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Fidelity and fidelity susceptibility based on Hilbert-Schmidt inner product

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We reinvestigate the fidelity based on Hilbert-Schmidt inner product and give a simplified form. The geometric meaning of the fidelity is clarified. We then give the analytic expression of the fidelity susceptibility in both Hilbert and Liouville space. By using the reconstruction of symmetric logarithmic derivative in Liouville space, we present the time derivative of fidelity susceptibility with the normalized density vector representation.

fidelity, fidelity susceptibility, Liouville space

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1 Introduction

The original concept of fidelity was proposed in the 1970s by Uhlmann and Alberti [1–4] through a series of work concerning the problem of transition probability in quantum mechanics. Jozsa et al. [5,6] then proposed a definition of fidelity by developing a general quantum analogue of Shannon's noiseless coding theorem. In his theorem, the original signal is a pure quantum state and the signal after the transmission is a mixed state. The transmission is performed through some channel. Fidelity is defined as the overall probability that the original signal could be transmitted into the target one [6]. Furthermore, Jozsa [5] realized that Uhlmann's transition probability is the extension of Schumacher's fidelity. Now Uhlmann's transition probability is well known in quantum information and usually called Uhlmann-Jozsa fidelity nowadays. Uhlmann-Jozsa fidelity is usually defined as:

$$F_{\mathrm{U}}(\rho, \sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}, \tag{1}$$

where ρ, σ are two density matrices. Both the density matrices ρ and σ can either be pure or mixed.

This fidelity has many properties, e.g., for the two pure states $|\psi\rangle$, $|\phi\rangle$, the fidelity reduces to the modulus of the inner production, namely, $F_{\mathrm{U}} = |\langle\psi|\phi\rangle|$. The fidelity keeps invariant under unitary transformations; the fidelity has the strong concavity, as well as the monotonicity under trace-preserving quantum operations [7]. Having these properties, the fidelity has a wide use in many fields as a quantitative measure of distinguishability. Because of its usefulness, several alternative forms of fidelity have been proposed [8, 9] and discussed.

Jozsa [5] proposed four axioms that any definition of a fidelity should satisfy. An alternative fidelity was recently proposed [8] based on the Hilbert-Schmidt inner product and perfectly satisfies all the four Jozsa axioms up to a normalization factor. It can be regarded as the well-defined operator fidelity for the two operators.

In recent years, fidelity susceptibility of Uhlmann-Jozsa fidelity has been studied in the field of quantum phase transitions (QPTs) [10, 11]. It is natural to introduce the Uhlmann-Jozsa fidelity as an indicator in the QPTs because Uhlmann-Jozsa fidelity is effectively the overlap between two states.

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Based on the importance of fidelity susceptibility, we calculate the fidelity susceptibility of fidelity based on the Hilbert-Schmidt inner product in both Hilbert space and Liouville space, as well as its time derivative using the normalized density vector.

This paper is organized as follows. In sect. 2, we give a brief introduction of this fidelity, simplify its form, and give the relative formula in Liouville space. Concurrently, we show the geometric meaning of this fidelity. In sect. 3, we show the results of fidelity susceptibility in both Hilbert and Liouville space. In sect. 4, with the reconstruction of SLD in Liouville space, we give the time derivative of the fidelity susceptibility in Liouville space with normalized density vector representation.

2 Fidelity based on Hilbert-Schmidt inner product

Let us consider an alternative quantum fidelity based on Hilbert-Schmidt inner product and proposed in ref. [8], which is defined as:

$$F_{\mathrm{HS}}(\rho_a, \rho_b) = \frac{|\mathrm{Tr}(\rho_a\rho_b)|}{\sqrt{\mathrm{Tr}(\rho_a^2)\,\mathrm{Tr}\left(\rho_b^2\right)}}, \tag{2}$$

where ρ_a, ρ_b are density matrices in the same Hilbert space. For any two density matrices ρ_a and ρ_b in the same Hilbert space, the spectral decomposition of ρ_a, ρ_b can be written as: $\rho_a = \sum_i \lambda_i^a |i\rangle\langle i|$, $\rho_b = \sum_j \lambda_j^b |j\rangle\langle j|$ where $\lambda_i^a, \lambda_j^b \geqslant 0$. Then, $\mathrm{Tr}(\rho_a\rho_b) = \sum_{ij}\lambda_i^a\lambda_j^b|\langle i|j\rangle|^2 \geqslant 0$. Thus, $\mathrm{Tr}(\rho_a\rho_b) \geqslant 0$. According to this inequality, the fidelity can be rewritten as:

$$F_{\mathrm{HS}}(\rho_a, \rho_b) = \frac{\mathrm{Tr}(\rho_a\rho_b)}{\sqrt{\mathrm{Tr}(\rho_a^2)\,\mathrm{Tr}\left(\rho_b^2\right)}}. \tag{3}$$

Jozsa axioms [5] are treated to be natural properties of any fidelity. It can be confirmed that this fidelity satisfies all these axioms, which indicates that it has the following properties [8]:

(1) The fidelity is normalized and the maximum 1 is attained if and only if $\rho_a = \rho_b$.

