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A REGULARITY CRITERION FOR THE NAVIER–STOKES EQUATIONS IN TERMS OF ONE DIRECTIONAL DERIVATIVE OF THE VELOCITY FIELD

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We consider the regularity criterion for the incompressible Navier–Stokes equations. We show that the weak solution is regular, provided

$$\frac{\partial u}{\partial x_3} \in L^{\frac{2}{1-r}}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{with } 0 \le r \le 1$$

for some $T > 0$, where $\dot{X}_r$ is the multiplier space. This extends a result of Kukavica and Ziane [14].

Keywords: Navier–Stokes equations; regularity criterion; *a priori* estimates.

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1. Introduction

Consider the Navier–Stokes equations in $\mathbb{R}^3$

$$\begin{aligned} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, &\quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} u = 0, &\quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = u_0(x), &\quad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p = p(x,t)$ is the scalar pressure and $u_0(x)$ with div $u_0 = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

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In the last century, Leray [16] and Hopf [12] constructed a weak solution u of (1.1) for arbitrary $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. The solution is called the Leray–Hopf weak solution. From that time on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier–Stokes equations. Different criteria for regularity of the weak solutions have been proposed. The Prodi–Serrin conditions (see [17–19]) states that any weak Leray–Hopf solution belonging to the class $L^\alpha((0,T);L^q(\mathbb{R}^3))$ with $2/\alpha + 3/q \le 1$, $2 < \alpha < \infty$, $3 < q < \infty$ is regular on $(0,T) \times \mathbb{R}^3$, while the limit case $q = 3$ was covered much later by Escauriaza, Seregin and Sverak [5]. In 1995, Beirão da Veiga [2] established a Serrin's type regularity criterion on the gradient of the velocity field: $\nabla u \in L^\alpha((0,T);L^q(\mathbb{R}^3))$ with $2/\alpha + 3/q = 2$ and $3/2 \le q \le \infty$. Note that the case $q = 3/2$ is a consequence of the Sobolev embedding theorem and [6]. Very recently, Zhou in [21] added a condition on the gradient of one velocity component with, say, $\nabla u_3 \in L^{\alpha,\gamma}$ for $2/\alpha + 3/\gamma = 3/2$, and a new regularity criterion added on any component of velocity was also established in [23]; see also [3, 13, 28]. Further criteria, concerning the gradient of one velocity component can be found in [27]. There are also some regularity criteria in terms of pressure, the pressure gradient and vorticity. We refer the readers who are interested in them to the literature [11, 22, 24, 25] and [7] for regularity issues in critical space. The regularity criteria with weighted form are exhibited in [26].

We note that, in [14], the authors proved that if the third derivative of the velocity $\partial u/\partial x_3$ belongs to the space $L^{s_0}(0,T;L^{r_0}(\mathbb{R}^3))$, where $2/s_0 + 3/r_0 \le 2$ and $9/4 \le r_0 \le 3$, then the solution is regular. This extends a result of Beirão da Veiga [2] by making a requirement only on one direction of the velocity instead of on the full gradient. The derivative $\partial u/\partial x_3$ can be substituted with any directional derivative of u. Our main purpose is to extend this criterion to the multiplier spaces and we show that the weak solution is regular, provided

$$\frac{\partial u}{\partial x_3} \in L^{\frac{2}{1-r}}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{for some } r \text{ with } 0 \le r \le 1. \tag{1.2}$$

2. Preliminaries and Main Result

In this section, we recall the definition of the multiplier space, which was introduced in ([8, 9]). The space $\dot{X}_r$ of pointwise multipliers which map L^2 into $\dot{H}^{-r}$ is defined in the following way:

Definition 2.1. For $0 \le r < 3/2$, the space $\dot{X}_r$ is defined as the space of $f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2} < \infty,$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of the space $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{\frac{r}{2}} u\|_{L^2}$.

The norm of $\dot{X}_r$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2}.$$

We have the homogeneity properties: $\forall x_0 \in \mathbb{R}^3$

$$\|f(. + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}$$
$$\|f(\lambda .)\|_{\dot{X}_r} \le \frac{1}{\lambda^r}\|f\|_{\dot{X}_r}, \quad \lambda > 0.$$

Lemma 2.2. *Let $0 \le r < 3/2$. Then,*

$$L^{\frac{3}{r}}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3)$$

holds.

