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Multichannel Sampling and Reconstruction of Bandlimited Signals in Fractional Fourier Domain

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***Abstract*—The classical multichannel sampling theorem for common bandlimited signals has been extended differently to fractional bandlimited signals associated with the fractional Fourier transform (FRFT). However, the implementation of those existing extensions is inefficient because of the effect of spectral leakage and hardware complexity. The purpose of this letter is to introduce a practical multichannel sampling theorem for fractional bandlimited signals. The theorem which is constructed by the ordinary convolution in the time domain can reduce the effect of spectral leakage and is easy to implement. The classical multichannel sampling theorem and the well-known sampling theorem for the FRFT are shown to be special cases of it. Some potential applications of this theorem are also presented. The validity of the theoretical derivations is demonstrated via simulations.**

***Index Terms*—Fractional bandlimited signal, fractional filter, fractional Fourier transform, multichannel sampling theorem.**

I. Introduction

The fractional Fourier transform (FRFT)-a generalization of the Fourier transform (FT)-has received much attention in recent years due to its numerous applications, including quantum physics, communications, optics and signal processing [1]–[9]. For more details of the FRFT, see [3]. The definition of the FRFT is as follows [2]:

$$F_\alpha(u) = \mathcal{F}^\alpha\left[f(t)\right](u) = \int_{-\infty}^{+\infty} f(t)\mathcal{K}_\alpha(u,t)dt \tag{1}$$

where the transform kernel is given by

$$\mathcal{K}_\alpha(u,t) = \begin{cases} A_\alpha e^{(j/2)(t^2+u^2)\cot\alpha - jut\csc\alpha}, & \text{if } \alpha \neq k\pi \\ \delta(t-u), & \text{if } \alpha = 2k\pi \\ \delta(t+u), & \text{if } \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha = \sqrt{(1-j\cot\alpha)/2\pi}$, $k \in \mathbb{Z}$. The inverse FRFT is the FRFT at angle $-\alpha$, given by

$$f(t) = \mathcal{F}^{-\alpha}\left[F_\alpha(u)\right](t) = \int_{-\infty}^{+\infty} F_\alpha(u)\overline{\mathcal{K}_\alpha(u,t)}du \tag{3}$$

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Fig. 1. Filtering the signal $f(t)$ in the fractional Fourier domain.

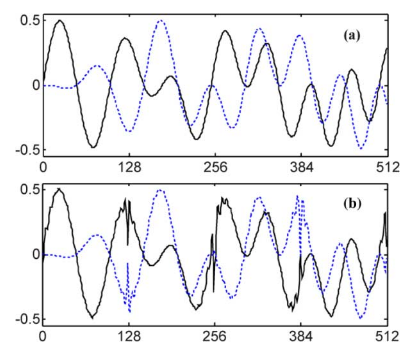

Fig. 2. FRFT domain filtering with $\alpha = \pi/4$. (a) Theoretical waveform of $g_1(t)$. (b) DFRFT out for $g_1(t)$, where DFRFT length $N = 128$.

where the bar denotes the complex conjugation. Whenever $\alpha = \pi/2$, (1) reduces to the FT.

Many properties of the FRFT including its multichannel sampling theorem have been currently derived as counterparts to the corresponding properties of the FT [4], [7]–[10]. The classical multichannel sampling theorem introduced by Papoulis [11] is a powerful extension of the Shannon sampling theory, revealing that a common bandlimited signal $f(t)$ could be restored exactly from several filtered versions $\{g_k(t)|k = 1,\ldots,M\}$ of $f(t)$ where each filtered version is sampled at a fraction of the Nyquist rate associated with $f(t)$. Due to the importance of the FRFT in signal processing, the concepts of the classical multichannel sampling have been extended to fractional bandlimited signals. However, the implementation of those extensions [12]–[14] is not practical. To be specific, in [12]–[14], the filtered version of the fractional bandlimited signal $f(t)$ is achieved by the method of transform domain filtering based on the discrete FRFT (DFRFT) shown in Fig. 1.