(2) The fidelity is invariant under swapping ρ_a and ρ_b.

(3) The fidelity is invariant under unitary transformation U on the state space.

(4) When one of the states is pure, which implies $\rho_a = |\psi\rangle\langle\psi|$, the fidelity reduces to $F_{\mathrm{HS}}(\rho_b, |\psi\rangle\langle\psi|) = \langle\psi|\rho_b|\psi\rangle/\mathrm{Tr}\left(\rho_b^2\right)$.

(5) The fidelity has the property of multiplication under tensor products of the density matrices: $F_{\mathrm{HS}}(\rho_a\otimes\sigma_a, \rho_b\otimes\sigma_b) = F_{\mathrm{HS}}(\rho_a, \rho_b)\,F_{\mathrm{HS}}(\sigma_a, \sigma_b)$.

Because this fidelity satisfies $F_{\mathrm{HS}} \leqslant 1$, and $\mathrm{Tr}\left(\rho^2\right) \leqslant 1$, then we have $\mathrm{Tr}(\rho_a\rho_b) \leqslant 1$. Therefore, $\mathrm{Tr}(\rho_a\rho_b) \in [0, 1]$.

Liouville space is a very important concept in quantum theory of relaxation as well as in quantum information theory. It was first termed by Fano [12] and important contributions were made by Fano et al. [12–21]. It is known that the density matrix ρ is a vector which would be denoted as $|\rho\rangle$ in $N^2\times N^2$ Liouville space (N is the dimension of the Hilbert space where the system is in) [22]. Thus, this fidelity in Liouville space could be expressed as:

$$F_{\mathrm{HS}}(|\rho_a\rangle, |\rho_b\rangle) = \frac{\langle\rho_a|\rho_b\rangle}{\sqrt{\langle\rho_a|\rho_a\rangle\langle\rho_b|\rho_b\rangle}}. \tag{4}$$

The density matrix is Hermitian, such that $\langle\rho_a|\rho_b\rangle = \langle\rho_b|\rho_a\rangle$, which implies that the inner product $\langle\rho_a|\rho_b\rangle$ is real. Next we construct the normalized density vector $|\widetilde{\rho}\rangle$ in Liouville space which defined as $|\widetilde{\rho}\rangle = \langle\rho|\rho\rangle^{-1/2}|\rho\rangle$, then the fidelity could be rewritten as:

$$F_{\mathrm{HS}}(|\widetilde{\rho}_a\rangle, |\widetilde{\rho}_b\rangle) = \langle\widetilde{\rho}_a|\widetilde{\rho}_b\rangle. \tag{5}$$

This formula has the same formula with the Uhlmann-Jozsa fidelity of pure states under the circumstances that the inner product between the two states is real. Actually, the normalized density vectors could be treated as states in $N^2\times N^2$ Hilbert space. Thus, it should not be unexpected that they have similar form.

In the following we show the geometric meaning of this fidelity. Considering the qubit system, in the Bloch representation, the density matrix could be written as:

$$\rho = \boldsymbol{r}\cdot\boldsymbol{\Lambda}, \tag{6}$$

where $\boldsymbol{r} = (1, x, y, z)$ and $\boldsymbol{\Lambda} = \frac{1}{2}\left(\mathbb{I}, ,\sigma_x, \sigma_y, \sigma_z\right)$. The real numbers x, y, z satisfies $x, y, z \in [0, 1]$. $\mathbb{I}$ is identity matrix and σ_i ($i = x, y, , z$) is corresponding Pauli matrix. Using this representation of density matrix, we could obtain the expression of the fidelity

$$F_{\mathrm{HS}}(\rho_a, \rho_b) = \frac{\boldsymbol{r}_a\cdot\boldsymbol{r}_b}{|\boldsymbol{r}_a|\cdot|\boldsymbol{r}_b|}. \tag{7}$$

It is known that the angle between two vectors in Euclidean space is defined as:

$$\cos\theta = \frac{\boldsymbol{r}_a\cdot\boldsymbol{r}_b}{|\boldsymbol{r}_a|\cdot|\boldsymbol{r}_a|}.$$