Proof. Let $f \in L^{\frac{3}{r}}(\mathbb{R}^3)$. By using the following well-known Sobolev embedding

$$\dot{H}^r(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$$

with $1/q = 1/2 - r/3$, we have

$$\|fg\|_{L^2} \le \|f\|_{L^{\frac{3}{r}}}\|g\|_{L^q} \le \|f\|_{L^{\frac{3}{r}}}\|g\|_{\dot{H}^r}.$$

Then, it follows that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2} \le C\|f\|_{L^{\frac{3}{r}}}.$$

□

Example 2.3. Due to the well-known inequality

$$\left\|\frac{g}{|x|}\right\|_{L^2} \le 2\|\nabla g\|_{L^2},$$

we see that $|x|^{-1} \in \dot{X}_1(\mathbb{R}^3)$.

Our main result can be stated as follows:

Theorem 2.4. *Suppose that $u_0 \in H^1(\mathbb{R}^3)$ with div $u_0 = 0$ in the sense of distribution. Let u be a weak solution to the Navier–Stokes equations corresponding to u_0 which satisfies the energy inequality. Let $T > 0$ and suppose that*

$$\frac{\partial u}{\partial x_3} \in L^{\frac{2}{1-r}}(0, T; \dot{X}_r(\mathbb{R}^3)) \quad \text{for some } r \text{ with } 0 \le r \le 1. \tag{2.1}$$

Then, $u(t, x)$ is as smooth as the data allow. Thus, in our case $u(t, x) \in C^\infty((0, T) \times \mathbb{R}^3)$ and u is unique in the class of all weak solutions satisfying the energy inequality.

3. Proofs

The proof of Theorem 2.4 is based on two major parts. The first part establishes bounds for $\|\frac{\partial u}{\partial x_3}\|_{L^2}^2$ and the time integral of $\|\nabla\frac{\partial u}{\partial x_3}\|_{L^2}^2$, while the second controls $\|\nabla u\|_{L^2}^2$ in terms of time integrals of $\|\nabla\frac{\partial u}{\partial x_3}\|_{L^2}^2$.

Lemma 3.1. *Suppose that* $u_0 \in H^1(\mathbb{R}^3)$ *with* div $u_0 = 0$ *in the sense of distribution. Let* u *be a weak solution to the Navier–Stokes equations corresponding to* u_0 *which satisfies the energy inequality. Suppose* (2.1) *holds. Then, for any* $t \leq T$, *we have*

$$\left\|\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2 \leq \left\|\frac{\partial u}{\partial x_3}(0,\cdot)\right\|_{L^2}^2 e^{\|u_0\|_{L^2}^2}\exp\left(C\int_0^t \left\|\frac{\partial u}{\partial x_3}(s,\cdot)\right\|_{\dot{X}_r}^{\frac{2}{1-r}} ds\right) \tag{3.1}$$

and

$$\int_0^t \left\|\nabla\frac{\partial u}{\partial x_3}(s,\cdot)\right\|_{L^2}^2 ds \leq C,$$

for some $C > 0$.

Proof. First, we differentiate the first equation of (1.1) about x_3, and then multiply the resulting equation by $\partial u/\partial x_3$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2 + \left\|\nabla\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2 &= -\int_{\mathbb{R}^3}\left(\frac{\partial u}{\partial x_3}\cdot\nabla u\right)\cdot\frac{\partial u}{\partial x_3}dx \\ &\leq \left\|\frac{\partial u}{\partial x_3}\cdot\frac{\partial u}{\partial x_3}\right\|_{L^2}\|\nabla u\|_{L^2}.\end{aligned}$$