This method is inefficient due to the effect of spectral leakage. Spectral leakage results from an assumption in the DFRFT algorithm that the time record exactly repeats throughout all time. In other words, when you use the DFRFT to measure the fractional frequency content of data, the transform assumes that the finite data set is one period of a chirp-periodic signal [15]. Thus, if the samples of the signal $f(t)$ shown in Fig. 1 are not chirp-periodic in the DFRFT's window of observation, a discontinuity occurs at the DFRFT's block boundary. For the case of a nonwindowed DFRFT, the discontinuity is abrupt and the spectral leakage can be significant. To illustrate this, Fig. 2 depicts the filtered version $g_1(t)$ of $f(t)$ that is composed of $g_1(t) = 0.5\exp(-j0.5t^2)\sin t$ and $g_2(t) = 2\exp(-j0.5t^2)\sin 40t$ (the solid lines are real parts and the dash lines are imaginary parts).

Comparing Fig. 2(a) with Fig. 2(b), we find that the filtered version $g_1(t)$ of $f(t)$ is distorted at the DFRFT's block boundary because of the effect of spectral leakage. Thus, the method of FRFT domain filtering shown in Fig. 1 is not practical in actual applications. In [12], each $g_k(t)$ can be also derived by the generalized convolution for the FRFT in the time domain, i.e.

$$\begin{aligned} g_k(t) &= f(t)\Theta h_k(t) \\ &= \frac{|\csc\alpha|}{2\pi}\iiint f(t')h_k(\tau)e^{(j/2)(t'^2+u^2+\tau^2-t^2)\cot\alpha} \\ &\qquad \times e^{ju(t-\tau-t')\csc\alpha}dt'dud\tau \end{aligned} \tag{4}$$

where Θ indicates the generalized convolution operator, and $h_k(t)$ is the impulse response of the fractional filter. Unfortunately, it is complicated to reduce the expression of the generalized convolution to a single integral form as in the ordinary convolution expression, so it is not easy to implement. Hence, those extensions derived in [12]–[14] of the classical multichannel sampling theorem are not practical.

The purpose of this letter is to propose a practical multichannel sampling theorem for fractional bandlimited signals, which differs from those introduced in [12]–[14]. The proposed theorem is advantageous over the existing ones [12]–[14] in that it can reduce the signal distortion, and is easy to implement with ordinary convolution structure.

The outline of this letter is organized as follows. In the next section, the concept of fractional bandlimited signals is briefly introduced and some useful formulas are given. In Section III, the practical multichannel sampling theorem is proposed, and some potential applications are presented. Conclusions appear at the end of the letter.

II. Preliminaries

A. Fractional Bandlimited Signal

A signal $f(t)$ is often called Ω_α-fractional bandlimited if its energy is finite and its FRFT $F_\alpha(u)$ vanishes outside the region $(-\Omega_\alpha, +\Omega_\alpha)$

$$F_\alpha(u) = 0, \quad \text{for } |u| \geq \Omega_\alpha, \tag{5}$$

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \int_{-\Omega_\alpha}^{+\Omega_\alpha} |F_\alpha(u)|^2\, du < \infty. \tag{6}$$

Then, the signal $f(t)$ can be resorted as follows:

$$f(t) = e^{-j\frac{t^2}{2}\cot\alpha}\sum_{n=-\infty}^{+\infty} f(nT_s)e^{j\frac{(nT_s)^2}{2}\cot\alpha} \times \frac{\sin\left[(t-nT_s)\Omega_\alpha\csc\alpha\right]}{(t-nT_s)\Omega_\alpha\csc\alpha} \tag{7}$$

where $T_{\mathrm{s}} = \pi\sin\alpha/\Omega_\alpha$. This basic result, originally due to Xia, is the well-known sampling theorem for the FRFT [9].