Thus, the fidelity has a strong connection with the angle θ, which is

$$F_{\mathrm{HS}}(\rho_a, \rho_b) = \cos\theta. \tag{8}$$

Generally, for an N-level system, the density matrix could be expressed using the generalized Bloch vectors [23],

$$\rho = \frac{1}{N}\mathbb{I} + \frac{1}{2}\sum_{j=1}^{N^2-1}\langle\lambda_j\rangle\lambda_j, \tag{9}$$

where $\mathbb{I}$ is the identity matrix, λ_j is called generalized Pauli matrices which is actually the generators of $SU(n)$, and satisfies the following relations [23]:

$$\mathrm{Tr}(\lambda_i) = 0, \quad \mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}, \tag{10}$$

and $\langle\lambda_j\rangle$ denotes the average value of λ_j which is defined as $\langle\lambda_j\rangle \equiv \mathrm{Tr}(\lambda_j\rho)$. Eq. (9) could also be rewritten in the form of $\rho = \boldsymbol{r} \cdot \boldsymbol{\Lambda}$, where $\boldsymbol{r} = (1, \langle\lambda_1\rangle, \cdots\langle\lambda_j\rangle, \cdots\langle\lambda_{N^2-1}\rangle)$ and $\boldsymbol{\Lambda} = (\frac{1}{N}\mathbb{I}, \frac{1}{2}\lambda_1, \cdots\frac{1}{2}\lambda_j, \cdots\frac{1}{2}\lambda_{N^2-1})$. Using this representation, we could obtain the expression of the fidelity like above,

$$F_{\rm HS}(\rho_a, \rho_b) = \cos\theta = \frac{\boldsymbol{r}_a \cdot \boldsymbol{r}_b}{|\boldsymbol{r}_a| \cdot |\boldsymbol{r}_a|}. \tag{11}$$

This equation implies that this fidelity is indeed the cosine function of the angle θ between two N^2 dimensional generalized Bloch vectors.

3 Fidelity susceptibility

Fidelity susceptibility is important and meaningful in the study of fidelity. It describes the distinguishability of two states when the inner parameters of one state are very close to those of the other state. In some fields, the fidelity susceptibility has applications. For instance, in QPTs, the fidelity susceptibility of Uhlmann-Jozsa fidelity is an effective tool in studying critical properties in many-body systems [11]. In the following, we will show the susceptibility of the fidelity based on Hilbert-Schmidt inner product.

Assuming that a parameter λ is involved in the density matrices and denoting $\rho_a = \rho(\lambda)$, $\rho_b = \rho(\lambda + \delta\lambda)$, the Taylor expansion of ρ_b till the second order $\delta^2\lambda$ is

$$\rho_b = \rho + \partial_\lambda\rho\delta\lambda + \frac{1}{2}\partial_\lambda^2\rho\delta^2\lambda. \tag{12}$$

Using this Taylor expansion, one could have

$$\mathrm{Tr}(\rho_a\rho_b) = [\mathrm{Tr}(\rho^2)]\left(1 + f_1\delta\lambda + f_2\delta^2\lambda\right), \tag{13}$$

and

$$\mathrm{Tr}(\rho_b^2) = [\mathrm{Tr}(\rho^2)]\left(1 + 2f_1\delta\lambda + f_3\delta^2\lambda\right), \tag{14}$$

where the coefficients f_1, f_2, f_3 read

$$f_1 = \frac{\mathrm{Tr}(\rho\partial_\lambda\rho)}{\mathrm{Tr}(\rho^2)},\ f_2 = \frac{1}{2}\frac{\mathrm{Tr}(\rho\partial_\lambda^2\rho)}{\mathrm{Tr}(\rho^2)},\ f_3 = \frac{\partial_\lambda\mathrm{Tr}(\rho\partial_\lambda\rho)}{\mathrm{Tr}(\rho^2)}.$$

Substituting eqs. (13) and (14) into eq. (3), the fidelity can be written as:

$$F_{\rm HS} = \frac{1 + f_1\delta\lambda + f_2\delta^2\lambda}{(1 + 2f_1\delta\lambda + f_3\delta^2\lambda)^{1/2}}. \tag{15}$$

