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Boundedness for iterated commutators on the mixed norm spaces ☆

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Abstract

This paper studies the iterated commutators on mixed norm spaces $L^2(\phi)$ characterizing the conjugate holomorphic symbols for which the corresponding iterated commutators are bounded by using the Bergman geometry, properties of holomorphic functions and related analysis.

Keywords: Mixed norm spaces; Iterated commutator; Normal function; Boundedness

1. Introduction

Let dV be the Lebesgue measure on the unit ball B of C^m normalized so that $V(B) = 1$, and $d\sigma$ be the normalized rotation invariant measure on the boundary ∂B of B so that $\sigma(\partial B) = 1$. The class of all holomorphic functions on B is denoted by $H(B)$ and $H^\infty(B)$ denotes the class as all bounded holomorphic functions on B. Let ϕ be a positive continuous function on $[0, 1)$. ϕ is called a normal function if there are two constants a and b: $0 < a < b$ such that

$$\frac{\phi(t)}{(1-t^2)^a} \text{ decreases for } t_0 \leqslant t < 1 \text{ and } \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^a} = 0,$$

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$$\frac{\phi(t)}{(1-t^2)^b} \text{ increases for } t_0 \leqslant t < 1 \text{ and } \lim_{t\to 1^-} \frac{\phi(t)}{(1-t^2)^b} = \infty.$$

Throughout this paper, we always suppose that $\{\phi, \psi\}$ is a normal pair, that is $\psi(t) = \frac{(1-t^2)^\alpha}{\phi(t)}$ for some $\alpha > b$. We will also need the following measure

$$dV_\phi(z) = \frac{\phi^2(|z|)}{1-|z|^2}\, dV(z), \quad z \in B.$$

Let $L^2(\phi)$ denote the space of all locally L^2 integrable functions f on B such that

$$\|f\|_\phi = \left\{ \int_B \left|f(z)\right|^2 dV_\phi(z) \right\}^{1/2} < \infty.$$

Let $A^2(\phi) = L^2(\phi) \cap H(B)$. Suppose $w \in B$, α is the constant mentioned in the above, and

$$k_w^{\alpha,m}(z) = \frac{1}{(1-\overline{w}z)^{\alpha+m}}, \quad \forall z \in B.$$

Then the linear operator P is defined as follows:

$$(Pf)(w) = c_{m,\alpha} \int_B f(z)\overline{k_w^{\alpha,m}(z)}\left(1-|z|^2\right)^{\alpha-1} dV(z), \quad f \in L^2(\phi) \cap L^2(\psi),$$

where $c_{m,\alpha} = \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)\Gamma(m+1)}$.

We note that [1] the operator P is bounded from both $L^2(\phi)$ onto $A^2(\phi)$ and $L^2(\psi)$ onto $A^2(\psi) = L^2(\psi) \cap H(B)$. Moreover $(Pf)(w) = f(w)$ for $f \in A^2(\phi) \cap A^2(\psi)$. Given a measurable function b on B, the multiplication operator M_b is defined by $M_b(f) = bf$. X.J. Zhang, J.B. Xiao, and Z.J. Hu in [2] characterized the multiplication operator M_b between the mixed spaces. The commutator (first order) with symbol b is the operator defined by $C_b = [M_b, P] = M_bP - PM_b$, the Hankel operator H_b with symbol b is given by

$$H_b(f) = (I-P)(bf), \quad f \in A^2(\phi),$$

we have $C_b = H_b$ on $A^2(\phi)$. And therefore, the study of the first order commutators is parallel to that of the Hankel operators. We refer the reader to [3–5] for results along this line. Let b_j be a measurable function, $j = 1, 2, \ldots, n$, if $\mathrm{b} = (b_1, b_2, \ldots, b_n)$, define the nth order iterated commutator with symbol b by

$$C_{\mathrm{b}} = \left[M_{b_n}, \ldots, \left[M_{b_2}, [M_{b_1}, P]\right]\ldots\right].$$

Unlike the commutators, the iterated commutators are no longer depending linearly on their symbols. Therefore most of the techniques used successfully in studying the first order commutators fail to work effectively for the nth order iterated commutators.