B. Some Useful Formulas

The relationship between the FRFT and the FT [10] is given below which will be used in the current subsection

$$\mathcal{F}^\alpha\left[f(t)\right](u) = \sqrt{2\pi}A_\alpha e^{(j/2)u^2\cot\alpha} \times \mathfrak{F}\left[f(t)e^{(j/2)t^2\cot\alpha}\right](u\csc\alpha) \tag{8}$$

where $\mathfrak{F}$ denotes the FT operator.

A formula is given below which will be used in Section III

$$z(t) \triangleq e^{-(j/2)t^2\cot\alpha}\cdot\left[\left(x(t)e^{(j/2)t^2\cot\alpha}\right) * h(t)\right] \tag{9}$$

where $*$ denotes the ordinary convolution operator. The formula can be easily deduced by the method of defining fractional operations introduced in [8]. It is easy to verify that

$$Z_\alpha(u) = \sqrt{2\pi}X_\alpha(u)H(u\csc\alpha). \tag{10}$$

The proof of (10) is as follows. According to (9), the definition of the FRFT and (8), we have

$$\begin{aligned} Z_\alpha(u) &= \int_{-\infty}^{+\infty} e^{-(j/2)t^2\cot\alpha}\cdot\left[\left(x(t)e^{(j/2)t^2\cot\alpha}\right) * h(t)\right] \\ &\qquad \times \mathcal{K}_\alpha(u,t)dt \\ &= \mathfrak{F}\left[\left(x(t)e^{(j/2)t^2\cot\alpha}\right) * h(t)\right](u\csc\alpha) \\ &\qquad \times A_\alpha e^{(j/2)u^2\cot\alpha}. \end{aligned} \tag{11}$$

By applying the ordinary convolution theorem [3] and (8) again, we obtain

$$\begin{aligned} Z_\alpha(u) &= \mathfrak{F}\left[x(t)e^{(j/2)t^2\cot\alpha}\right](u\csc\alpha) \\ &\qquad \times \sqrt{2\pi}A_\alpha e^{(j/2)u^2\cot\alpha}\mathfrak{F}\left[h(t)\right] \\ &= \sqrt{2\pi}X_\alpha(u)H(u\csc\alpha) \end{aligned} \tag{12}$$

where H denotes the FT of h. This completes the proof of (10). Whenever $\alpha = \pi/2$, (10) reduces to the ordinary convolution theorem of the FT [3]. The proof is easy and omitted.

Equations (9) and (10) state that a modified ordinary convolution in the time domain is equivalent to a simple multiplication in the fractional Fourier domain. This convolution-multiplication property of the FRFT is useful for fractional filter design. For instance, if we are interested only in the FRFT $X_\alpha(u)$ in the region $(-u_{\mathrm{m}}, +u_{\mathrm{m}})$ of a signal $x(t)$, we can chose a transfer function $H(u\csc\alpha)$ which is constant over the region $(-u_{\mathrm{m}}, +u_{\mathrm{m}})$, and zero or of rapid decay outside that region. Particularly, the (9) is useful for deriving the practical multichannel sampling theorem for fractional bandlimited signals as will be shown later on.

III. A Practical Multichannel Sampling Theorem for the FRFT and Its Potential Applications

A. A Practical Multichannel Sampling Theorem

In the following, we propose a practical multichannel sampling theorem, revealing that a Ω_α-fractional bandlimited $f(t)$ can be recovered in terms of the samples $\{g_k(n\Delta)|k = 1,\ldots,M; n = -\infty,\ldots,+\infty\}$ of M linear functions $g_k(t)$ sampled at the slower rate as follows:

$$\Delta = 2\pi\sin\alpha/\sigma_\alpha, \quad \sigma_\alpha = 2\Omega_\alpha/M. \tag{13}$$