Using the Taylor expansion $[1 + f(x)]^{-\frac{1}{2}} = 1 - \frac{1}{2}f(x) + \frac{3}{8}f^2(x)$, the expansion of eq. (15) till $\delta^2\lambda$ could be obtained as:

$$F_{\rm HS} = 1 - \frac{1}{2}\mathcal{F}_\lambda\delta^2\lambda, \tag{16}$$

where $\mathcal{F}_\lambda$ is the fidelity susceptibility we expect and the expression of it is

$$\begin{aligned}\mathcal{F}_\lambda &= f_3 - 2f_2 - f_1^2 \\ &= \frac{1}{[\mathrm{Tr}(\rho^2)]^2}\left\{\mathrm{Tr}[(\partial_\lambda\rho)^2]\mathrm{Tr}\left(\rho^2\right) - [\mathrm{Tr}(\rho\partial_\lambda\rho)]^2\right\}.\end{aligned} \tag{17}$$

In Liouville space, the fidelity susceptibility could be rewritten as:

$$\mathcal{F}_\lambda = \frac{1}{\langle\rho|\rho\rangle^2}\left[\langle\partial_\lambda\rho|\partial_\lambda\rho\rangle\langle\rho|\rho\rangle - \langle\rho|\partial_\lambda\rho\rangle^2\right]. \tag{18}$$

As $\partial_\lambda\rho$ is Hermitian and $\mathrm{Tr}(\rho\partial_\lambda\rho) = \mathrm{Tr}[(\partial_\lambda\rho)\rho]$, we have

$$\langle\rho|\partial_\lambda\rho\rangle = \langle\partial_\lambda\rho|\rho\rangle, \tag{19}$$

which indicates that $\langle\rho|\partial_\lambda\rho\rangle$ is real and $\langle\rho|\partial_\lambda\rho\rangle^2 = |\langle\rho|\partial_\lambda\rho\rangle|^2$. Based on the Cauchy-Schwarz inequality, there is $\langle\partial_\lambda\rho|\partial_\lambda\rho\rangle\langle\rho|\rho\rangle \geqslant |\langle\rho|\partial_\lambda\rho\rangle|^2$. Thus, the fidelity susceptibility is nonnegative, which means $\mathcal{F}_\lambda \geqslant 0$.

Next, we consider the normalized density vector representation in Liouville space. As we mentioned before, the normalized density vectors could be treated as states in $N^2 \times N^2$ Hilbert space, then the fidelity susceptibility should be written as:

$$\mathcal{F}_\lambda = \langle\partial_\lambda\widetilde{\rho}|\partial_\lambda\widetilde{\rho}\rangle - |\langle\widetilde{\rho}|\partial_\lambda\widetilde{\rho}\rangle|^2. \tag{20}$$

Recalling the definition of $|\widetilde{\rho}\rangle$, which is $|\widetilde{\rho}\rangle = \langle\rho|\rho\rangle^{-\frac{1}{2}}|\rho\rangle$, one has

$$\begin{aligned}|\partial_\lambda\widetilde{\rho}\rangle &= \partial_\lambda(\langle\rho|\rho\rangle^{-\frac{1}{2}}|\rho\rangle) \\ &= -\langle\rho|\rho\rangle^{-\frac{3}{2}}\langle\partial_\lambda\rho|\rho\rangle|\rho\rangle + \langle\rho|\rho\rangle^{-\frac{1}{2}}|\partial_\lambda\rho\rangle.\end{aligned} \tag{21}$$

From eq. (21) we could obtain that

$$\begin{aligned}\langle\widetilde{\rho}|\partial_\lambda\widetilde{\rho}\rangle &= -\langle\rho|\rho\rangle^{-1}\langle\partial_\lambda\rho|\rho\rangle + \langle\rho|\rho\rangle^{-1}\langle\rho|\partial_\lambda\rho\rangle \\ &= 0,\end{aligned} \tag{22}$$

during which the equation $\langle\rho|\partial_\lambda\rho\rangle = \langle\partial_\lambda\rho|\rho\rangle$ has been used. Thus, it is not expected that the fidelity susceptibility reads

$$\mathcal{F}_\lambda = \langle\partial_\lambda\widetilde{\rho}|\partial_\lambda\widetilde{\rho}\rangle. \tag{23}$$

We give a brief check of this expression. Based on eq. (21), we can obtain

$$\begin{aligned}\langle\partial_\lambda\widetilde{\rho}|\partial_\lambda\widetilde{\rho}\rangle &= \langle\rho|\rho\rangle^{-1}\langle\partial_\lambda\rho|\partial_\lambda\rho\rangle - \langle\rho|\rho\rangle^{-2}\langle\rho|\partial_\lambda\rho\rangle^2 \\ &= \frac{1}{\langle\rho|\rho\rangle^2}\left[\langle\partial_\lambda\rho|\partial_\lambda\rho\rangle\langle\rho|\rho\rangle - \langle\rho|\partial_\lambda\rho\rangle^2\right],\end{aligned} \tag{24}$$

which is identical with the expression of the fidelity susceptibility.

