Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 340 (2008) 1326–1335
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Well-posedness of the Cauchy problem for the fractional power dissipative equation in critical Besov spaces

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Received 1 July 2007
Available online 3 October 2007
Submitted by J. Xiao

Abstract

In this paper we study the Cauchy problem for the semilinear fractional power dissipative equation $u_t + (-\Delta)^\alpha u = F(u)$ for the initial data u_0 in critical Besov spaces $\dot{B}^\sigma_{2,r}$ with $\sigma \triangleq \frac{n}{2} - \frac{2\alpha-d}{b}$, where $\alpha > 0$, $F(u) = P(D)u^{b+1}$ with $P(D)$ being a homogeneous pseudo-differential operator of order $d \in [0, 2\alpha)$ and $b > 0$ being an integer. Making use of some estimates of the corresponding linear equation in the frame of mixed time–space spaces, the so-called "mono-norm method" which is different from the Kato's "double-norm method," Fourier localization technique and Littlewood–Paley theory, we get the well-posedness result in the case $\sigma > -\frac{n}{2}$.

Keywords: Dissipative equation; Cauchy problem; Well-posedness; Besov spaces; Fourier localization; Littlewood–Paley theory

1. Introduction

In this paper we study the Cauchy problem for the semilinear fractional power dissipative equation

$$\begin{cases} u_t + (-\Delta)^\alpha u = F(u), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

for the initial data $u_0(x)$ in critical Besov spaces $\dot{B}^\sigma_{2,r}$ with $\sigma \triangleq \frac{n}{2} - \frac{2\alpha-d}{b}$, where $\alpha > 0$, $F(u) = P(D)u^{b+1}$ with $P(D)$ being a homogeneous pseudo-differential operator of order $d \in [0, 2\alpha)$ and $b > 0$ being an integer.

The evolution equation in (1.1) models several classical equations, for example:

1. The semilinear fractional power dissipative equation

$$u_t + (-\Delta)^\alpha u = \mu |u|^b u$$

with μ being a constant.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.09.060

2. The generalized convection–diffusion equation

$$u_t + (-\Delta)^\alpha u = \mathbf{a} \cdot \nabla\big(|u|^b u\big), \quad \mathbf{a} \in \mathbb{R}^n \setminus \{0\}.$$

3. The generalized Navier–Stokes equation

$$u_t + (-\Delta)^\alpha u + u \cdot \nabla u + \nabla P = 0, \quad \operatorname{div} u = 0.$$

4. The subcritical dissipative quasi-geostrophic equation

$$\begin{cases} \theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = (u_1, u_2) = \nabla^\perp \psi, \quad (-\Delta)^{\frac{1}{2}} \psi = \theta, \end{cases} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2,$$

where $\frac{1}{2} < \alpha \leqslant 1$.

The case $\alpha = 1$ for the Cauchy problem (1.1) corresponds to the semilinear heat equation and has been studied extensively, see e.g. [6–11,13–15,20–24,26,27]. For the generalized Navier–Stokes equation, see [2,32]. For the Q-G equation, please refer to [3,5,28–31,33]. About some results for the general case, refer to [8,12,16–18]. Recently, the well-posedness in Lebesgue space for general case has been studied in [19] by using "double-norm method" and some time–space estimates.

In this paper, making use of Fourier localization technique and Littlewood–Paley theory, we will firstly prove some estimates of the corresponding linear equation in the frame of mixed time–space spaces, then make use of "mono-norm method" which is different from the Kato's "double-norm method" to investigate the well-posedness of Cauchy problem (1.1) for general $\alpha > 0$ in critical Besov spaces $\dot{B}^\sigma_{2,r}$.

That $\dot{B}^\sigma_{2,r}$ is the critical space when $\sigma = \frac{n}{2} - \frac{2\alpha-d}{b}$ is due to the scaling invariance in $\dot{B}^\sigma_{2,r}$. That is, if $u(t,x)$ is a solution, then $u_\lambda(t,x) = \lambda^{\frac{2\alpha-d}{b}} u(\lambda^{2\alpha}t, \lambda x)$ is also a solution of the equation and $\|u_\lambda(t,\cdot)\|_{\dot{B}^\sigma_{2,r}} = \lambda^{\sigma-\frac{n}{2}+\frac{2\alpha-d}{b}} \|u(\lambda^{2\alpha}t,\cdot)\|_{\dot{B}^\sigma_{2,r}}$. It must be noticed that when $r = \infty$, the Besov space $\dot{B}^\sigma_{2,\infty}$ contains self-similar initial data in the sense that $u_0(x)$ satisfies $\lambda^{\frac{2\alpha-d}{b}} u_0(\lambda x) = u_0(x)$ for any $\lambda > 0$, thus the following Theorem 1.1 implies the existence of self-similar solutions to the Cauchy problem (1.1).

In this paper, our main results are the following theorems (some notation used there is referred to Section 2).