For this purposes, we first give M linear systems with system functions $\{h_k(t)|k = 1,\ldots,M\}$ and let the Ω_α-fractional bandlimited signal $f(t)$ be applied to the input of these systems. From (10), the resulting outputs are M functions

$$G_{k\alpha}(u) = \sqrt{2\pi}F_\alpha(u)H_k(u\csc\alpha), \quad -\Omega_\alpha < u < +\Omega_\alpha \tag{14}$$

where $G_{k\alpha}$ and F_α denote the FRFT of g_k and f, respectively, and H_k indicates the FT of h_k. It follows from (9) that

$$g_k(t) = e^{-(j/2)t^2\cot\alpha}\cdot\left[\left(f(t)e^{(j/2)t^2\cot\alpha}\right) * h_k(t)\right]. \tag{15}$$

Since the derivation of $g_k(t)$ is realized in the time domain with ordinary convolution structure, it is advantageous over the existing method [12]–[14] which is shown in Fig. 1 in that it can

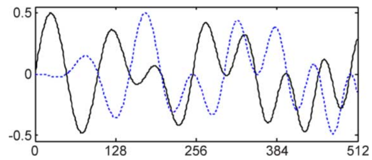

Fig. 3. Using the modified ordinary convolution to derive the filtered version $g_1(t)$ of $f(t)$ given in Fig. 2.

reduce the signal distortion, and it is easier to implement than the alternative method [12] given in (4). Fig. 3 depicts the filtered version $g_1(t)$ of $f(t)$ given in Fig. 2 by using the modified ordinary convolution in (15).

Obviously, the distortion of the filtered version can be avoided if we use the modified ordinary convolution instead of FRFT domain filtering, see Fig. 2(b) and Fig. 3.

To derive the practical multichannel sampling theorem for fractional bandlimited signals, according to (9) and (10), we form the following system of equations:

$$\sum_{k=1}^{M}\sqrt{2\pi}H_k\left((u+r\sigma_\alpha)\csc\alpha\right)Y_k(u\csc\alpha,t)=e^{jtr\sigma_\alpha\csc\alpha},\quad r=0,1,\ldots,M-1 \tag{16}$$

where t is any number, $u\in(-\Omega_\alpha,-\Omega_\alpha+\sigma_\alpha)$, r is some integer, and $H_k(u)$ denotes the FT of $h_k(t)$. This system defines M functions $\{Y_k(u\csc\alpha,t)|k=1,\ldots,M\}$ of u and t because the coefficients $H_k((u+r\sigma_\alpha)\csc\alpha)$ of the system (16) depend on u, and the right side depends on t. Although functions $\{H_k(u\csc\alpha)|k=1,\ldots,M\}$ are general, they cannot be completely arbitrary: they must meet the condition that the determinant of the coefficients $H_k((u+r\sigma_\alpha)\csc\alpha)$ of (16) differs from zero for every $u\in(-\Omega_\alpha,-\Omega_\alpha+\sigma_\alpha)$. We will also make an assumption that the solutions $\{Y_k(u\csc\alpha,t)|k=1,\ldots,M\}$ of (16), considered as functions of u, can be expanded into a Fourier series in the region $(-\Omega_\alpha,-\Omega_\alpha+\sigma_\alpha)$. The practical multichannel sampling theorem can be concluded as follows:

Theorem 1: If $f(t)$ is a Ω_α-fractional bandlimited signal, it can then be restored by the interpolation formula:

$$f(t)=e^{-(j/2)t^2\cot\alpha}\sum_{n=-\infty}^{+\infty}\sum_{k=1}^{M}g_k(n\Delta)\times e^{(j/2)(n\Delta)^2\cot\alpha}y_k(t-n\Delta) \tag{17}$$

where

$$y_k(t)=\sigma_\alpha^{-1}\int_{-\Omega_\alpha}^{-\Omega_\alpha+\sigma_\alpha}Y_k(u\csc\alpha,t)e^{jtu\csc\alpha}du,\quad k=1,\ldots,M \tag{18}$$

and $g_k(t)$ is given in (15).