Using the same method as above, we could obtain the expression of fidelity susceptibility with a group of parameters λ_i which are brought in at the initial time of the evolution,

$$\begin{aligned}\mathcal{F}_{ij}^\lambda = \frac{1}{[\mathrm{Tr}(\rho^2)]^2}\Big\{&\mathrm{Tr}[(\partial_{\lambda_i}\rho)(\partial_{\lambda_j}\rho)]\mathrm{Tr}\left(\rho^2\right) \\ &-\mathrm{Tr}[\rho(\partial_{\lambda_i}\rho)]\mathrm{Tr}[\rho(\partial_{\lambda_j}\rho)]\Big\}.\end{aligned} \tag{25}$$

Here, the fidelity susceptibility $\mathcal{F}^\lambda$ is a matrix with the elements $\mathcal{F}_{ij}^\lambda$. In Liouville space, it could be written as:

$$\mathcal{F}_{ij}^\lambda = \frac{1}{\langle\rho|\rho\rangle^2}[\langle\partial_{\lambda_i}\rho|\partial_{\lambda_j}\rho\rangle\langle\rho|\rho\rangle - \langle\rho|\partial_{\lambda_i}\rho\rangle\langle\rho|\partial_{\lambda_j}\rho\rangle]. \tag{26}$$

Also, like the single parameter case, considering the normalized density vector representation in Liouville space, the element of the fidelity susceptibility could be rewritten as:

$$\mathcal{F}^{\lambda}_{ij} = \langle \partial_{\lambda_i}\widetilde{\rho}|\partial_{\lambda_j}\widetilde{\rho}\rangle. \tag{27}$$

If we choose $i = j$, we obtain the fidelity susceptibility of single parameter λ_i, which indicates that the ith diagonal elements of the fidelity susceptibility matrix with multiple parameters have a distinct physical meaning, which is the fidelity susceptibility of parameter λ_i.

4 Time derivative of fidelity susceptibility

The study on open systems has been carried on for some time. It is well known that the time evolutions of many open systems could be described by master equation of Lindblad form [24–26]. Considering two states, both of them are dependent on parameter λ and time t, which are independent variables, and the difference on the parameter λ brought in the initial time between these two states is small. To reflect the evolution of the distinguishability between these two states, whose evolutions are determined by the Lindblad master equation, fidelity susceptibility is a useful quantitative measure. Sometimes we need to understand the changing rate of the fidelity susceptibility with time, then the time derivative of fidelity susceptibility needs to be considered.

In the following, we will show the calculation of the time derivative of the fidelity susceptibility in Liouville space. It should be noticed that the density matrices we consider below are nonsingular.

4.1 SLD construction in Liouville space

In $N \times N$ Hilbert space, the symmetric logarithmic derivative (SLD) is defined as [27]:

$$\partial_\lambda \rho = \frac{1}{2}(\rho L + L\rho), \tag{28}$$

where λ is the parameter of the density matrix and brought in at the initial state. As ρ is expanded to be a vector $|\rho\rangle$ in Liouville space, the SLD in Liouville space has to be redefined. According to the SLD definition in Hilbert space, it is reasonable for us to define that

$$|\partial_\lambda \rho\rangle = \frac{1}{2}\mathcal{L}|\rho\rangle, \tag{29}$$

where $\mathcal{L}$ is an operator (matrix) in Liouville space and the relation between $\mathcal{L}$ and the SLD is

$$\mathcal{L}_{ij,km} = L_{mj}\delta_{ik} + L_{ik}\delta_{mj}. \tag{30}$$

From the elements of $\mathcal{L}$ one may find $\mathcal{L}$ is a Hermitian matrix. Next we consider the normalized density vector which is defined as $|\widetilde{\rho}\rangle = \langle\rho|\rho\rangle^{-\frac{1}{2}}|\rho\rangle$, and from eq. (21), we have

$$|\partial_\lambda \widetilde{\rho}\rangle = \frac{1}{2}\widetilde{\mathcal{L}}|\widetilde{\rho}\rangle, \tag{31}$$

where $\widetilde{\mathcal{L}} = \mathcal{L} - \langle\langle\mathcal{L}\rangle\rangle$ and $\langle\langle\mathcal{L}\rangle\rangle$ is defined as:

$$\langle\langle\mathcal{L}\rangle\rangle \equiv \langle\widetilde{\rho}|\mathcal{L}|\widetilde{\rho}\rangle = \frac{\langle\rho|\mathcal{L}|\rho\rangle}{\langle\rho|\rho\rangle}. \tag{32}$$

