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# Normality criteria of meromorphic functions sharing one value

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ABSTRACT

Let $k$ be a positive integer and $\mathscr{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least $k+1$. If for each pair $(f, g)$ in $\mathscr{F}$, $ff^{(k)}$ and $gg^{(k)}$ share a non-zero complex number $a$ ignoring multiplicity, then $\mathscr{F}$ is normal in $D$.



## 1. Introduction and main results

Let $D$ be a domain in $\mathbb{C}$ and let $\mathscr{F}$ be meromorphic functions defined in the domain $D$. Then $\mathscr{F}$ is said to be normal in $D$, in the sense of Montel, if any sequence $\{f_n\} \subset \mathscr{F}$ contains a subsequence $\{f_{n_j}\}$ such that $f_{n_j}$ converges spherically locally uniformly in $D$, to a meromorphic function or $\infty$.

Let $F$ and $G$ be two non-constant meromorphic functions, we say that $F$ and $G$ share $a$ IM if $F-a$ and $G-a$ have the same zeros (ignoring multiplicity). When $a=\infty$ the zeros of $F-a$ mean the poles of $F$ (see [8] and [30]).

In 1959, W.K. Hayman [10] proved that if $F$ is a transcendental meromorphic function, then $F^nF'$ assumes every finite non-zero complex number infinitely often for any positive integer $n \geqslant 3$. He conjectured [11] that this remains valid for $n=1$ and for $n=2$. Further, the case of $n=2$ was confirmed by E. Mues [17] in 1979. The case $n=1$ was respectively considered and settled by J. Clunie [7]; W. Bergweiler and A. Eremenko [3]; H.H. Chen and M.L. Fang [4].

Related to above problem on value distribution, Hayman [11] proposed a conjecture on normal family as follows: If each $f \in \mathscr{F}$ satisfies $f^nf' \neq a$ for a positive integer $n$ and a non-zero complex number $a$, then $\mathscr{F}$ is normal. This conjecture has been shown to be true by Yang and Zhang [32] (for $n \geqslant 5$ and for $n \geqslant 2$ in case that $\mathscr{F}$ is a family of holomorphic functions), Gu [14] (for $n=3, 4$), Oshkin [18] (for holomorphic functions, $n=1$; cf. [15]), and Pang [20] (for $n \geqslant 2$ in general; cf. [9]). As indicated by X.C. Pang [20] (or see [4,34]), the conjecture for $n=1$ is a consequence of Chen–Fang's theorem and his theorem which is a generalization of Zalcman's lemma (cf. [33]). Thus, the Hayman's conjecture on normal family is also verified completely.

From the view of shared values, Q.C. Zhang [36] proved that $\mathscr{F}$ is also normal when each pair $(f,g)$ of $\mathscr{F}$ is such that $f^nf'$ and $g^ng'$ share a finite non-zero complex number $a$ IM for $n \geqslant 2$ (or see [35]). By definition, two meromorphic functions $F$ and $G$ are said to share $a$ IM (ignoring multiplicity) if $F^{-1}(a) = G^{-1}(a)$ (see [8]). There are examples showing that this result is not true if $n=1$. Further, more results related to this problem have been obtained, see W. Hennekemper [12], Y.F. Wang and M.L. Fang [26], W. Schwick [24], Y.T. Li, Y.X. Gu [16] respectively.

To understand above Hayman's problem on value distribution well, many authors studied the functions of the form $F(F^{(k)})^n$. In the case of $n \geqslant 2$, C.C. Yang, L. Yang and Y.F. Wang [29] obtained that if $F$ is transcendental entire function, then

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the only possible Picard value of $F(F^{(k)})^n$ is the value zero, further, Z.F. Zhang and G.D. Song [37], A. Alotaibi [1] proved that $F(F^{(k)})^n$ take every finite non-zero value infinitely ofen when $F$ is transcendental meromorphic. According to Block's principle (cf. [2,23]), corresponding to above problem on value distribution, there are analogues in normal family theory. Really, we confirmed it in [13], namely, if $\mathscr{F}$ is a family of meromorphic functions in a domain $D$ of the plane $\mathbb{C}$ such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least $k(\geqslant 2)$. If for each pair $(f,g) \in \mathscr{F}$, $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share $a$ IM, then $\mathscr{F}$ is normal in $D$, where $n, k \geqslant 2$.

