algorithm introduced in Ref.[3] can be used to compute $P_{\mu B_G}(f-u)$. It indeed amounts to finding:

$$\min\{\|\mu\mathrm{div}(p)-(f-u)\|^2_2 : p/|p_{i,j}|\le 1\forall_{i,j}=1,\cdots,N\}.$$

This problem can be solved by a fixed point method: $p^0=0$ and

$$p^{n+1}_{i,j}=\frac{p^n_{i,j}+\tau(\nabla(\mathrm{div}(p^n)-(f-u-\omega)/\mu))_{i,j}}{1+\tau|(\nabla(\mathrm{div}(p^n)-(f-u-\omega)/\mu))_{i,j}|} \quad (24)$$

2. Description of algorithm

From the above analysis, we know that the proposed algorithm should be the synthesis of a projection algorithm, a redundant wavelet (or curvelet) soft threshold and two coupled PDE. Thus, one has

(1) Initialization: $u^0=v^0=\omega^0=0$, $b^0=1$

(2) Iterations:

Solve Eq.(19), with u,v and b fixed:

$$\omega^{n+1}=P_{\delta B_E}(f-u^n-v^n)=f-u^n-v^n-WST(f-u^n-v^n,2\delta)$$

Solve Eq.(17), with u,ω and b fixed

$$v^{n+1}=P_{\mu B_G}(f-u^n-\omega^{n+1})$$

Solve Eq.(21), with v,ω and b fixed, *i.e.* Eq.(23)

Solve Eq.(20), with u,v and ω fixed, *i.e.* Eq.(22)

(3) Stopping test.

IV. Experimental Results

In this section we present numerical results of image decomposition using Aujol's method and the new model. All edge maps in Aujol's method were obtained by applying edge detectors based on MATLAB to the structure u.

Fig.1 shows an original intercepting image of Barbara and noisy image, which contains rich textures and non-textured parts. The noise is additive white noise with standard deviation $\sigma=20$. The Peak signal to noise (PSNR) is 23.1426. In this experiment, we use "db8" wavelet with eight vanishing moments. Fig.2 shows results of image decomposition using

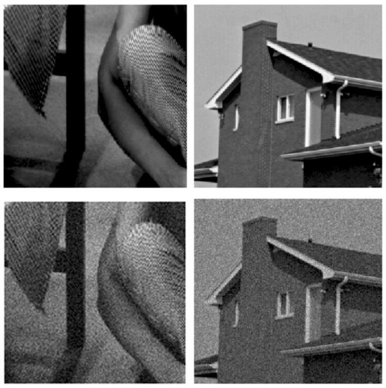

Fig. 1. Original image and noisy image

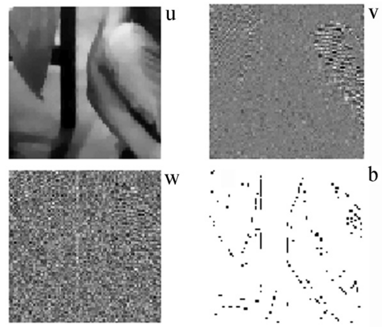

Fig. 2. Decomposition results from Aujol's method and Laplacian-Gaussian operator

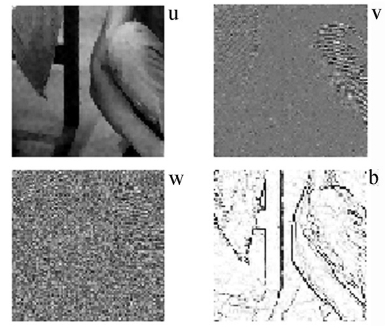

Fig. 3. Decomposition results from the proposed method

Aujol's method, which contains the structure u, the texture v, the noise ω. The edge b is detected using the Laplacian-Gaussian operator on the structure u. Fig.3 shows respectively the results of image decomposition into the structure u, the texture v, the noise ω and the edges b from the proposed method. To overcome artifacts effect of the structure u in Fig.2, we use redundant wavelet transform. We still use "db8" wavelet with eight vanishing moments. One can see that the proposed method yields better decomposition results than Aujol's method. In addition, $u+v$ shows denoising result. Fig.4 gives denoising result respectively from Aujol's method and the proposed model. We can see that the proposed method yields better denoising results than Aujol's method.