Proof: It follows from (13) that

$$(t+\Delta)r\sigma_\alpha\csc\alpha=tr\sigma_\alpha\csc\alpha+2\pi r. \tag{19}$$

Thus, the right side of (16) consists of periodic functions of t with period Δ. Since the coefficients $H_k((u+r\sigma_\alpha)\csc\alpha)$ of (16) are independent of t, the solutions $Y_k(u\csc\alpha,t)$ are run

$$Y_k(u\csc\alpha,t+\Delta)=Y_k(u\csc\alpha,t). \tag{20}$$

Then, according to (18) and (20), we have

$$\begin{aligned}y_k(t-n\Delta)&=\sigma_\alpha^{-1}\int_{-\Omega_\alpha}^{-\Omega_\alpha+\sigma_\alpha}Y_k(u\csc\alpha,t-n\Delta)\times e^{j(t-n\Delta)u\csc\alpha}du\\&=\sigma_\alpha^{-1}\int_{-\Omega_\alpha}^{-\Omega_\alpha+\sigma_\alpha}Y_k(u\csc\alpha,t)e^{jtu\csc\alpha}\times e^{-jn\Delta u\csc\alpha}du\end{aligned} \tag{21}$$

which states that $y_k(t-n\Delta)$ is the nth Fourier series expansion of the function $Y_k(u\csc\alpha,t)e^{jtu\csc\alpha}$ in the region $(-\Omega_\alpha,-\Omega_\alpha+\sigma_\alpha)$. Hence

$$Y_k(u\csc\alpha,t)e^{jtu\csc\alpha}=\sum_{n=-\infty}^{+\infty}y_k(t-n\Delta)e^{jn\Delta u\csc\alpha} \tag{22}$$

for $u\in(-\Omega_\alpha,-\Omega_\alpha+\sigma_\alpha)$. Then, multiplying the first equation of (16) with $r=0$ by $e^{jtu\csc\alpha}$ and using (22) results in

$$e^{jtu\csc\alpha}=\sum_{k=1}^{M}\sqrt{2\pi}H_k(u\csc\alpha)\sum_{n=-\infty}^{+\infty}y_k(t-n\Delta)e^{jn\Delta u\csc\alpha} \tag{23}$$

for every $u\in(-\Omega_\alpha,-\Omega_\alpha+\sigma_\alpha)$.

Now along the lines of [11], it can be shown that (23) also holds good for every $u\in(-\Omega_\alpha,+\Omega_\alpha)$, i.e.

$$e^{jtu\csc\alpha}=\sum_{k=1}^{M}\sqrt{2\pi}H_k(u\csc\alpha)\sum_{n=-\infty}^{+\infty}y_k(t-n\Delta)e^{jn\Delta u\csc\alpha},\quad -\Omega_\alpha<u<+\Omega_\alpha. \tag{24}$$

Since $f(t)$ is a Ω_α-fractional bandlimited signal, it follows form (3) that

$$f(t)=A_{-\alpha}e^{-(j/2)t^2\cot\alpha}\int_{-\Omega_\alpha}^{+\Omega_\alpha}F_\alpha(u)e^{-(j/2)u^2\cot\alpha+jtu\csc\alpha}du. \tag{25}$$

Substituting (24) into (25) yields

$$f(t)=e^{-(j/2)t^2\cot\alpha}\sum_{k=1}^{M}\sum_{n=-\infty}^{+\infty}y_k(t-n\Delta)A_{-\alpha}\int_{-\Omega_\alpha}^{+\Omega_\alpha}\sqrt{2\pi}F_\alpha(u)H_k(u\csc\alpha)e^{-(j/2)u^2\cot\alpha+jn\Delta u\csc\alpha}du. \tag{26}$$

Next, from (14) and the inverse FRFT, we have

$$g_k(t)e^{(j/2)t^2\cot\alpha}=A_{-\alpha}\int_{-\Omega_\alpha}^{+\Omega_\alpha}\sqrt{2\pi}F_\alpha(u)H_k(u\csc\alpha)\times e^{-(j/2)u^2\cot\alpha+jtu\csc\alpha}du \tag{27}$$

from which it follows that

$$g_k(n\Delta)e^{(j/2)(n\Delta)^2\cot\alpha}=A_{-\alpha}\int_{-\Omega_\alpha}^{+\Omega_\alpha}\sqrt{2\pi}F_\alpha(u)H_k(u\csc\alpha)\times e^{-(j/2)u^2\cot\alpha+jn\Delta u\csc\alpha}du. \tag{28}$$