Theorem 1.1. *Let* $1 \leqslant r \leqslant +\infty$, $\sigma \triangleq \frac{n}{2} - \frac{2\alpha-d}{b}$. *Suppose* $\sigma > -\frac{n}{2}$ *and* $u_0 \in \dot{B}^\sigma_{2,r}$, *then there exits* $T > 0$ *such that the Cauchy problem* (1.1) *has a unique solution* $u(t) \in \mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}(I; \dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})$ *and*

$$u \in \mathcal{L}^\infty\big(I; \dot{B}^\sigma_{2,r}\big) \cap \mathcal{L}^{\frac{2\alpha}{2\alpha-d}}\big(I; \dot{B}^{\sigma+2\alpha-d}_{2,r}\big), \tag{1.2}$$

where $I = [0, T)$.

If in addition $r < +\infty$, *then* $u \in \mathcal{C}(I; \dot{B}^\sigma_{2,r})$.

Denoting the maximum lifespan by $T^\star_{u_0}$, *we also have the following results*:

1. *There exists a constant* $c > 0$ *such that, when* $\|u_0\|_{\dot{B}^\sigma_{2,r}} \leqslant c$, *we have* $T^\star_{u_0} = +\infty$.
2. *If* u *and* v *are two solutions of the Cauchy problem* (1.1) *with initial data* u_0 *and* v_0, *then there exists a constant* $C > 0$ *such that*

$$\|u - v\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}(I; \dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})} \leqslant C\|u_0 - v_0\|_{\dot{B}^\sigma_{2,r}}. \tag{1.3}$$

Theorem 1.2 *(Blow-up criterion). Under the assumption of Theorem* 1.1, *if* $T^\star_{u_0} < +\infty$, *then*

$$\|u\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}([0,T^\star_{u_0}); \dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})} = +\infty. \tag{1.4}$$

Remark 1.1. Recall the basic facts:

1. When $\sigma \geqslant 0$ and $1 \leqslant r \leqslant 2$, $\dot{B}^{\sigma}_{2,r} \hookrightarrow L^{\frac{nb}{2\alpha-d}}$.
2. When $-\frac{n}{2} < \sigma < 0$ and $1 \leqslant r < 2$, $L^{\frac{nb}{2\alpha-d}} \not\hookrightarrow \dot{B}^{\sigma}_{2,r}$ and $\dot{B}^{\sigma}_{2,r} \not\hookrightarrow L^{\frac{nb}{2\alpha-d}}$.
3. When $\sigma > 0$ and $r > 2$, $L^{\frac{nb}{2\alpha-d}} \not\hookrightarrow \dot{B}^{\sigma}_{2,r}$ and $\dot{B}^{\sigma}_{2,r} \not\hookrightarrow L^{\frac{nb}{2\alpha-d}}$.
4. When $-\frac{n}{2} < \sigma \leqslant 0$ and $r \geqslant 2$, $L^{\frac{nb}{2\alpha-d}} \hookrightarrow \dot{B}^{\sigma}_{2,r}$.

Therefore the Besov spaces $\dot{B}^{\sigma}_{2,r}$ in this paper are different from the Lebesgue space $L^{\frac{nb}{2\alpha-d}}$ in [19].

This paper is arranged as following:

In Section 2, we introduce some definitions and properties about homogeneous Besov spaces and Littlewood–Paley decomposition. In Section 3, making use of Fourier localization technique and Littlewood–Paley theory, we will prove some estimates of linear fractional power dissipative equation in the frame of mixed time–space spaces. In Section 4, we make use of the results derived in Section 3, "mono-norm method," Fourier localization technique and Littlewood–Paley theory to prove the well-posedness in critical Besov spaces, and we will also prove the blow-up criterion.

2. Besov spaces and Littlewood–Paley decomposition

The proof of the results presented in this paper is based on a dyadic partition of unity in Fourier variables, the so-called *homogeneous Littlewood–Paley decomposition*.

Let (χ, φ) be a couple of smooth functions valued in [0, 1] such that χ is supported in the ball $\{\xi \in \mathbb{R}^n \mid |\xi| \leqslant \frac{4}{3}\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^n \mid \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ and

$$\chi(\xi) + \sum_{q\in\mathbb{N}} \varphi\big(2^{-q}\xi\big) = 1, \quad \forall \xi \in \mathbb{R}^n;$$

$$\sum_{q\in\mathbb{Z}} \varphi\big(2^{-q}\xi\big) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Denoting $\varphi_q(\xi) = \varphi(2^{-q}\xi)$ and $h_q = \mathcal{F}^{-1}\varphi_q$, we define the dyadic blocks as

$$\dot{\Delta}_q u \triangleq \varphi\big(2^{-q}D\big)u = \int_{\mathbb{R}^n} h_q(y)u(x-y)\,\mathrm{d}y, \quad \forall q \in \mathbb{Z}.$$