A straightforward computation shows that the explicit formula for C_{b} is

$$C_{\mathrm{b}}(f)(w) = c_{m,\alpha} \int_B \prod_{j=1}^n \left(b_j(w) - b_j(z)\right)\overline{k_w^{\alpha,m}(z)}\left(1-|z|^2\right)^{\alpha-1} f(z)\, dV(z), \tag{1}$$

$\forall f \in L^\infty$. Because L^∞ is dense in $L^2(\phi)$, we take (1) as the definition of the iterated commutator C_{b} on $L^2(\phi)$, if $b_j \in H(D)$, $j = 1, 2, \ldots, n$, we have the following identity

$$C_{\bar{\mathrm{b}}}\left(k_z^{\alpha,m}\right)(w) = \prod_{j=1}^n \left(\overline{b_j(w)} - \overline{b_j(z)}\right) k_z^{\alpha,m}(w), \tag{2}$$

if $b_1 = b_2 = \cdots = b_n = h$, we write $C_{\rm b} = C_h^n$, then

$$C_h^n(f)(w) = c_{m,\alpha} \int_B \big(h(w) - h(z)\big)^n \overline{k_w^{\alpha,m}(z)} \big(1 - |z|^2\big)^{\alpha-1} f(z)\, dV(z), \tag{3}$$

$$C_{\bar{h}}^n\big(k_z^{\alpha,m}\big)(w) = \big(\bar{h}(w) - \bar{h}(z)\big)^n k_z^{\alpha,m}(w). \tag{4}$$

The main result of this paper is Theorem 9 stated below as Theorem.

Theorem. *Suppose $h \in H(B)$. Then, $C_{\bar{h}}^n$ is bounded on $L^2(\phi)$ if and only if $h \in \mathcal{B}$.*

Corollary. *Suppose $h \in H(B)$. Then $C_{\bar{h}}^n$ is bounded on $L^2(dV_\lambda)$ if and only if $h \in \mathcal{B}$, where $dV_\lambda(z) = C_\lambda(1 - |z|^2)^\lambda\, dV(z)$, $\lambda > -1$.*

Proof. In Theorem, take $\phi(t) = C_\lambda^{1/2}(1 - t^2)^{(\lambda+1)/2}$, we have

$$dV_\phi(z) = \frac{\phi^2(|z|)}{1 - |z|^2}\, dV(z) = C_\lambda\big(1 - |z|^2\big)^\lambda\, dV(z) = dV_\lambda(z).$$

It is easy to see that the corollary holds. □

Thus the Theorem is a generalization of Theorem 3.3 in [6]. In particular, the Theorem partially answers a question posed at the end of [6], $L^2(\phi)$ is more general than $L^2(dV_\lambda)$. We still do not know how to characterize the antiholomorphic symbols $\mathrm{b} = (b_1, b_2, \ldots, b_n)$ such that the iterated commutator $C_{\rm b}$ is bounded.

We will use the symbol C denote a positive constant which does not depend on variables z, w and may depend on some parameters, not necessarily the same at each occurrence.

2. Preliminaries

We begin with the following quantity which was introduced by R.M. Timoney [7] in order to give a definition of the Bloch semi-norm that is invariant under biholomorphic mappings.

$$Q_f(z) = \sup\left\{\frac{|\sum_{k=1}^m w_k \frac{\partial f}{\partial z_k}(z)|}{\sqrt{\langle B(z)w, w\rangle}}:\ w \in C^m,\ w \neq 0\right\}.$$

Here

$$B(z) = \frac{1}{m+1}\left(\frac{\partial^2}{\partial \overline{z_i}\partial z_j} \log K(z,z)\right)_{m\times m}$$

is the Bergman matrix at z and

$$K(z,w) = \frac{1}{(1 - \langle z, w\rangle)^{m+1}}$$

is the Bergman kernel of B. The Bloch space $\mathcal{B}$ (introduced by R.M. Timoney, [7]) is the set of holomorphic functions f on B for which

$$\|f\|_{\mathcal{B}} = \sup\big\{Q_f(z):\ z \in B\big\} < \infty.$$

R.M. Timoney has proved that the norms $\|f\|_1 = \sup\{(1 - |z|^2)|\nabla f(z)|:\ z \in B\}$ and $\|f\|_{\mathcal{B}}$ are equivalent, where $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \ldots, \frac{\partial f}{\partial z_n}(z))$ is the complex gradient of f.

Lemma 1. [7] *If $h \in \mathcal{B}$, then there exists a constant C such that*

$$\left|h(z) - h(w)\right| \leqslant C\|h\|_{\mathcal{B}}\beta(z, w), \quad \forall z, w \in B,$$

where $\beta(z, w)$ is Bergman metric on B, which is Möbius invariant.