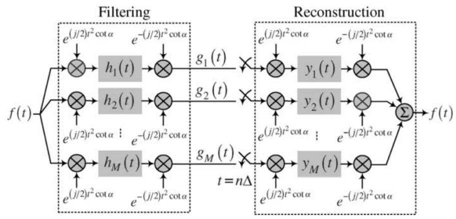

Fig. 4. Configuration of the proposed multichannel sampling.

Inserting (28) into (26) yields (17). This completes the proof of *Theorem 1*.

Corollary 1: If $M = 1$ and $H_1(u \csc \alpha) = (2\pi)^{-1/2}$, then Theorem 1 reduces to the well-known sampling theorem for the FRFT introduced by Xia [9] shown in (7).

Proof: Since $M = 1$ and $H_1(u \csc \alpha) = (2\pi)^{-1/2}$, it follows from (15) and (16) that

$$\mathrm{g}(t) = f(t), \quad Y_1(u \csc \alpha, t) = 1. \tag{29}$$

Inserting (29) into (16) results in

$$y_1(t) = \frac{\sin(t\Omega_\alpha \csc \alpha)}{t\Omega_\alpha \csc \alpha}. \tag{30}$$

By putting (30) and (29) into (17), the (7) can be established. This completes the proof of *Corollary 1*.

Corollary 2: Whenever $\alpha = \pi/2$, *Theorem 1* collapses to the classical multichannel sampling theorem earlier introduced by Papoulis in [11].

The proof of *Corollary 2* is easy and omitted. Moreover, by designing different fractional filters, reconstruction methods for other sampling strategies including the nonuniform sampling theorem [13] and the derivative sampling theorem [14] can be also achieved according to the results in this letter.

B. Potential Applications

The proposed multichannel sampling theorem states that fractional bandlimited signals can be restored exactly in terms of linear fractional filtering operations in the time domain (see Fig. 4) Specially, if the original sampled signal is not directly achieved, but the filtered version can be accessible, then the proposed theorem reveals that the original signal can be still restored.

The immediate application of the proposed theorem can be found when processing some specific class of signals which are not bandlimited in the Fourier domain but bandlimited in the fractional Fourier domain or whose energy is better concentrated in the fractional Fourier domain than that in the Fourier domain. For an example, the chirp signal which is widely used in engineering systems such as radar, sonar and communications [3], [5], [7]. In particular, the waveforms transmitted in chirp ultra-wideband (UWB) communication systems [16] are very short chirp signals with bandwidth on the order of several gigahertz, thus designing a single analog-to-digital converter (ADC) to operate at the waveform Nyquist rate is not practical, and parallel ADC architectures with each ADC operating at a fraction of the Nyquist rate need to be employed (see Fig. 4). Other potential applications can be found in multichannel data acquisition environment, such as flexible interleaving/multiplexing ADC for fractional bandlimited signals, the multicarrier multiplexing system based on the FRFT [5], the sampling and reconstruction of fractional multiband signals and the subcoding in the fractional Fourier domain. Moreover, the proposed theorem can be also useful in image superresolution problems and other problems of signal processing as discussed in [12]–[14].

IV. Conclusion

In this letter, we have proposed a practical multichannel sampling theorem for fractional bandlimited signals. The proposed theorem is advantageous over the existing ones [12]–[14] in that it can reduce the signal distortion resulting form the effect of spectral leakage, and is easy to implement with ordinary convolution structure in the time domain. The classical multichannel sampling theorem for common bandlimited signals and the well-known sampling theorem for the FRFT are shown to be special cases of it. In particular, by designing different fractional filters, other interpolation expansions for other elaborate sampling schemes can be also derived. Some potential applications of the proposed theorem are also presented. The validity of the theoretical derivations is demonstrated via simulations.

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