We shall also use the following low-frequency cut-off:

$$\dot{S}_q u \triangleq \chi\big(2^{-q}D\big)u.$$

Definition 2.1. Let $\mathcal{S}'_h$ be the space of temperate distributions u such that

$$\lim_{q\to-\infty} \dot{S}_q u = 0, \quad \text{in } \mathcal{S}'.$$

The formal equality

$$u = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u \tag{2.1}$$

holds in $\mathcal{S}'_h$ and is called the *homogeneous Littlewood–Paley decomposition*. It has nice properties of quasi-orthogonality

$$\dot{\Delta}_q \dot{\Delta}_{q'} u \equiv 0 \quad \text{if } |q - q'| \geqslant 2. \tag{2.2}$$

Let us now define the homogeneous Besov spaces

Definition 2.2. For $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ and $u \in \mathcal{S}'_h$, we set

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \left(\sum_{q\in\mathbb{Z}} 2^{qsr} \|\dot{\Delta}_q u\|^r_{L^p}\right)^{\frac{1}{r}} \quad \text{if } r < +\infty,$$

and

$$\|u\|_{\dot{B}^s_{p,\infty}} \triangleq \sup_{q\in\mathbb{Z}} 2^{qs} \|\dot{\Delta}_q u\|_{L^p}.$$

We then define the *homogeneous Besov spaces* as

$$\dot{B}^s_{p,r} \triangleq \left\{u \in \mathcal{S}'_h \;\middle|\; \|u\|_{\dot{B}^s_{p,r}} < +\infty\right\}.$$

The above definition does not depend on the choice of the couple (χ,φ). We can further remark that if $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $r = 1$, then $\dot{B}^s_{p,r}$ is a Banach space.

About complete study of Besov spaces, please refer to [1,4,25]. Let us just recall some basic properties.

Proposition 2.1. *The following properties hold* (*refer to* [25]):

1. $\dot{B}^s_{2,2} = \dot{H}^s$.
2. *Generalized derivatives*: *Let f be a smooth function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree m. Assume that $s - m < \frac{n}{p}$ or $s - m = \frac{n}{p}$ and $r = 1$, then $f(D)$ is continuous from $\dot{B}^s_{p,r}$ to $\dot{B}^{s-m}_{p,r}$.*
3. *If r is finite, then $\mathcal{C}^\infty_c \cap \dot{B}^s_{p,r}$ is densely embedded in $\dot{B}^s_{p,r}$.*
4. *Sobolev embedding*: *If $p_1 \leqslant p_2$ and $r_1 \leqslant r_2$, then $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.*
5. *Real interpolation*: $\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leqslant \|u\|^\theta_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}$, *for $\theta \in [0,1]$.*

We have the following continuity properties for the product of two functions (refer to [25]).

Proposition 2.2. *If $1 \leqslant p, r \leqslant \infty$, $s_1, s_2 < \frac{n}{p}$ and $s_1 + s_2 > 0$, there exists a positive constant $C = C(s_1, s_2, p, r, n)$ such that*

$$\|uv\|_{\dot{B}^{s_1+s_2-\frac{n}{p}}_{p,r}} \leqslant C\|u\|_{\dot{B}^{s_1}_{p,r}} \|v\|_{\dot{B}^{s_2}_{p,r}}. \tag{2.3}$$

For the time–space used in Theorem 1.1, we have the following definition.

Definition 2.3. Let $s \in \mathbb{R}$, $1 \leqslant p, r, \rho \leqslant +\infty$ and $I = [0,T)$, $T \in (0,+\infty]$. We set

$$\|u\|_{\mathcal{L}^\rho(I;\dot{B}^s_{p,r})} \triangleq \left(\sum_{q\in\mathbb{Z}} 2^{qsr} \|\dot{\Delta}_q u\|^r_{L^\rho(I;L^p)}\right)^{\frac{1}{r}} \tag{2.4}$$

and denote by $\mathcal{L}^\rho(I;\dot{B}^s_{p,r})$ the set of distributions of $\mathcal{S}'(I\times\mathbb{R}^n)$ with finite $\|\cdot\|_{\mathcal{L}^\rho(I;\dot{B}^s_{p,r})}$ norm.

Let us remark that by virtue of Minkowski inequality, we have

$$\|u\|_{\mathcal{L}^\rho(I;\dot{B}^s_{p,r})} \leqslant \|u\|_{L^\rho(I;\dot{B}^s_{p,r})} \quad \text{if } \rho \leqslant r,$$

and

$$\|u\|_{L^\rho(I;\dot{B}^s_{p,r})} \leqslant \|u\|_{\mathcal{L}^\rho(I;\dot{B}^s_{p,r})} \quad \text{if } \rho \geqslant r.$$

3. Some estimates of linear equation

In this section we will investigate some time–space estimates of solution to the Cauchy problem of the following linear fractional power dissipative equation:

$$\begin{cases} u_t + (-\Delta)^\alpha u = f(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $u_0 \in \dot{B}^s_{p,r}$ and $f \in \mathcal{L}^\rho(I; \dot{B}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,r})$.

At first, let us prove estimates for the semi-group of the fractional power dissipative equation restricted to functions with compact supports away from the origin in Fourier variables.