In the following, we make an agreement that for a matrix A, $\langle\langle A\rangle\rangle \equiv \langle\widetilde{\rho}|A|\widetilde{\rho}\rangle$. It can be readily seen that $\widetilde{\mathcal{L}}$ is also a Hermitian matrix because $\langle\langle\mathcal{L}\rangle\rangle$ is a real number. Following on one may define $\varrho = |\widetilde{\rho}\rangle\langle\widetilde{\rho}|$ as a "super" density matrix in Liouville space. With this definition we obtain

$$\partial_\lambda \varrho = \frac{1}{2}\left(\widetilde{\mathcal{L}}\varrho + \varrho\widetilde{\mathcal{L}}\right). \tag{33}$$

It is similar with the definition of the SLD in Hilbert space. Substituting eq. (31) into eq. (23), the fidelity susceptibility in Liouville space could be rewritten as:

$$\mathcal{F}_\lambda = \frac{1}{4}\mathrm{Tr}(\varrho\widetilde{\mathcal{L}}^2). \tag{34}$$

4.2 Dynamics in Liouville space

Let us investigate the dynamics described by master equation in Liouville space. In $N \times N$ Hilbert space, the time-local master equation reads

$$\partial_t \rho = \mathcal{K}^H(t)\rho, \tag{35}$$

where $\mathcal{K}^H(t)$ is a superoperator acting on the reduced density matrix and can be written as [24–26]:

$$\mathcal{K}^H(t)\rho = -\mathrm{i}[H, \rho] + \sum_a \gamma_a(t)\left[A_a\rho A_a^\dagger - \frac{1}{2}\left\{A_a^\dagger A_a, \rho\right\}\right],$$

with H the Hamiltonian of the open system and $\{\cdot, \cdot\}$ denotes anticommutator. Using the same method above, one could obtain the form of master equation in Liouville space, the process of which is similar with the derivation of the Redfield equation [28],

$$|\partial_t \rho\rangle = \mathcal{K}^L(t)|\rho\rangle. \tag{36}$$

Here $\mathcal{K}^L(t)$ is defined as $\mathcal{K}^L(t) = -iK^I + \sum_a \gamma_a K^a$ and the definition of the elements of K^I is

$$K^I_{ij,km} \equiv H_{ik}\delta_{jm} - H^*_{jm}\delta_{ik}, \tag{37}$$

and the definition of the elements of K^a is

$$K^a_{ij,km} \equiv A^a_{ik}A^{a*}_{jm} - \frac{1}{2}\sum_n\left(A^{a*}_{ni}A^a_{nk}\delta_{mj} + A^{a*}_{nm}A^a_{nj}\delta_{ki}\right).$$

According to the definition of K^I, one could see that K^I is a Hermitian matrix. Like the SLD construction in Liouville space, we consider the normalized density vector. Taking the time derivative on both sides of $|\widetilde{\rho}\rangle = \langle\rho|\rho\rangle^{-1/2}|\rho\rangle$, we obtain

$$\begin{aligned}|\partial_t\widetilde{\rho}\rangle &= [-\frac{1}{2}\langle\rho|\rho\rangle^{-1}\langle\rho|\mathcal{K}^{L\dagger} + \mathcal{K}^L|\rho\rangle + \mathcal{K}^L]\langle\rho|\rho\rangle^{-\frac{1}{2}}|\rho\rangle \\ &= [\mathcal{K}^L - \frac{1}{2}\langle\widetilde{\rho}|\mathcal{K}^{L\dagger} + \mathcal{K}^L|\widetilde{\rho}\rangle]|\widetilde{\rho}\rangle. \end{aligned} \tag{38}$$

Thus, the master equation reads

$$|\partial_t\widetilde{\rho}\rangle = \widetilde{\mathcal{K}}^L(t\,;\,\lambda)\,|\widetilde{\rho}\rangle, \tag{39}$$

where

$$\widetilde{\mathcal{K}}^L(t\,;\,\lambda) = \mathcal{K}^L(t) - \frac{1}{2}\langle\langle\mathcal{K}^L(t) + \mathcal{K}^{L\dagger}(t)\rangle\rangle. \tag{40}$$