We recall the Forelli–Rudin estimate.

Lemma 2. [8] *Suppose $z \in B$, c is real, $t > -1$ and*

$$I_{c,t}(z) = \int\limits_B \frac{(1-|w|^2)^t}{|1-z\overline{w}|^{m+1+t+c}}\, dV(w).$$

Then as $|z| \to 1^-$ we have

$$I_{c,t}(z) \sim \frac{1}{(1-|z|^2)^c}$$

for $c > 0$, and

$$I_{c,t}(z) \sim \log\frac{1}{1-|z|^2}$$

for $c = 0$, and

$$I_{c,t}(z) \sim 1$$

for $c < 0$.

Lemma 3. [9] *Suppose $\lambda \in B$, $p \in (0, \infty)$, and $r \in (0, 1)$. Then there exist constants C_i $(i = 1, 2)$ depending only on a, b, p, r and m such that*

$$C_1\bigl(1-|\lambda|^2\bigr)^m\phi^p\bigl(|\lambda|\bigr) \leqslant V_\phi^p\bigl(D(\lambda, r)\bigr) \leqslant C_2\bigl(1-|\lambda|^2\bigr)^m\phi^p\bigl(|\lambda|\bigr),$$

where $D(\lambda, r) = \{z \in B\colon |\rho_\lambda(z)| < r\}$ is the pseudohyperbolic ball with center λ and radius r, ρ_λ denotes the involutive automorphism of B satisfying $\rho_\lambda(0) = \lambda$, $\rho_\lambda(\lambda) = 0$.

If $\tau \in D(\lambda, r)$, then [2]

$$1-|\tau| \sim 1-|\lambda|, \qquad 1-|\tau|^2 \sim 1-|\lambda|^2, \qquad |1-\tau\bar{\lambda}| \sim 1-|\lambda|^2,$$

thus, by the definition of normal function, we have $\phi(|\tau|) \sim \phi(|\lambda|)$, and

$$\frac{(1-|\lambda|^2)^{m+1}}{|1-\tau\bar{\lambda}|^{2m+2}}\, dV(\tau) \leqslant \frac{C}{V_\phi^p(D(\lambda, r))}\frac{\phi^p(|\tau|)}{1-|\tau|^2}\, dV(\tau). \tag{5}$$

The following results give a new characterization of Bloch space $\mathcal{B}$.

Lemma 4. *Let $h \in H(B)$, $1 < p < \infty$. Then $h \in \mathcal{B}$ if and only if*

$$\sup_{\lambda \in B}\left\{\int\limits_B \left|(h\circ\rho_\lambda)(z) - h(\lambda)\right|^p \frac{\phi^p(|z|)}{1-|z|^2}\, dV(z)\right\}^{1/p} < \infty.$$

Proof. Assume that $h \in \mathcal{B}$. By Lemma 1, there exists a constant C such that

$$\left|h(z) - h(w)\right| \leqslant C\|h\|_{\mathcal{B}}\beta(z, w), \quad \forall z, w \in B.$$

Since the following explicit formula for the Bergman distance:

$$\beta(0, z) = \left(\frac{m+1}{8}\right)^{1/2} \log \frac{1+|z|}{1-|z|},$$

the Möbius invariance property of the Bloch space implies

$$\|h\|_{\mathcal{B}} = \left\|(h \circ \rho_\lambda)(z) - h(\lambda)\right\|_{\mathcal{B}},$$

so

$$\left|(h \circ \rho_\lambda)(z) - h(\lambda)\right| \leqslant C\|h\|_{\mathcal{B}} \log \frac{1}{1-|z|},$$

this gives

$$\begin{aligned}
&\int_B \left|(h \circ \rho_\lambda)(z) - h(\lambda)\right|^p \frac{\phi^p(|z|)}{1-|z|^2}\, dV(z) \\
&\quad \leqslant C\|h\|_{\mathcal{B}}^p \int_B \left(\log \frac{1}{1-|z|}\right)^p \frac{\phi^p(|z|)}{1-|z|^2}\, dV(z) \\
&\quad \leqslant C\|h\|_{\mathcal{B}}^p \int_0^1 \left(\log \frac{1}{1-r}\right)^p \left(1-r^2\right)^{ap-1} r^{2m-1}\, dr \\
&\quad \leqslant C\|h\|_{\mathcal{B}}^p.
\end{aligned}$$