In the case of $FF^{(k)}$, L. Yang and C.C. Yang [31] proposed the conjecture: If $F$ is transcendental, then $FF^{(k)}$ assumes every finite non-zero complex number infinitely often for any positive integer $k$. C.C. Yang and P.C. Hu [28] obtained a part of answer. In 2006, J.P. Wang [25, Theorem 3] proved that this conjecture holds when $F$ has only zeros of multiplicity at least $k+1$ ($k \geqslant 2$). It is natural to study normality criteria corresponding to the case of $FF^{(k)}$. In this paper, we discuss the problem and prove the following main result:

**Theorem 1.1.** *Take a positive integer $k$ and a non-zero complex number $a$. Let $\mathscr{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least $k+1$. For each pair $(f,g) \in \mathscr{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share $a$ IM, then $\mathscr{F}$ is normal in $D$.*

Obviously, for the case of multiplicity at least $k+1$, Theorem 1.2 answered the unsolved problem of [13] corresponding to the case of $n=1$, and partially solve Fang's conjecture. The following example shows that the multiplicity restriction on zeros of $f$ is sharp in Theorem 1.1 when $k=1$.

**Example 1.2.** Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and take a non-zero complex number $a$ with $|a| < \frac{1}{2}$. We consider the family

$$\mathscr{F} = \left\{ f_m(z) = m\left(z - \frac{1}{2}\right) + \frac{a}{m} \;\middle|\; m = 1, 2, \ldots \right\}.$$

Obviously, each $f_m \in \mathscr{F}$ has only a simple zero, and for distinct positive integers $m, l$, $f_m f_m'$ and $f_l f_l'$ share $a$ IM. However, the family $\mathscr{F}$ is not normal at $z = \frac{1}{2}$.

However, when $k \geqslant 2$, for a family of holomorphic functions, we can show the following result:

**Theorem 1.3.** *Take a positive integer $k \geqslant 2$ and a non-zero complex number $a$. Let $\mathscr{F}$ be a family of holomorphic functions in the domain $D \subset \mathbb{C}$ such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least $k$. For each pair $(f,g) \in \mathscr{F}$, if $ff^{(k)}$ and $gg^{(k)}$ share $a$ IM, then $\mathscr{F}$ is normal in $D$.*

In [22], X.C. Pang and L. Zalcman proved that if $f$ is a transcendental entire function and has only zeros of multiplicity at least $k$, then $f^n f^{(k)}$ take every non-zero complex number infinitely often, where $n, k$ are positive integers. Obviously, according to the proof of Theorem 1.1, Theorem 1.3 follows from Pang–Zaclman's result above. Moreover, the condition that $f$ has only zeros of multiplicity at least $k$ in Theorem 1.3 is sharp, we show this claim by the following example:

**Example 1.4.** Take $D = \{z \in \mathbb{C} \mid |z| < 1\}$, an integer $k \geqslant 2$, and a non-zero complex number $a$. We consider the family

$$\mathscr{F} = \left\{ f_m(z) = mz^{k-1} \;\middle|\; m = 1, 2, \ldots \right\}.$$

Obviously, each $f_m \in \mathscr{F}$ has only a zero of multiplicity $k-1$, and for distinct positive integers $m$, $l$, we have $f_m f_m^{(k)}$ and $f_l f_l^{(k)}$ share $a$ IM. But, the family $\mathscr{F}$ is not normal at $z = 0$.

The condition $a \neq 0$ in Theorem 1.1 and Theorem 1.3 is necessary. For example, we consider the following families:

**Example 1.5.** Define $D$ as in Example 1.4, and write

$$\mathscr{F} = \left\{ f_m(z) = mz^{k+1} \;\middle|\; m = 1, 2, \ldots \right\}$$

or

$$\mathscr{F} = \left\{ f_m(z) = e^{mz} \;\middle|\; m = 1, 2, \ldots \right\}.$$

Obviously, any $f_m \in \mathscr{F}$ has only zeros of multiplicity at least $k+1$. For distinct positive integers $m$, $l$, $f_m f_m^{(k)}$ and $f_l f_l^{(k)}$ share 0 IM. However, the families $\mathscr{F}$ are not normal at $z = 0$.