Lemma 3.1. *Let ϕ be a smooth function supported in the shell $\{\xi \in \mathbb{R}^n \mid R_1 \leqslant |\xi| \leqslant R_2,\ 0 < R_1 < R_2\}$. There exist two positive constants κ and C_1 depending only on ϕ such that for all $1 \leqslant p \leqslant \infty$, $\tau \geqslant 0$ and $\lambda > 0$, we have*

$$\big\|\phi(\lambda^{-1}D)e^{-\tau(-\Delta)^\alpha}u\big\|_{L^p} \leqslant C_1 e^{-\kappa\tau\lambda^{2\alpha}}\big\|\phi(\lambda^{-1}D)u\big\|_{L^p}. \tag{3.2}$$

Proof. Let $\tilde{\phi}$ be a smooth function supported in the shell $\{\xi \in \mathbb{R}^n \mid R_1' \leqslant |\xi| \leqslant R_2'\}$ for some $0 < R_1' < R_1$ and $R_2' > R_2$ such that $\tilde{\phi} \equiv 1$ in a neighborhood of $\operatorname{supp}\phi$. We have

$$\begin{aligned} \mathcal{F}\big(\phi(\lambda^{-1}D)e^{-\tau(-\Delta)^\alpha}u\big)(\xi) &= \phi(\lambda^{-1}\xi)e^{-\tau|\xi|^{2\alpha}}\mathcal{F}(u)(\xi) \\ &= \tilde{\phi}(\lambda^{-1}\xi)e^{-\tau|\xi|^{2\alpha}}\phi(\lambda^{-1}\xi)\mathcal{F}(u)(\xi) \\ &= \big(\tilde{\phi}(\lambda^{-1}\xi)e^{-\tau|\xi|^{2\alpha}}\big)\mathcal{F}\big(\phi(\lambda^{-1}D)u\big)(\xi). \end{aligned}$$

Thus we have

$$\phi(\lambda^{-1}D)e^{-\tau(-\Delta)^\alpha}u = g_\lambda(\tau,\cdot) * \phi(\lambda^{-1}D)u,$$

where

$$g_\lambda(\tau,x) \triangleq (2\pi)^{-n}\int_{\mathbb{R}^n} \tilde{\phi}(\lambda^{-1}\xi)e^{-\tau|\xi|^{2\alpha}}e^{ix\cdot\xi}\,\mathrm{d}\xi.$$

According to Young equality, we have

$$\big\|\phi(\lambda^{-1}D)e^{-\tau(-\Delta)^\alpha}u\big\|_{L^p} \leqslant \big\|g_\lambda(\tau,\cdot)\big\|_{L^1}\big\|\phi(\lambda^{-1}D)u\big\|_{L^p}.$$

Let $g(\tau,x) \triangleq (2\pi)^{-n}\int_{\mathbb{R}^n}\tilde{\phi}(\xi)e^{-\tau|\xi|^{2\alpha}}e^{ix\cdot\xi}\,\mathrm{d}\xi$, by simple computation we have

$$\begin{aligned} g_\lambda(\tau,x) &= \lambda^n(2\pi)^{-n}\int_{\mathbb{R}^n}\tilde{\phi}(\lambda^{-1}\xi)e^{-\tau\lambda^{2\alpha}|\lambda^{-1}\xi|^{2\alpha}}e^{i\lambda x\cdot\lambda^{-1}\xi}\,\mathrm{d}(\lambda^{-1}\xi) \\ &= \lambda^n g\big(\tau\lambda^{2\alpha},\lambda x\big), \end{aligned}$$

thus $\|g_\lambda(\tau,\cdot)\|_{L^1} = \|\lambda^n g(\tau\lambda^{2\alpha},\lambda x)\|_{L^1} = \|g(\tau\lambda^{2\alpha},\cdot)\|_{L^1}$. Therefore it is sufficient to prove that there exist two positive constants κ and C_1 such that

$$\big\|g(\tau,\cdot)\big\|_{L^1} \leqslant C_1 e^{-\kappa\tau}. \tag{3.3}$$

In fact, we have

$$\begin{aligned} g(\tau,x) &= (2\pi)^{-n}\big(1+|x|^2\big)^{-n}\int_{\mathbb{R}^n}\big(1+|x|^2\big)^n\tilde{\phi}(\xi)e^{-\tau|\xi|^{2\alpha}}e^{ix\cdot\xi}\,\mathrm{d}\xi \\ &= (2\pi)^{-n}\big(1+|x|^2\big)^{-n}\int_{\mathbb{R}^n}\tilde{\phi}(\xi)e^{-\tau|\xi|^{2\alpha}}(\mathrm{Id}-\Delta_\xi)^n e^{ix\cdot\xi}\,\mathrm{d}\xi \\ &= (2\pi)^{-n}\big(1+|x|^2\big)^{-n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}(\mathrm{Id}-\Delta_\xi)^n\big(\tilde{\phi}(\xi)e^{-\tau|\xi|^{2\alpha}}\big)\,\mathrm{d}\xi. \end{aligned}$$

From the last equality and the fact that the integration may be restricted to $\operatorname{supp}\tilde{\phi}$, we conclude that there exist two positive constants κ and C_2 such that

$$\left|g(\tau,x)\right| \leqslant C_2\left(1+|x|^2\right)^{-n} e^{-\kappa\tau}.$$

Thus we can get (3.3). □

Let us now state our result for the linear fractional power dissipative equation (3.1).