On the other hand, writing $V_\phi^p(D(\lambda, r)) = \int_{D(\lambda,r)} \frac{\phi^p(|w|)}{1-|w|^2}\, dV(w)$, and

$$\hat{h}_{D(\lambda,r)} = \frac{1}{V_\phi^p(D(\lambda, r))} \int_{D(\lambda,r)} h(w) \frac{\phi^p(|w|)}{1-|w|^2}\, dV(w),$$

we obtain

$$\begin{aligned}
&\left|h(\lambda) - \hat{h}_{D(\lambda,r)}\right| \\
&\quad \leqslant \frac{1}{V_\phi^p(D(\lambda, r))} \int_{D(\lambda,r)} \left|h(w) - h(\lambda)\right| \frac{\phi^p(|w|)}{1-|w|^2}\, dV(w) \\
&\quad \leqslant \left(\frac{1}{V_\phi^p(D(\lambda, r))} \int_{D(\lambda,r)} \left|h(w) - h(\lambda)\right|^p \frac{\phi^p(|w|)}{1-|w|^2}\, dV(w)\right)^{1/p}.
\end{aligned}$$

By the triangle inequality for the L^p integral,

$$\begin{aligned}
&\left(\frac{1}{V_\phi^p(D(\lambda, r))} \int_{D(\lambda,r)} \left|h(z) - \hat{h}_{D(\lambda,r)}\right|^p \frac{\phi^p(|z|)}{1-|z|^2}\, dV(z)\right)^{1/p} \\
&\quad \leqslant 2\left(\frac{1}{V_\phi^p(D(\lambda, r))} \int_{D(\lambda,r)} \left|h(z) - h(\lambda)\right|^p \frac{\phi^p(|z|)}{1-|z|^2}\, dV(z)\right)^{1/p}. \qquad (6)
\end{aligned}$$

Note the identity

$$1-\left|\rho_\lambda(w)\right|^2=\frac{(1-|\lambda|^2)(1-|w|^2)}{|1-w\bar{\lambda}|^2}$$

and

$$dV\left(\rho_\lambda(w)\right)=\frac{(1-|\lambda|^2)^{m+1}}{|1-w\bar{\lambda}|^{2m+2}}\,dV(w).$$

By the monotonicity of ϕ and Lemma 3, we get that

$$\begin{aligned}
&\frac{1}{V_\phi^p(D(\lambda,r))}\int\limits_{D(\lambda,r)}\left|h(z)-h(\lambda)\right|^p\frac{\phi^p(|z|)}{1-|z|^2}\,dV(z)\\
&\quad\leqslant\frac{C}{V_\phi^p(D(\lambda,r))}\int\limits_{D(\lambda,r)}\left|h(z)-h(\lambda)\right|^p\frac{\phi^p(|\lambda|)}{1-|z|^2}\,dV(z)\\
&\quad=\frac{C\phi^p(|\lambda|)}{V_\phi^p(D(\lambda,r))}\int\limits_{D(0,r)}\left|(h\circ\rho_\lambda)(w)-h(\lambda)\right|^p\frac{1}{1-|\rho_\lambda(w)|^2}\frac{(1-|\lambda|^2)^{m+1}}{|1-w\bar{\lambda}|^{2m+2}}\,dV(w)\\
&\quad=\frac{C(1-|\lambda|^2)^m\phi^p(|\lambda|)}{V_\phi^p(D(\lambda,r))}\int\limits_{D(0,r)}\left|(h\circ\rho_\lambda)(w)-h(\lambda)\right|^p\frac{1}{(1-|w|^2)|1-w\bar{\lambda}|^{2m}}\,dV(w)\\
&\quad\leqslant C\int\limits_{D(0,r)}\left|(h\circ\rho_\lambda)(w)-h(\lambda)\right|^p\frac{\phi^p(|w|)}{1-|w|^2}\,dV(w)\\
&\quad\leqslant C\int\limits_{B}\left|(h\circ\rho_\lambda)(w)-h(\lambda)\right|^p\frac{\phi^p(|w|)}{1-|w|^2}\,dV(w).
\end{aligned}\tag{7}$$