Furthermore, according to the ideas of [4], we may obtain another result as following:

**Theorem 1.6.** *Take a positive integer $k$ and a non-zero complex number $a$. Let $\mathscr{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$ such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least $k$ and such that $f^{(k)}$ has no simple zeros. For each element $f$ of $\mathscr{F}$, if $f(z)f^{(k)}(z) = a$ implies $|f^{(k)}(z)| \leqslant A$ for a positive number $A$, then $\mathscr{F}$ is normal in $D$.*

## 2. Preliminary lemmas

To prove Theorem 1.1, we will need the following *Zalcman's lemma* (cf. [34]):

**Lemma 2.1.** *Take a positive integer $k$. Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that zeros of each $f \in \mathcal{F}$ are of multiplicity at least $k$. If $\mathcal{F}$ is not normal at a point $z_0 \in \Delta$, then for $0 \leqslant \alpha < k$, there exist a sequence $\{z_n\} \subset \Delta$ of complex numbers with $z_n \to z_0$; a sequence $\{f_n\}$ of $\mathcal{F}$; and a sequence $\{\rho_n\}$ of positive numbers with $\rho_n \to 0$ such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ locally uniformly (with respect to the spherical metric) to a non-constant meromorphic function $g(\xi)$ on $\mathbb{C}$. Moreover, the zeros of $g(\xi)$ are of multiplicity at least $k$, and the function $g(\xi)$ may be taken to satisfy the normalization $g^\sharp(\xi) \leqslant g^\sharp(0) = 1$ for any $\xi \in \mathbb{C}$. In particular, $g(\xi)$ has at most order 2.*

This is Pang's generalization (cf. [19,21,27]) of the Main Lemma in [33] (where $\alpha$ is taken to be 0), with improvements due to Schwick [24] and Chen and Gu [5]. In Lemma 2.1, the *order* of $g$ is defined by using the Nevanlinna's characteristic function $T(r, g)$:

$$\operatorname{ord}(g) = \limsup_{r\to\infty} \frac{\log T(r, g)}{\log r}.$$

Here $g^\sharp$ denotes the *spherical derivative*

$$g^\sharp(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|^2}.$$

**Lemma 2.2.** *Let $k$ be a positive integer and $a \neq 0$ be a finite complex number. If $f$ is a rational function but not a polynomial and $f$ has only zeros of multiplicity at least $k+1$, then $ff^{(k)} - a$ has at least two distinct zeros.*

**Proof.** First of all, assume, to the contrary, that $ff^{(k)} - a$ has exactly one zero. We set

$$f = \frac{A(z-\alpha_1)^{m_1}(z-\alpha_2)^{m_2}\cdots(z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1}(z-\beta_2)^{n_2}\cdots(z-\beta_t)^{n_t}}, \tag{2.1}$$

where $A$ is a non-zero constant. Since all zeros of $f$ have at least multiplicity $k+1$, we have $m_i \geqslant k+1$ $(i = 1, 2, \ldots, s)$, $n_j \geqslant 1$ $(j = 1, 2, \ldots, t)$. For simplicity, we denote

$$m_1 + m_2 + \cdots + m_s = M \geqslant (k+1)s, \tag{2.2}$$

$$n_1 + n_2 + \cdots + n_t = N \geqslant t. \tag{2.3}$$

By (2.1), we obtain

$$f^{(k)} = \frac{A(z-\alpha_1)^{m_1-k}(z-\alpha_2)^{m_2-k}\cdots(z-\alpha_s)^{m_s-k}g(z)}{(z-\beta_1)^{n_1+k}(z-\beta_2)^{n_2+k}\cdots(z-\beta_t)^{n_t+k}}, \tag{2.4}$$

in which $g$ is a polynomial of degree at most $k(s+t-1)$. Thus (2.1) together with (2.4) imply

$$ff^{(k)} = \frac{A^2(z-\alpha_1)^{2m_1-k}(z-\alpha_2)^{2m_2-k}\cdots(z-\alpha_s)^{2m_s-k}g(z)}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}} = \frac{P(z)}{Q(z)}, \tag{2.5}$$

where $P$ and $Q$ are polynomials which have no common factor. Since $ff^{(k)} - a = 0$ has only a zero $z_0$, then from (2.5) we deduce that

$$ff^{(k)} = a + \frac{B(z-z_0)^l}{(z-\beta_1)^{2n_1+k}(z-\beta_2)^{2n_2+k}\cdots(z-\beta_t)^{2n_t+k}} = \frac{P(z)}{Q(z)}, \tag{2.6}$$

where $l$ is a positive integer, $B$ is a non-zero constant. Obviously, we find $z_0 \neq \alpha_i$ $(i = 1, \ldots, s)$ due to $a \neq 0$.