Theorem 3.2. *Let $0<T\leqslant+\infty$, $I=[0,T)$, $s\in\mathbb{R}$ and $1\leqslant\rho,p,r\leqslant+\infty$. Assume that $u_0\in\dot{B}^s_{p,r}$ and $f\in\mathcal{L}^\rho(I;\dot{B}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,r})$. Then the Cauchy problem (3.1) has a unique solution $u\in\mathcal{L}^\infty(I;\dot{B}^s_{p,r})\cap\mathcal{L}^\rho(I;\dot{B}^{s+\frac{2\alpha}{\rho}}_{p,r})$ and there exists a constant $C_3>0$ depending only on n such that $\forall\rho_1\in[\rho,+\infty]$, we have*

$$\|u\|_{\mathcal{L}^{\rho_1}(I;\dot{B}^{s+\frac{2\alpha}{\rho_1}}_{p,r})} \leqslant C_3\left(\|u_0\|_{\dot{B}^s_{p,r}}+\|f\|_{\mathcal{L}^\rho(I;\dot{B}^{s+\frac{2\alpha}{\rho}-2\alpha}_{p,r})}\right). \tag{3.4}$$

If in addition $r<+\infty$, then $u\in\mathcal{C}(I;\dot{B}^s_{p,r})$.

Proof. Since u_0 and f are temperate distributions, Eq. (3.1) has a unique solution u in $\mathcal{S}'(I\times\mathbb{R}^n)$, which satisfies

$$\hat{u}(t,\xi)=e^{-t|\xi|^{2\alpha}}\widehat{u_0}(\xi)+\int_0^t e^{-(t-\tau)|\xi|^{2\alpha}}\hat{f}(\tau,\xi)\,\mathrm{d}\tau. \tag{3.5}$$

Because $u_0\in\mathcal{S}'_h(\mathbb{R}^n)$ and $f\in\mathcal{S}'_h(I\times\mathbb{R}^n)$, we easily get $u\in\mathcal{S}'_h(I\times\mathbb{R}^n)$. Now, applying $\dot{\Delta}_q$ to (3.1) yields

$$\dot{\Delta}_q u(t)=e^{-t(-\Delta)^{2\alpha}}\dot{\Delta}_q u_0+\int_0^t e^{-(t-\tau)(-\Delta)^{2\alpha}}\dot{\Delta}_q f(\tau)\,\mathrm{d}\tau. \tag{3.6}$$

Thus we get

$$\left\|\dot{\Delta}_q u(t)\right\|_{L^p}\leqslant\left\|e^{-t(-\Delta)^{2\alpha}}\dot{\Delta}_q u_0\right\|_{L^p}+\int_0^t\left\|e^{-(t-\tau)(-\Delta)^{2\alpha}}\dot{\Delta}_q f(\tau)\right\|_{L^p}\mathrm{d}\tau. \tag{3.7}$$

By virtue of Lemma 3.1, we have for some $\kappa>0$,

$$\left\|\dot{\Delta}_q u(t)\right\|_{L^p}\lesssim e^{-\kappa 2^{2\alpha q}t}\|\dot{\Delta}_q u_0\|_{L^p}+\int_0^t e^{-\kappa 2^{2\alpha q}(t-\tau)}\left\|\dot{\Delta}_q f(\tau)\right\|_{L^p}\mathrm{d}\tau. \tag{3.8}$$

By Young equality, we get

$$\left\|\dot{\Delta}_q u(t)\right\|_{L^{\rho_1}(I;L^p)}\lesssim\left(\frac{1-e^{-\kappa 2^{2\alpha q}\rho_1 T}}{\kappa 2^{2\alpha q}\rho_1}\right)^{\frac{1}{\rho_1}}\|\dot{\Delta}_q u_0\|_{L^p}+\left(\frac{1-e^{-\kappa 2^{2\alpha q}\rho_2 T}}{\kappa 2^{2\alpha q}\rho_2}\right)^{\frac{1}{\rho_2}}\left\|\dot{\Delta}_q f(\tau)\right\|_{L^\rho(I;L^p)}, \tag{3.9}$$

where $1+\frac{1}{\rho_1}=\frac{1}{\rho_2}+\frac{1}{\rho}$.