From (6) and (7), we further have

$$\begin{aligned}
&\frac{1}{V_\phi^p(D(\lambda,r))}\int\limits_{D(\lambda,r)}\left|h(z)-\hat{h}_{D(\lambda,r)}\right|^p\frac{\phi^p(|z|)}{1-|z|^2}\,dV(z)\\
&\quad\leqslant C\int\limits_{B}\left|(h\circ\rho_\lambda)(w)-h(\lambda)\right|^p\frac{\phi^p(|w|)}{1-|w|^2}\,dV(w).
\end{aligned}\tag{8}$$

Since $\forall a\in B$,

$$\left|f(a)\right|^p\leqslant\frac{C}{(1-|a|)^{m+1}}\int\limits_B\left|f(z)\right|^p\,dV(z).$$

Let $f_r(z)=f(rz)$ for $r\in(0,1)$, then $\forall\zeta\in\partial B$,

$$\left|f(r\zeta)\right|^p\leqslant C\int\limits_{D(0,r)}\left|f(z)\right|^p\,dV(z).$$

By the Taylor expansion of $f\in H(B)$, we have

$$\int\limits_{\partial B}f(r\zeta)\bar{\zeta}^\alpha\,d\sigma(\zeta)=\frac{(D^\alpha f)(0)}{\alpha!}\omega_\alpha r^{|\alpha|},$$

where $\omega_\alpha = \int_{\partial B} |\zeta^\alpha|^2 \, d\sigma(\zeta) = \frac{(m-1)\alpha!}{(m+|\alpha|-1)!}$. Taking $\alpha = (0, \ldots, 1, \ldots, 0)$, the kth coordinate is 1, and the rest are 0,

$$\left|\frac{\partial f}{\partial z_k}(0)\right|^p \leqslant C \sup_{\zeta \in \partial B} \left|f(r\zeta)\right|^p \leqslant C \int\limits_{D(0,r)} \left|f(z)\right|^p \, dV(z) \quad (k = 1, \ldots, m).$$

By the elementary inequality, when $a > 0$, $b > 0$,

$$(a+b)^p \leqslant \begin{cases} a^p + b^p, & 0 < p < 1, \\ 2^{p-1}(a^p + b^p), & p \geqslant 1, \end{cases}$$

we have

$$\left|\nabla f(0)\right|^p \leqslant C \int\limits_{D(0,r)} \left|f(z)\right|^p \, dV(z).$$

Let $F = h \circ \rho_\lambda - \hat{h}_{D(\lambda,r)}$, then $(1 - |\lambda|^2)|\nabla h(\lambda)| \leqslant |\nabla F(0)|$. Replace f by F in the above inequality, and change the variable w by $\tau = \rho_\lambda(w)$, we obtain

$$\begin{aligned} \left((1 - |\lambda|^2)\left|\nabla h(\lambda)\right|\right)^p &\leqslant \left|\nabla F(0)\right|^p \\ &\leqslant C \int\limits_{D(0,r)} \left|h \circ \rho_\lambda(w) - \hat{h}_{D(\lambda,r)}\right|^p \, dV(w) \\ &\leqslant C \int\limits_{D(\lambda,r)} \left|h(\tau) - \hat{h}_{D(\lambda,r)}\right|^p \frac{(1 - |\lambda|^2)^{m+1}}{|1 - \tau\bar{\lambda}|^{2m+2}} \, dV(\tau) \\ &\leqslant \frac{C}{V_\phi^p(D(\lambda, r))} \int\limits_{D(\lambda,r)} \left|h(\tau) - \hat{h}_{D(\lambda,r)}\right|^p \frac{\phi^p(|\tau|)}{1 - |\tau|^2} \, dV(\tau) \\ &\leqslant C \int\limits_B \left|(h \circ \rho_\lambda)(w) - h(\lambda)\right|^p \frac{\phi^p(|w|)}{1 - |w|^2} \, dV(w). \end{aligned}$$

The above inequalities come from (8). This implies $\sup_{\lambda \in B}\{(1 - |\lambda|^2)|\nabla h(\lambda)|\} < \infty$, that is $h \in \mathcal{B}$. $\square$

Lemma 5. [9] *Let $w \in B$ and $k_w^{\alpha,m}$ be the weighted Bergman reproducing kernel. Then there exist $C_j > 0$ $(j = 1, 2, 3, 4)$ such that*

$$C_1 \leqslant \left(1 - |w|^2\right)^{\frac{m}{2}} \psi\left(|w|\right) \left\|k_w^{\alpha,m}\right\|_\phi \leqslant C_2,$$

$$C_3 \leqslant \left(1 - |w|^2\right)^{\frac{m}{2}} \phi\left(|w|\right) \left\|k_w^{\alpha,m}\right\|_\psi \leqslant C_4.$$