In the second example we decomposed a house image with rich edges. Fig.1 shows the original image and noisy image. The noise is additive white noise with standard deviation $\sigma = 20$. The PSNR is 23.7210. Fig.5 shows decomposition results of Aujol's method using "db8" discrete wavelet, which contains the structure u, the texture v, the noise ω. And edges are detected by the zerocross operator on the structure u. Fig.6 and Fig.7 give decomposition results respectively from the proposed model using redundant wavelet shrinkage and curvelet shrinkage. We can see that the proposed method yields better decomposition results than Aujol's method. Fig.8 gives denoising results $u+v$ respectively using Aujol's method and the proposed model. Again, our model yields better results than Aujol's method.

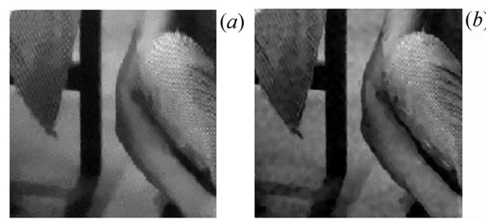

Fig. 4. Denoising result using Aujol's method (PSNR=26.7303) and the proposed model (PSNR=27.6232). (a) u+v denoise result of dual method; (b) u+v denoise result of new method

V. Conclusion

In this paper we present an edge-preserving variational model which can split an image into four components: structure、texture、noise and edge. Our decomposition model is composed of three terms: the edge-preserving regularization for the geometrical component and the edge, a negative Sobolev norm for the texture, and a negative Besov norm for the noise. We developed the new minimization algorithm, which is the synthesis of a projection algorithm, a redundant wavelet (or curvelet) soft threshold and two coupled Partial differential equations (PDE's). Extended experiments have shown the effectiveness and efficiency of the proposed model.

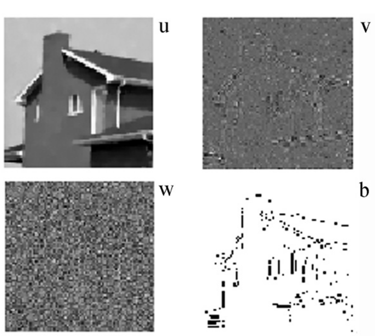

Fig. 5. Decomposition results from Aujol's method and zerocross operator

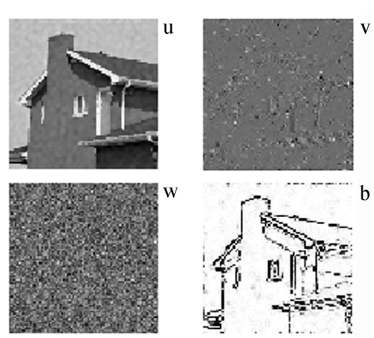

Fig. 6. Decomposition results from the proposed method with redundant wavelet shrinkage

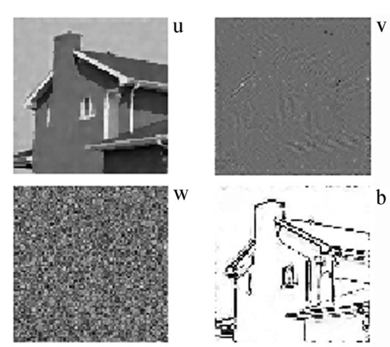

Fig. 7. Decomposition results from the proposed method with curvelet shrinkage

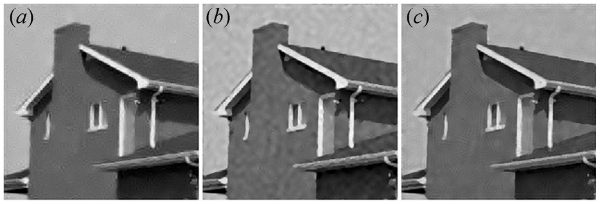

Fig. 8. Denoising result using Aujol's method (PSNR=28.5292) and the proposed model (middle-redundant wavelet shrinkage: PSNR=29.7808 and right-curvelet shrinkage: PSNR=30.2108). (*a*) denoise result from dual method; (*b*) denoise result from new method; (*c*) denoise result from new method

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