Finally, taking the $l^r(\mathbb{Z})$ norm, we conclude that

$$\begin{aligned}\|u\|_{\mathcal{L}^{\rho_1}(I;\dot{B}^{s+\frac{2\alpha}{\rho_1}}_{p,r})}\lesssim{}&\left[\sum_{q\in\mathbb{Z}}\left(\frac{1-e^{-\kappa 2^{2\alpha q}\rho_1 T}}{\kappa\rho_1}\right)^{\frac{r}{\rho_1}}\left(2^{qs}\|\dot{\Delta}_q u_0\|_{L^p}\right)^r\right]^{\frac{1}{r}}\\&+\left[\sum_{q\in\mathbb{Z}}\left(\frac{1-e^{-\kappa 2^{2\alpha q}\rho_2 T}}{\kappa\rho_2}\right)^{\frac{r}{\rho_2}}\left(2^{q(s+\frac{2\alpha}{\rho}-2\alpha)}\left\|\dot{\Delta}_q f(\tau)\right\|_{L^\rho(I;L^p)}\right)^r\right]^{\frac{1}{r}}.\end{aligned}$$

Thus, we get that $u\in\mathcal{L}^\infty(I;\dot{B}^s_{p,r})\cap\mathcal{L}^\rho(I;\dot{B}^{s+\frac{2\alpha}{\rho}}_{p,r})$ and satisfies the inequality (3.4).

That $u \in \mathcal{C}(I; \dot{B}^s_{p,r})$ in the case where r is finite may be easily deduced from the density of $\mathcal{S} \cap \dot{B}^s_{p,r}$ in $\dot{B}^s_{p,r}$ (see Proposition 2.1). $\square$

4. Well-posedness in critical Besov spaces

In this section we make use of the results derived in Section 3, "mono-norm method," Fourier localization technique and Littlewood–Paley theory to prove the well-posedness in critical Besov spaces $\dot{B}^{\sigma}_{2,r}$ with $\sigma \triangleq \frac{n}{2} - \frac{2\alpha - d}{b}$, and we will also prove the blow-up criterion.

Lemma 4.1. *Let $Q(u_1, u_2, \ldots, u_{b+1}) = P(D) \prod_{j=1}^{b+1} u_j$. Then for the $(b+1)$-linear map $Q(u_1, u_2, \ldots, u_{b+1})$, when $\sigma > -\frac{n}{2}$, there exists a constant C_4 such that*

$$\left\| Q(u_1, u_2, \ldots, u_{b+1}) \right\|_{\dot{B}^{\sigma-d}_{2,r}} \leqslant C_4 \prod_{j=1}^{b+1} \|u_j\|_{\dot{B}^{\sigma + \frac{2\alpha-d}{b+1}}_{2,r}} \tag{4.1}$$

and

$$\left\| Q(u_1, u_2, \ldots, u_{b+1}) \right\|_{\mathcal{L}^{\frac{2\alpha}{2\alpha-d}}(I; \dot{B}^{\sigma-d}_{2,r})} \leqslant C_4 \prod_{j=1}^{b+1} \|u_j\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}(I; \dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})}. \tag{4.2}$$

Proof. According to Proposition 2.2, under the assumption that $\sigma > -\frac{n}{2}$, we may easily get the proof of (4.1). The proof of (4.2) is referred to [25]. $\square$

Now we give a lemma which proof can be found in [20].

Lemma 4.2. *Let X be a Banach space and let $B : X \times X \times \cdots \times X \to X$ be an m-linear continuous operator satisfying*

$$\left\| B(u_1, u_2, \ldots, u_m) \right\|_X \leqslant K \prod_{j=1}^{m} \|u_j\|_X \quad \text{for all } u_1, u_2, \ldots, u_m \in X, \tag{4.3}$$

for some constant $K > 0$. Let $R > 0$ be such that $m(2R)^{m-1}K < 1$. Then for every $y \in X$ with $\|y\|_X \leqslant R$ the equation

$$u = y + B(u, u, \ldots, u) \tag{4.4}$$

has a unique solution $u \in X$ satisfying that $\|u\|_X \leqslant 2R$ and $\|u\|_X \leqslant \frac{m}{m-1}\|y\|_X$. Moreover, the solution u depends continuously on y in the sense that, if $\|z\|_X \leqslant R$ and $v = z + B(v, v, \ldots, v)$, $\|v\|_X \leqslant 2R$, then

$$\|u - v\|_X \leqslant \frac{1}{1 - m(2R)^{m-1}K} \|y - z\|_X. \tag{4.5}$$

From now on, we begin to prove Theorem 1.1.

Proof of Theorem 1.1. *Step* 1. The case for small u_0.

From (1.1), we have

$$\begin{aligned} u &= e^{-t(-\Delta)^\alpha} u_0 + \int_0^t e^{-(t-t')(-\Delta)^\alpha} Q(u, u, \ldots, u)\, dt' \\ &\triangleq e^{-t(-\Delta)^\alpha} u_0 + B(u, u, \ldots, u). \end{aligned} \tag{4.6}$$

Let $\mathscr{X}(I) \triangleq \mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}(I; \dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})$, now we consider the $(b+1)$-linear map $B(u_1, u_2, \ldots, u_{b+1})$. According to Theorem 3.2 and Lemma 4.1, we get

$$\begin{aligned}\big\|B(u_1,u_2,\dots,u_{b+1})\big\|_{\mathscr{X}(I)} &\leqslant \big\|Q(u_1,u_2,\dots,u_{b+1})\big\|_{\mathcal{L}^{\frac{2\alpha}{2\alpha-d}}(I;\dot B^{\sigma-d}_{2,r})}\\ &\leqslant C_4\prod_{j=1}^{b+1}\|u_j\|_{\mathscr{X}(I)}.\end{aligned}\tag{4.7}$$

By Lemma 4.2, we know that, if we can prove $\|e^{-t(-\Delta)^\alpha}u_0\|_{\mathscr{X}(I)}\leqslant R$ with R satisfying $(b+1)(2R)^bC_4<1$, then (4.6) has a unique solution in $B_{2R}(0)$, where $B_{2R}(0)$ is a closed Ball with center 0 and radius $2R$ in $\mathscr{X}(I)$.