Lemma 6. *Suppose $2b - 2\alpha < 2\beta < 4a - 2b - \alpha - m + 1$. Then*

$$\begin{aligned} I &= \int\limits_B \beta(z, w)^n \left|k_w^{\alpha,m}(z)\right| \left(1 - |z|^2\right)^{2\beta} \, dV_\psi(z) \\ &\leqslant C\left(1 - |w|^2\right)^{2\beta-\alpha} \psi^2\left(|w|\right), \quad w \in B. \end{aligned}$$

Proof. Since $2b - 2\alpha < 2\beta < 4a - 2b - \alpha - m + 1$, we can choose $\eta < 0$ such that $t = 2(\beta + \alpha - b) - 1 + \eta > -1$, $c = 2\beta + \alpha - 4a + 2b - \eta + m - 1 < 0$. By explicit formula for the Bergman metric β we can find a constant $C > 0$ satisfying

$$\beta(0,u)^n \leqslant C\big(1-|u|^2\big)^\eta, \quad u \in B.$$

Since $\phi(t)/(1-t^2)^a$ decreases and $\phi(t)/(1-t^2)^b$ increases, we have

$$C_1 \frac{(1-|u|^2)^{2b}\phi^2(|w|)}{|1-u\overline{w}|^{4a}} \leqslant \phi^2\big(|\rho_w(u)|\big) \leqslant C_2 \frac{(1-|u|^2)^{2a}\phi^2(|w|)}{|1-u\overline{w}|^{4b}}.$$

In the following integral, let $z = \rho_w(u)$, using the identity

$$(1-u\overline{w})\big(1-\rho_w(u)\overline{w}\big) = 1-|w|^2,$$

we find

$$\begin{aligned}
I &= \int_B \beta(z,w)^n \big|k_w^{\alpha,m}(z)\big| \frac{(1-|z|^2)^{2(\beta+\alpha)-1}}{\phi^2(|z|)}\,dV(z) \\
&= \int_B \beta(0,u)^n \left|\frac{1-u\overline{w}}{1-|w|^2}\right|^{\alpha+m} \left(\frac{(1-|w|^2)(1-|u|^2)}{|1-u\overline{w}|^2}\right)^{2(\beta+\alpha)-1} \\
&\quad \times \frac{(1-|w|^2)^{m+1}}{\phi^2(|\rho_w(u)|)|1-u\overline{w}|^{2m+2}}\,dV(u) \\
&= \big(1-|w|^2\big)^{\alpha+2\beta} \int_B \beta(0,u)^n \frac{(1-|u|^2)^{2(\beta+\alpha)-1}}{|1-u\overline{w}|^{4(\beta+\alpha)-2-\alpha-m+2m+2}\phi^2(|\rho_w(u)|)}\,dV(u) \\
&\leqslant C\big(1-|w|^2\big)^{\alpha+2\beta} \frac{1}{\phi^2(|w|)} \int_B \frac{(1-|u|^2)^{2(\beta+\alpha)-2b-\eta-1}}{|1-u\overline{w}|^{4\beta+3\alpha+m-4a}}\,dV(u) \\
&\leqslant C\big(1-|w|^2\big)^{2\beta-\alpha}\psi^2\big(|w|\big), \quad w \in B.
\end{aligned}$$

The last inequality is true because of Lemma 2 for $t = 2(\beta+\alpha-b) - 1 + \eta$, $c = 2\beta + \alpha - 4a + 2b - \eta + m - 1$, $2 + t + c = 4\beta + 3\alpha + m - 4a$. □

Lemma 7. *Suppose $-\alpha < 2\beta < 1 - m$, then*

$$\begin{aligned}
J &= \int_B \beta(z,w)^n \big|k_w^{\alpha,m}(z)\big|\big(1-|w|^2\big)^{\alpha+2\beta-1}\,dV(w) \\
&\leqslant C\big(1-|z|^2\big)^{2\beta}.
\end{aligned}$$