Differentiating (2.5), we have

$$\left[ff^{(k)}\right]' = \frac{(z-\alpha_1)^{2m_1-k-1}(z-\alpha_2)^{2m_2-k-1}\cdots(z-\alpha_s)^{2m_s-k-1}g_1(z)}{(z-\beta_1)^{2n_1+k+1}\cdots(z-\beta_t)^{2n_t+k+1}}, \tag{2.7}$$

in which $g_1(z)$ is a polynomial satisfying

$$s+t-1 \leqslant \deg(g_1) \leqslant (k+1)(s+t-1).$$

Subsequently, (2.6) yields

$$\left[ff^{(k)}\right]' = \frac{(z-z_0)^{l-1} g_2(z)}{(z-\beta_1)^{2n_1+k+1} + \cdots + (z-\beta_t)^{2n_t+k+1}}, \tag{2.8}$$

where $g_2(z) = B(l-2N-kt)z^t + B_1 z^{t-1} + \cdots + B_t$ is a polynomial.

Now we distinguish two cases.

**Case 1.** $l \neq 2N+kt$. By using (2.6), we have $\deg(P) \geqslant \deg(Q)$. Thus (2.5) implies

$$\sum_{i=1}^{s}(2m_i - k) + \deg(g) \geqslant \sum_{j=1}^{t}(2n_j + k),$$

and hence $M \geqslant N + \frac{k}{2}$, that is, $M > N$. From (2.7) and (2.8), noting that $z_0 \neq \alpha_i$, we find

$$\sum_{i=1}^{s}(2m_i - k - 1) \leqslant \deg(g_2) = t.$$

It follows that $2M - (k+1)s \leqslant t$, or equivalently, $2M \leqslant (k+1)s + t$. Combining this inequality with (2.2) and (2.3), we obtain

$$2M \leqslant (k+1)s + t \leqslant (k+1)\left(\frac{M}{k+1}\right) + N < 2M,$$

which is impossible.

**Case 2.** $l = 2N + kt$. Next we distinguish two subcases: $M > N$ and $M \leqslant N$.

When $M > N$, similar to the Case 1, it follows that $2M < 2M$ from (2.7) and (2.8). This is a contradiction.

If $M \leqslant N$, by comparing (2.7) with (2.8), we may give the following inequality

$$l-1 \leqslant \deg(g_1) \leqslant (k+1)(s+t-1),$$

and hence

$$2N = l - kt \leqslant \deg(g_1) + 1 \leqslant (k+1)s + t - k < (k+1)s + t < (k+1)\left(\frac{M}{k+1}\right) + N \leqslant 2N.$$

Obviously this is a contradiction.

Finally, assume, to the contrary, that $ff^{(k)} - a$ has no zero, then $l = 0$ for (2.6). Proceeding as in the proof for Case 1, we also obtain a contradiction.

Hence, Lemma 2.2 is proved complectly. □

**Lemma 2.3.** *Take a positive integers $k$ and a non-zero complex number $a$. If $f$ is a non-constant meromorphic function such that $f$ has only zeros of multiplicity at least $k+1$, then $ff^{(k)} - a$ has at least two distinct zeros.*

**Proof.** If $f$ is a polynomial, we find immediately that $ff^{(k)}$ has multiple zeros since $f$ has only zeros of multiplicity at least $k+1$, and hence $ff^{(k)} - a$ has at least one zero. Suppose, to the contrary, that $ff^{(k)} - a$ has only a zero $z_0$, then there exist a non-zero constant $A$ and an integer $l \geqslant 2$ such that

$$f(z)f^{(k)}(z) = a + A(z-z_0)^l,$$

which, however, has only simple zeros since $a \neq 0$. This is a contradiction. Hence $ff^{(k)} - a$ must has at least two zeros.

When $f$ is a rational but not a polynomial function, it follows from Lemma 2.1.

If $f$ is transcendental, we obtain directly that $ff^{(k)} - a$ has infinitely many zeros due to [4] and [25].

The proof of Lemma 2.3 is completed. □

**Lemma 2.4.** *Take a positive integer $k$ and a non-zero complex number $a$. Let $f$ be a non-constant rational function such that all zeros of $f$ have multiplicity at least $k$. Then $ff^{(k)} - a$ has at least one zero.*

**Proof.** We first consider the case for $f$ is a non-constant polynomial, since all zeros of $f$ have multiplicity at least $k$, $ff^{(k)}$ is also a non-constant polynomial. Therefore, $ff^{(k)}-a$ has at least one zero.