Due to Hölder's inequality and the following ones ($0 \leq r \leq 1$)

$$\|w\|_{\dot{H}^r} = \frac{1}{(2\pi)^{\frac{d}{2}}}\||\xi|^r\widehat{w}\|_{L^2} \leq \|w\|_{L^2}^{1-r}\|\nabla w\|_{L^2}^r, \tag{3.2}$$

it is easy to see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2 + \left\|\nabla\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2 \\ &\quad\leq \left\|\frac{\partial u}{\partial x_3}\right\|_{\dot{X}_r}\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot{H}^r}\|\nabla u\|_{L^2} \\ &\quad\leq C\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot{X}_r}\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2}^{1-r}\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^{r}\|\nabla u\|_{L^2} \\ &\quad\leq C\left(\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot{X}_r}^{\frac{2}{2-r}}\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2}^{2(\frac{1-r}{2-r})}\|\nabla u\|_{L^2}^{\frac{2}{2-r}}\right)^{1-\frac{r}{2}}\left(\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^2\right)^{\frac{r}{2}}.\end{aligned}$$

By Young's inequality a few times, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2+\left\|\nabla\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2\\
&\quad\le C\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot X_r}^{\frac{2}{2-r}}\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2}^{2(\frac{1-r}{2-r})}\|\nabla u\|_{L^2}^{\frac{2}{2-r}}+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^2\\
&\quad=C\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2}^{2(\frac{1-r}{2-r})}\left[\left(\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot X_r}^{\frac{2}{1-r}}\right)^{\frac{1-r}{2-r}}(\|\nabla u\|_{L^2}^2)^{\frac{1}{2-r}}\right]+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^2\\
&\quad\le C\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2}^{2(\frac{1-r}{2-r})}\left[\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot X_r}^{\frac{2}{1-r}}+\|\nabla u\|_{L^2}^2\right]+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^2.
\end{aligned}$$

Hence,

$$\frac{d}{dt}\left\|\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2+\left\|\nabla\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2\le C\left\|\frac{\partial u}{\partial x_3}\right\|_{L^2}^2\left[\left\|\frac{\partial u}{\partial x_3}\right\|_{\dot X_r}^{\frac{2}{1-r}}+\|\nabla u\|_{L^2}^2\right]$$

since $(1-r)/(2-r)<1$. Thanks to Gronwall's inequality, we obtain

$$\begin{aligned}
\left\|\frac{\partial u}{\partial x_3}(t,\cdot)\right\|_{L^2}^2&\le\left\|\frac{\partial u}{\partial x_3}(0,\cdot)\right\|_{L^2}^2\\
&\qquad\times\exp\left(C\int_0^t\left(\left\|\frac{\partial u}{\partial x_3}(s,\cdot)\right\|_{\dot X_r}^{\frac{2}{1-r}}+\|\nabla u(s,\cdot)\|_{L^2}^2\right)ds\right)\\
&\le\left\|\frac{\partial u}{\partial x_3}(0,\cdot)\right\|_{L^2}^2\exp\left(C\int_0^t\left\|\frac{\partial u}{\partial x_3}(s,\cdot)\right\|_{\dot X_r}^{\frac{2}{1-r}}ds+\|u_0\|_{L^2}^2\right)\\
&\le\left\|\frac{\partial u}{\partial x_3}(0,\cdot)\right\|_{L^2}^2e^{\|u_0\|_{L^2}^2}\exp\left(C\int_0^t\left\|\frac{\partial u}{\partial x_3}(s,\cdot)\right\|_{\dot X_r}^{\frac{2}{1-r}}ds\right)
\end{aligned}$$

and

$$\int_0^t\left\|\nabla\frac{\partial u}{\partial x_3}(s,\cdot)\right\|_{L^2}^2ds\le C,$$

where C depends on $\|u_0\|_{L^2}$, $\|\frac{\partial u}{\partial x_3}(s,\cdot)\|_{L^{\frac{2}{1-r}}(0,t;\dot X_r(\mathbb{R}^3))}$. □

Now, we establish bounds for H^1 estimates. For convenience, we recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space $\mathbb{R}^3$ (see e.g. [1, 4, 10, 15]). There exits a constant $C_\alpha > 0$ such that

$$\begin{aligned}\|f\|_{L^\alpha}^{2\alpha} &\le C_\alpha \|f\|_{L^2}^{6-\alpha}\|\partial_{x_1} f\|_{L^2}^{\alpha-2}\|\partial_{x_2} f\|_{L^2}^{\alpha-2}\|\partial_{x_3} f\|_{L^2}^{\alpha-2}\\ &\le C_\alpha \|f\|_{L^2}^{6-\alpha}\|f\|_{H^1}^{3(\alpha-2)},\end{aligned}\tag{3.3}$$

for all $f \in H^1(\mathbb{R}^3)$ and every $\alpha \in [2,6]$.