Proof. Since $-\alpha < 2\beta < 1 - m$, we can choose $\eta < 0$ such that $\alpha + 2\beta - 1 + \eta > -1$, $2\beta - \eta + m - 1 < 0$, and so we can find a constant C satisfying

$$\beta(0,u)^n \leqslant C\big(1-|u|^2\big)^\eta, \quad u \in B,$$

let $w = \rho_z(u)$, we have

$$J = \int_B \beta(0,u)^n \left|\frac{1-z\bar{u}}{1-|z|^2}\right|^{\alpha+m} \left(\frac{(1-|z|^2)(1-|u|^2)}{|1-z\bar{u}|^2}\right)^{\alpha+2\beta-1} \frac{(1-|z|^2)^{m+1}}{|1-z\bar{u}|^{2m+2}}\,dV(u)$$

$$= \left(1-|z|^2\right)^{2\beta} \int_B \beta(0,u)^n \frac{(1-|u|^2)^{\alpha+2\beta-1}}{|1-z\bar{u}|^{\alpha+4\beta+m}}\,dV(u)$$

$$\leqslant C\left(1-|z|^2\right)^{2\beta} \int_B \frac{(1-|u|^2)^{\alpha+2\beta+\eta-1}}{|1-z\bar{u}|^{\alpha+4\beta+m}}\,dV(u).$$

In Lemma 2, take $t=\alpha+2\beta+\eta-1$, $c=2\beta-\eta+m-1$, $2+t+c=\alpha+4\beta+m$, hence $J \leqslant C(1-|z|^2)^{2\beta}$. $\square$

Lemma 8.

$$\left\|C_{\bar{h}}^n k_z^{\alpha,m}\right\|_\phi^2 \geqslant C\left\|k_z^{\alpha,m}\right\|_\phi^2 \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n}\left(1-|u|^2\right)^{\eta-1}\,dV(u),$$

where $\eta = 2(\alpha+b+1)-1>0$.

Proof. By Lemma 5 and the change-of-variable formula, we obtain

$$\begin{aligned}
&\left\|C_{\bar{h}}^n k_z^{\alpha,m}\right\|_\phi^2 \\
&\quad= \int_B \left|C_{\bar{h}}^n k_z^{\alpha,m}(w)\right|^2 dV_\phi(w) \\
&\quad= \int_B \left|h(z)-h(w)\right|^{2n}\left|k_z^{\alpha,m}(w)\right|^2 \frac{\phi^2(|w|)}{1-|w|^2}\,dV(w) \\
&\quad= \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n} \frac{\phi^2(|\rho_z(u)|)(1-|z|^2)^{m+1}}{|1-z\overline{\rho_z(u)}|^{2\alpha+2m}(1-|\rho_z(u)|^2)|1-z\bar{u}|^{2m+2}}\,dV(u) \\
&\quad= \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n} \frac{(1-z\bar{u})^{2\alpha+2m}|1-z\bar{u}|^2(1-|z|^2)^{m+1}\phi^2(|\rho_z(u)|)}{(1-|z|^2)^{2\alpha+2m}(1-|z|^2)(1-|u|^2)|1-z\bar{u}|^{2m+2}}\,dV(u) \\
&\quad= \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n}\left(1-|z|^2\right)^{-2\alpha-m} \frac{\phi^2(|\rho_z(u)|)}{(1-|u|^2)|1-z\bar{u}|^{-2\alpha}}\,dV(u) \\
&\quad\geqslant C\left(1-|z|^2\right)^{-2\alpha-m}\phi^2\left(|z|\right) \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n} \frac{(1-|u|^2)^{-1}(1-|u|^2)^{2b}}{|1-z\bar{u}|^{-2\alpha}|1-z\bar{u}|^{4a}}\,dV(u) \\
&\quad\geqslant C\left\|k_z^{\alpha,m}\right\|_\phi^2 \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n}\left(1-|u|^2\right)^{2(\alpha+b)}\,dV(u) \\
&\quad= C\left\|k_z^{\alpha,m}\right\|_\phi^2 \int_B \left|(h\circ\rho_z)(u)-h(z)\right|^{2n}\left(1-|u|^2\right)^{\eta-1}\,dV(u). \qquad \square
\end{aligned}$$

3. Boundedness for iterated commutators

Before we prove our result, recall the following definition, for some $\alpha > b$, let $\psi(t) = \frac{(1-t^2)^\alpha}{\phi(t)}$, then $\{\phi, \psi\}$ is called a normal pair. We can now prove the main result of this paper.