Compared with eq. (36), a significant difference between $\widetilde{\mathcal{K}}^L$ and $\mathcal{K}^L$ is that $\widetilde{\mathcal{K}}^L$ depends not only on the time t but also on the parameter λ. With eq. (39), we could obtain

$$\partial_t\varrho = \widetilde{\mathcal{K}}^L(t\,;\,\lambda)\,\varrho + \varrho\widetilde{\mathcal{K}}^{L\dagger}(t\,;\,\lambda)\,. \tag{41}$$

4.3 Calculation of the time derivative

Next we will use the same calculation procedure in ref. [29] below. First, from eq. (33) we could have the time derivative of $\partial_\lambda\varrho$, which is

$$\partial_t\partial_\lambda\varrho = \frac{1}{2}[(\partial_t\widetilde{\mathcal{L}})\varrho + \widetilde{\mathcal{L}}(\partial_t\varrho) + (\partial_t\varrho)\widetilde{\mathcal{L}} + \varrho(\partial_t\widetilde{\mathcal{L}})]. \tag{42}$$

Next there is

$$\begin{aligned}&\mathrm{Tr}[\varrho(\partial_t\widetilde{\mathcal{L}})\widetilde{\mathcal{L}} + \varrho\widetilde{\mathcal{L}}(\partial_t\widetilde{\mathcal{L}})]\\ &= \mathrm{Tr}(2\widetilde{\mathcal{L}}\partial_t\partial_\lambda\varrho) - \mathrm{Tr}[2\,(\partial_t\varrho)\,\widetilde{\mathcal{L}}^2].\end{aligned}$$

Taking the time derivative on both sides of eq. (34), we have

$$\partial_t\mathcal{F}_\lambda = \frac{1}{4}\mathrm{Tr}[(\partial_t\varrho)\widetilde{\mathcal{L}}^2 + \varrho(\partial_t\widetilde{\mathcal{L}})\widetilde{\mathcal{L}} + \varrho\widetilde{\mathcal{L}}(\partial_t\widetilde{\mathcal{L}})]. \tag{43}$$

Combining the two equations above, we obtain

$$\partial_t\mathcal{F}_\lambda = \frac{1}{4}\mathrm{Tr}[(2\widetilde{\mathcal{L}}\partial_\lambda - \widetilde{\mathcal{L}}^2)\partial_t\varrho]. \tag{44}$$

Substituting eq. (41) into eq. (44), we have

$$\begin{aligned}\partial_t\mathcal{F}_\lambda &= \frac{1}{4}\mathrm{Tr}[2\widetilde{\mathcal{L}}\partial_\lambda(\widetilde{\mathcal{K}}^L\varrho) + 2\widetilde{\mathcal{L}}\partial_\lambda(\varrho\widetilde{\mathcal{K}}^{L\dagger}) - \widetilde{\mathcal{L}}^2\varrho\widetilde{\mathcal{K}}^{L\dagger}]\\ &= \frac{1}{4}\mathrm{Tr}[2\widetilde{\mathcal{L}}\widetilde{\mathcal{K}}^L\partial_\lambda\varrho - \widetilde{\mathcal{L}}^2\widetilde{\mathcal{K}}^L\varrho]\\ &\quad+\frac{1}{4}\mathrm{Tr}[2\widetilde{\mathcal{L}}\,(\partial_\lambda\varrho)\,\widetilde{\mathcal{K}}^{L\dagger} - \widetilde{\mathcal{L}}^2\varrho\widetilde{\mathcal{K}}^{L\dagger}]\\ &\quad+\frac{1}{4}\mathrm{Tr}[2\widetilde{\mathcal{L}}(\partial_\lambda\widetilde{\mathcal{K}}^L)\varrho + 2\widetilde{\mathcal{L}}\varrho\partial_\lambda\widetilde{\mathcal{K}}^{L\dagger}].\end{aligned} \tag{45}$$

From the definition of $\widetilde{\mathcal{K}}^L$ in eq. (40), it is not difficult to see that $\partial_\lambda\widetilde{\mathcal{K}}^L = \frac{1}{2}\partial_\lambda\langle\langle\mathcal{K}^L(t) + \mathcal{K}^{L\dagger}(t)\rangle\rangle$ is a c-number, which can be put out of the trace, and it is also easy to find out that $\mathrm{Tr}(\varrho\widetilde{\mathcal{L}}) = \langle\widetilde{\rho}|\widetilde{\mathcal{L}}|\widetilde{\rho}\rangle = \langle\widetilde{\rho}|\mathcal{L}|\widetilde{\rho}\rangle - \langle\langle\mathcal{L}\rangle\rangle = 0$. Thus, we obtain the expression of the time derivative of fidelity susceptibility, which reads

$$\begin{aligned}\partial_t\mathcal{F}_\lambda &= \frac{1}{4}\mathrm{Tr}[2\widetilde{\mathcal{L}}\widetilde{\mathcal{K}}^L\partial_\lambda\varrho - \widetilde{\mathcal{L}}^2\widetilde{\mathcal{K}}^L\varrho\\ &\quad+2\widetilde{\mathcal{L}}(\partial_\lambda\varrho)\widetilde{\mathcal{K}}^{L\dagger} - \widetilde{\mathcal{L}}^2\varrho\widetilde{\mathcal{K}}^{L\dagger}].\end{aligned} \tag{46}$$