In fact, according to Theorem 3.2, there exists a constant $c>0$ such that when $\|u_0\|_{\dot B^\sigma_{2,r}}\leqslant c$, we have $\|e^{-t(-\Delta)^\alpha}u_0\|_{\mathscr{X}(I)}\leqslant R$. Therefore, (4.6) has a unique global solution ($T=\infty$) such that

$$\|u\|_{\mathscr{X}(I)}\leqslant\frac{b+1}{b}\big\|e^{-t(-\Delta)^\alpha}u_0\big\|_{\mathscr{X}(I)}\leqslant\frac{b+1}{b}R\leqslant 2R.\tag{4.8}$$

Step 2. The case for large u_0.

According to absolute continuity of norm, there exists $N\in\mathbb{N}$ such that

$$\begin{aligned}\widehat{u_0}(\xi)&=\widehat{u_0}(\xi)\chi_{|\xi|\geqslant 2^N}(\xi)+\widehat{u_0}(\xi)\chi_{|\xi|\leqslant 2^N}(\xi)\\ &\triangleq\widehat{u_{0h}}+\widehat{u_{0l}}\end{aligned}\tag{4.9}$$

and

$$\|u_{0h}\|_{\dot B^\sigma_{2,r}}\leqslant\frac12 c.\tag{4.10}$$

Thus we have

$$\big\|e^{-t(-\Delta)^\alpha}u_0\big\|_{\mathscr{X}(I)}\leqslant\frac12 R+\big\|e^{-t(-\Delta)^\alpha}u_{0l}\big\|_{\mathscr{X}(I)}.\tag{4.11}$$

But

$$\begin{aligned}\big\|e^{-t(-\Delta)^\alpha}u_{0l}\big\|_{\mathscr{X}(I)}&\leqslant 2^{N\frac{2\alpha-d}{b+1}}\big\|e^{-t(-\Delta)^\alpha}u_{0l}\big\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}(I;\dot B^\sigma_{2,r})}\\ &\leqslant 2^{N\frac{2\alpha-d}{b+1}}T^{\frac{2\alpha-d}{2(b+1)\alpha}}C_3\|u_0\|_{\dot B^\sigma_{2,r}},\end{aligned}\tag{4.12}$$

thus if we choose T to satisfy

$$2^{N\frac{2\alpha-d}{b+1}}T^{\frac{2\alpha-d}{2(b+1)\alpha}}C_3\|u_0\|_{\dot B^\sigma_{2,r}}\leqslant\frac12 R,\tag{4.13}$$

that is

$$T\leqslant\left(\frac{R}{2^{1+N\frac{2\alpha-d}{b+1}}C_3\|u_0\|_{\dot B^\sigma_{2,r}}}\right)^{\frac{2(b+1)\alpha}{2\alpha-d}},\tag{4.14}$$

then by Lemma 4.2 we can conclude that (4.6) has a unique solution in the closed ball $B_{2R}(0)$ in $\mathscr{X}(I)$.

Step 3. Now let us prove the regularity.

$u\in\mathscr{X}(I)$ is the solution of (1.1), then by Lemma 4.1 we can get

$$Q(u,u,\dots,u)\in\mathcal{L}^{\frac{2\alpha}{2\alpha-d}}\big(I;\dot B^{\sigma-d}_{2,r}\big),\tag{4.15}$$

therefore by Theorem 3.2 we have

$$u\in\mathcal{L}^\infty\big(I;\dot B^\sigma_{2,r}\big)\cap\mathcal{L}^{\frac{2\alpha}{2\alpha-d}}\big(I;\dot B^{\sigma+2\alpha-d}_{2,r}\big),\tag{4.16}$$

and if $r<+\infty$, then $u\in\mathcal{C}(I;\dot B^\sigma_{2,r})$.