Theorem 9. *Suppose $h \in H(B)$. Then $C^n_{\bar{h}}$ is bounded on $L^2(\phi)$ if and only if $h \in \mathcal{B}$.*

Proof. Given $m \in N$, we can choose positive real numbers a and b, $a < b$ such that

$$m < 4a - 2b + 1,$$

then take α: $\alpha > \max\{2b, m-1\}$. We first show that $h \in \mathcal{B}$ assuming that $C^n_{\bar{h}}$ is bounded on $L^2(\phi)$. If $C^n_{\bar{h}}$ is bounded on $L^2(\phi)$, then by Lemma 8, we obtain

$$\|C^n_{\bar{h}}\|_\phi \geqslant \frac{\|C^n_{\bar{h}} k_z^{\alpha,m}\|_\phi}{\|k_z^{\alpha,m}\|_\phi} \geqslant C\left\{\int_B \left|(h\circ\rho_z)(u) - h(z)\right|^{2n}\left(1-|u|^2\right)^{\eta-1}\,dV(u)\right\}^{1/2},$$

where $\eta = 2(\alpha + b + 1) - 1$. Using Lemma 4, we have $h \in \mathcal{B}$.

We now finish the proof by assuming that h is in the Bloch space and proving that $C^n_{\bar{h}}$ is bounded on $L^2(\phi)$. Let $h \in \mathcal{B}$, take

$$\beta: \quad -\alpha < 2\beta < 4a - 2b - \alpha - m + 1,$$

then the conditions of Lemmas 6 and 7 hold. By Hölder's inequality, Lemmas 1 and 6, we have

$$\begin{aligned}
&\left|C^n_{\bar{h}} f(w)\right|^2 \\
&\quad\leqslant C\left(\int_B \beta(z,w)^n \left|k_w^{\alpha,m}(z)\right|\left(1-|z|^2\right)^{\alpha-1}\left|f(z)\right|dV(z)\right)^2 \\
&\quad= C\left(\int_B \beta(z,w)^n \left|k_w^{\alpha,m}(z)\right|\frac{\phi(|z|)\psi(|z|)|f(z)|}{1-|z|^2}\,dV(z)\right)^2 \\
&\quad\leqslant C\int_B \beta(z,w)^n \left|k_w^{\alpha,m}(z)\right|\left|f(z)\right|^2\left(1-|z|^2\right)^{-2\beta}\,dV_\phi(z) \\
&\qquad\times\int_B \beta(z,w)^n \left|k_w^{\alpha,m}(z)\right|\left(1-|z|^2\right)^{2\beta}\,dV_\psi(z) \\
&\quad\leqslant C\int_B \beta(z,w)^n \left|k_w^{\alpha,m}(z)\right|\left|f(z)\right|^2\left(1-|z|^2\right)^{-2\beta}\left(1-|w|^2\right)^{2\beta-\alpha}\psi^2\left(|w|\right)dV_\phi(z).
\end{aligned}$$

Hence from Fubini's theorem and Lemma 7, it follows that

$$\begin{aligned}
\|C^n_{\bar{h}} f\|^2_\phi &= \int_B \left|C^n_{\bar{h}} f(w)\right|^2 dV_\phi(w) \\
&\leqslant C\int_B\left(\int_B \beta(z,w)^n\left|k_w^{\alpha,m}(z)\right|\left(1-|w|^2\right)^{2\beta-\alpha}\psi^2\left(|w|\right)dV_\phi(w)\right) \\
&\qquad\times\left|f(z)\right|^2\left(1-|z|^2\right)^{-2\beta}\,dV_\phi(z)
\end{aligned}$$

$$= C\int_B \left(\int_B \beta(z,w)^n \left|k_w^{\alpha,m}(z)\right| \left(1-|w|^2\right)^{2\beta+\alpha-1} dV(w) \right)$$
$$\times \left|f(z)\right|^2 \left(1-|z|^2\right)^{-2\beta} dV_\phi(z)$$
$$\leqslant C\int_B \left(1-|z|^2\right)^{2\beta} \left|f(z)\right|^2 \left(1-|z|^2\right)^{-2\beta} dV_\phi(z) = C\|f\|_\phi^2,$$

so

$$\left\|C_h^n\right\|_\phi \leqslant C,$$

and the proof is complete. □

Remark. When $\beta \geqslant 0$, Fan–Wu's sufficient condition (Lemma 3.1 in [6]) for a general measurable symbol $\mathrm{b} = (b_1, b_2, \ldots, b_n)$ is again valid with $d\mu_\alpha$ replaced by dV_ϕ. In fact, the proof of the 'if' part of Theorem 9 of the paper under review again applies to this general case.

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