Substituting eq. (33) into the equation above, we have

$$\begin{aligned}\partial_t\mathcal{F}_\lambda &= \frac{1}{4}\mathrm{Tr}(\widetilde{\mathcal{L}}\widetilde{\mathcal{K}}^L\widetilde{\mathcal{L}}\varrho + \widetilde{\mathcal{L}}\widetilde{\mathcal{K}}^{L\dagger}\widetilde{\mathcal{L}}\varrho)\\ &= \frac{1}{4}\langle\widetilde{\rho}|\widetilde{\mathcal{L}}(\widetilde{\mathcal{K}}^L + \widetilde{\mathcal{K}}^{L\dagger})\widetilde{\mathcal{L}}|\widetilde{\rho}\rangle\\ &= \frac{1}{4}\langle\langle\widetilde{\mathcal{L}}(\widetilde{\mathcal{K}}^L + \widetilde{\mathcal{K}}^{L\dagger})\widetilde{\mathcal{L}}\rangle\rangle.\end{aligned} \tag{47}$$

Based on the definition of $\widetilde{\mathcal{K}}^L$, we know that

$$\begin{aligned}&\widetilde{\mathcal{K}}^L + \widetilde{\mathcal{K}}^{L\dagger}\\ &= \sum_a \gamma_a[(K^a + K^{a\dagger}) - \langle\langle K^a + K^{a\dagger}\rangle\rangle].\end{aligned} \tag{48}$$

Therefore, the time derivative can be written as:

$$\partial_t\mathcal{F}_\lambda = \frac{1}{4}\sum_a\gamma_a\mathcal{M}_a, \tag{49}$$

where $\mathcal{M}_a = \langle\langle\widetilde{\mathcal{L}}\left(K^a + K^{a\dagger}\right)\widetilde{\mathcal{L}}\rangle\rangle - \langle\langle K^a + K^{a\dagger}\rangle\rangle\langle\langle\widetilde{\mathcal{L}}^2\rangle\rangle$.

From the above, we obtain the analytic expression of the time derivative of fidelity susceptibility in Liouville space with normalized density vector representation. The time derivative can be treated as a flow of the fidelity susceptibility and is determined by a group of subflow $\gamma_a\mathcal{M}_a$. It is similar but not exactly the same with the Fisher information flow [29]. One could find that under the circumstance $\gamma_a \geqslant 0$, if $\mathcal{M}_a$ satisfies $\mathcal{M}_a < 0$, there is $\partial_t\mathcal{F}_\lambda \leqslant 0$. Also we know the fidelity reads $F = 1 - \frac{1}{2}\mathcal{F}_\lambda\delta^2\lambda$, then we obtain $\partial_t F = -\frac{1}{2}\partial_t\mathcal{F}_\lambda\delta^2\lambda \geqslant 0$. The inequality $\partial_t F \geqslant 0$ strongly implies no matter what the initial states are, the distinguishability between these two states is reducing and the lost information has been flowing into the reservior.

5 Conclusion

In summary, we have re-investigated the properties of the alternative fidelity based on Hilbert-Schmidt inner product and simplified its form. Being similar with the Uhlmann-Jozsa fidelity, the fidelity based on Hilbert-Schmidt inner product has a distinct geometric meaning. If we express the density matrix using Bloch vector representation, the fidelity is actually the cosine function of the angle between the two generalized Bloch vectors. Following on, the analytic expression of the fidelity susceptibility in both Hilbert and Liouville space is shown.

In Liouville space with normalized density vector representation, the fidelity susceptibility is a special form of Uhlmann-Jozsa fidelity susceptibility for pure states. The key observation is that the inner product of two normalized density vectors is a real number. Also, like the SLD in Hilbert space, we construct the relative SLD in Liouville space. Using the SLD, we then give the time derivative of fidelity susceptibility with normalized density vector representation in Liouville space. The results of the alternative fidelity and fidelity susceptibility developed in this investigation are believed to have further applications in the study of QPT, quantum chao, and quantum information theory.

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