Step 4. Let u, v be two solutions of (1.1) in $\mathscr{X}(I)$ for initial data u_0 and v_0, then $w=u-v$ satisfies

$$\begin{cases}w_t+(-\Delta)^\alpha w=F(u)-F(v), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^n,\\ w(0,x)=w_0(x)=u_0(x)-v_0(x), & x\in\mathbb{R}^n.\end{cases}\tag{4.17}$$

According to Theorem 3.2 and Lemma 4.1, we have

$$\|w\|_{\mathscr{X}(I)} \leqslant C_3\big(\|w_0\|_{\dot{B}^{\sigma}_{2,r}} + \big\|F(u) - F(v)\big\|_{\mathcal{L}^{\frac{2\alpha}{2\alpha-d}}(I;\dot{B}^{\sigma-d}_{2,r})}\big)$$
$$\leqslant C_3\left(\|w_0\|_{\dot{B}^{\sigma}_{2,r}} + C_4\sum_{j=0}^{b}\|u\|^{j}_{\mathscr{X}(I)}\|v\|^{b-j}_{\mathscr{X}(I)}\|w\|_{\mathscr{X}(I)}\right). \tag{4.18}$$

Denoting $Z(T) \triangleq C_3C_4\sum_{j=0}^{b}\|u\|^{j}_{\mathscr{X}(I)}\|v\|^{b-j}_{\mathscr{X}(I)}$, we have

$$\|w\|_{\mathscr{X}(I)} \leqslant C_3\|w_0\|_{\dot{B}^{\sigma}_{2,r}} + Z(T)\|w\|_{\mathscr{X}(I)}. \tag{4.19}$$

Lebesgue dominated convergence theorem insures that Z is a continuous nondecreasing function which vanishes at zero. Hence for small enough T_1 we have $Z(T_1) \leqslant \frac{1}{2}$ and

$$\|w\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}([0,T_1);\dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})} \leqslant 2C_3\|w_0\|_{\dot{B}^{\sigma}_{2,r}}. \tag{4.20}$$

Now a standard connectivity argument like as $[0,T_1), [T_1,2T_1), \ldots$ enable us to conclude that there exists a constant $C>0$ such that

$$\|w\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}([0,T);\dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})} \leqslant C\|w_0\|_{\dot{B}^{\sigma}_{2,r}}. \tag{4.21}$$

Thus (1.3) is proved. □

Remark 4.1. According to Proposition 2.1, $\dot{B}^{\sigma}_{2,2} = \dot{H}^{\sigma}$, thus when $r=2$, Theorem 1.1 implied the well-posedness in Sobolev space.

Finally let us prove the blow-up criterion.

Proof of Theorem 1.2. We will prove that if the solution $u(t)$ satisfies

$$\|u\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}([0,T);\dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})} < +\infty, \tag{4.22}$$

then $T^{\star}_{u_0} > T$. ($\Rightarrow$ If $T^{\star}_{u_0} < +\infty$, then $\|u\|_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}([0,T^{\star}_{u_0});\dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})} = +\infty$.)

According to Theorem 3.2 and Lemma 4.1, we can get

$$\|u\|_{\mathcal{L}^{\infty}([0,T);\dot{B}^{\sigma}_{2,r})} \leqslant C_3\big(\|u_0\|_{\dot{B}^{\sigma}_{2,r}} + \big\|Q(u,u,\ldots,u)\big\|_{\mathcal{L}^{\frac{2\alpha}{2\alpha-d}}([0,T);\dot{B}^{\sigma-d}_{2,r})}\big)$$
$$\leqslant C_3\big(\|u_0\|_{\dot{B}^{\sigma}_{2,r}} + C_4\|u\|^{b+1}_{\mathcal{L}^{\frac{2(b+1)\alpha}{2\alpha-d}}([0,T);\dot{B}^{\sigma+\frac{2\alpha-d}{b+1}}_{2,r})}\big)$$
$$< +\infty. \tag{4.23}$$

Therefore there exists $N \in \mathbb{N}$ such that $\forall t \in [0,T)$,

$$\hat{u}(\xi) = \hat{u}(\xi)\chi_{|\xi|\geqslant 2^N}(\xi) + \hat{u}(\xi)\chi_{|\xi|\leqslant 2^N}(\xi)$$
$$\triangleq \widehat{u_h} + \widehat{u_l} \tag{4.24}$$

and

$$\|u_h\|_{\dot{B}^{\sigma}_{2,r}} \leqslant \frac{1}{2}c. \tag{4.25}$$

Now, taking $\forall t \in [0,T)$ as initial time, we can choose $\widetilde{T}$ to satisfy

$$\widetilde{T} - t \leqslant \left(\frac{R}{2^{1+N\frac{2\alpha-d}{b+1}}C_3\|u(t)\|_{\dot{B}^{\sigma}_{2,r}}}\right)^{\frac{2(b+1)\alpha}{2\alpha-d}}, \tag{4.26}$$

thus we have

$$\widetilde{T} \leqslant t + \left(\frac{R}{2^{1+N\frac{2\alpha-d}{b+1}} C_3 \sup\limits_{0\leqslant t\leqslant T} \|u(t)\|_{\dot{B}^{\sigma}_{2,r}}} \right)^{\frac{2(b+1)\alpha}{2\alpha-d}}. \tag{4.27}$$

Let $t \to T$, then $\widetilde{T}$ is larger than T. Thus the conclusion is proved. □

Acknowledgments

We are deeply grateful to the referees and the associated editor for their invaluable comments and suggestions which helped improve the paper greatly.

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