Lemma 3.2. *Suppose that $u_0 \in H^1(\mathbb{R}^3)$ with* div $u_0 = 0$ *in the sense of distribution. Let u be a weak solution to the Navier–Stokes equations corresponding to u_0 which satisfies the energy inequality. Suppose* (2.1) *holds. Then, for any $t \le T$, we have*

$$\|\nabla u(t,\cdot)\|_{L^2}^2 + \int_0^t \|\Delta u(s,\cdot)\|_{L^2}^2\, ds \le C, \tag{3.4}$$

where C depends on T, $\|\nabla u_0\|_{L^2}$ and $\|\frac{\partial u}{\partial x_3}\|_{L^{\frac{2}{1-r}}(0,T;\dot{X}_r(\mathbb{R}^3))}$.

Proof. Taking the inner product of the equation (1.1) with $-\Delta u$ in L^2 and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t,\cdot)\|_{L^2}^2 + \|\Delta u(t,\cdot)\|_{L^2}^2 = \int_{\mathbb{R}^3} u\cdot\nabla u\cdot\Delta u\, dx \le \|\nabla u\|_{L^3}^3.$$

By (3.3), we have

$$\begin{aligned}\|\nabla u\|_{L^3}^3 &\le C\left(\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla_h\nabla u\|_{L^2}^{\frac{1}{3}}\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^{\frac{1}{6}}\right)^3\\ &= C\left[(\|\nabla_h\nabla u\|_{L^2}^2)^{\frac{1}{6}}\left(\|\nabla u\|_{L^2}^3\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}\right)^{\frac{1}{6}}\right]^3\\ &= C(\|\nabla_h\nabla u\|_{L^2}^2)^{\frac{1}{2}}\left(\|\nabla u\|_{L^2}^3\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}\right)^{\frac{1}{2}},\end{aligned}$$

where $\nabla_h = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. Hence, by Young's inequality and the Cauchy inequality, we obtain

$$\begin{aligned}\|\nabla u\|_{L^3}^3 &\le \frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^3\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}\\ &\le \frac{1}{4}\|\nabla_h\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\left(\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).\end{aligned}$$

As a result we get

$$\frac{d}{dt}\|\nabla u(t,\cdot)\|_{L^2}^2 + \|\Delta u(t,\cdot)\|_{L^2}^2 \le C\|\nabla u\|_{L^2}^2\left(\left\|\nabla\frac{\partial u}{\partial x_3}\right\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).$$

Then, Gronwall's inequality coupled with Lemma 3.1 yields

$$\|\nabla u(t,\cdot)\|_{L^2}^2 + \int_0^t \|\Delta u(s,\cdot)\|_{L^2}^2\, ds \le C. \qquad \square$$

We are now in a position to prove our main result.

Proof. First note that since the initial velocity field $u_0 \in L^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, then it is well known that there is a $T_0 > 0$ such that there is a unique strong solution

$$u \in L^\infty(0, T_0; H^1(\mathbb{R}^3)) \cap L^2(0, T_0; H^2(\mathbb{R}^3))$$

to the Navier–Stokes equations (1.1) (see [20]). According to the result about uniqueness [20], our weak solution identifies with the strong solution in $(0, T_0)$. If

$$\frac{\partial u}{\partial x_3} \in L^{\frac{2}{1-r}}(0, T; \dot{X}_r(\mathbb{R}^3)),$$

then Lemmas 3.1 and 3.2 imply

$$\sup_{0 \le t \le T} \|u(t,\cdot)\|_{H^1} + \int_0^T \|\nabla u(t,\cdot)\|_{L^2}^2 + \int_0^T \|\Delta u(s,\cdot)\|_{L^2}^2 ds \le C(\|u_0\|_{H^1}).$$

Thus, the local strong solution u can be extended to time T, and also identifies with the weak solution. Moreover the classical regularity criteria [20] implies that u is a regular solution on $[0, T]$. This completes the proof of Theorem 2.4. □

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