Secondly, we study the case that $f$ is a non-constant rational function but not a polynomial. We suppose, to the contrary, that $ff^{(k)}-a$ has no zero. As in the proof of Lemma 2.2, we can also obtain (2.1), (2.3), (2.4), (2.5) and (2.7) respectively. Moreover, we also get analogues of (2.2) and (2,6), in which the condition $M \geqslant (k+1)s$ in (2.2) is replaced by $M \geqslant ks$, and $l$ in (2.6) takes 0. Consequently, it follows that

$$\left[ff^{(k)}\right]' = \frac{g_3(z)}{(z-\beta_1)^{2n_1+k+1}+\cdots+(z-\beta_t)^{2n_t+k+1}}, \tag{2.9}$$

in which $g_3(z) = -BNz^{t-1}+\cdots$ is a polynomial. Compare (2.7) with (2.9). As a result, we obtain

$$2M-(k+1)s+\deg(g_1) = \deg(g_3) = t-1.$$

Noting that $\deg(g_1) \geqslant s+t-1$ in (2.7), we find consequently

$$M \leqslant \frac{1}{2}ks,$$

but on the other hand $M \geqslant ks$, hence $s=0$. Therefore, from (2.4) and (2.5) we obtain

$$ff^{(k)} = \frac{A^2 g(z)}{(z-b_1)^{2n_1+k}(z-b_2)^{2n_2+k}\cdots(z-b_t)^{2n_t+k}} = \frac{P(z)}{Q(z)},$$

where $g$ is a polynomial of degree at most $k(t-1)$. Since $l$ in (2.6) takes 0, it follows that $\deg(P) = \deg(Q)$, which implies

$$2N+kt = \deg(g) \leqslant k(t-1),$$

and hence $2N \leqslant -k$. This is impossible.

Thus $ff^{(k)}-a$ has at least one zero. This proves Lemma 2.4. □

**Lemma 2.5.** *(See [6].) Let $f$ a transcendental meromorphic function, and let $P_f(z)$, $Q_f(z)$ be two differential polynomials of $f$. If $f^n P_f = Q_f$ holds and the degree of $Q_f$ is at most $n$, then $m(r, P_f) = S(r, f)$.*

Let $F$ be a non-constant meromorphic function in the whole plane and $k$ a positive integer. We denote by $N_{(k)}(r, \frac{1}{F})$ the Nevanlinna's counting function for zeros of $F$ with multiplicity $k$, and by $\overline{N}_{(k)}(r, \frac{1}{F})$ the corresponding one ignoring multipilicity. Then we show the following lemma:

**Lemma 2.6.** *Take a positive integer $k$ and a non-zero complex number $a$. Let $f$ be a transcendental meromorphic function such that all zeros of $f$ have multiplicity at least $k$ and such that $f^{(k)}$ has no simple zeros. Then $ff^{(k)}-a$ has infinitely many zeros.*

**Proof.** Obviously, Lemma 2.6 for the case $k=1$ is a direct consequence of [3] and [4], thus it is sufficient to discuss the case $k \geqslant 2$. Denote

$$F = ff^{(k)}-a, \tag{2.10}$$

and

$$A = \frac{f'}{f}f^{(k)} + f^{(k+1)} - f^{(k)}\frac{F'}{F}. \tag{2.11}$$

We suppose, to the contrary, that $F = ff^{(k)}-a$ has only finitely many zeros, since $f$ is transcendental, then

$$N\left(r, \frac{1}{F}\right) = S(r, f). \tag{2.12}$$

By using (2.10), we obtain

$$T(r, F) = O\big(T(r, f)\big). \tag{2.13}$$

Furthermore, (2.11) implies

$$fA = -\frac{F'}{F}. \tag{2.14}$$

Notice that the zeros of $f$ with multiplicity $\geqslant k+1$ must be the zeros of $f^{(k)}$. Thus by the second fundamental theorem, we have

$$\begin{aligned} N\big(r, ff^{(k)}\big) &\leqslant T\big(r, ff^{(k)}\big) \\ &\leqslant \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{ff^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F}\right) + s(r, f) \\ &\leqslant \overline{N}(r, f) + \overline{N}_{(k)}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f). \end{aligned} \tag{2.15}$$

Since $f^{(k)}$ has no simple zeros and the multiple zeros of $f^{(k)}$ must be the zeros of $A$, from (2.15) we deduce that

$$N\big(r, ff^{(k)}\big) - \overline{N}(r, f) \leqslant \overline{N}_{(k)}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{A}\right) + S(r, f).$$

Moreover, note that $2N(r, f) \leqslant N(r, ff^{(k)}) - \overline{N}(r, f)$. Thus from above we get

$$N(r, f) \leqslant \frac{1}{2}\overline{N}_{(k)}\left(r, \frac{1}{f}\right) + \frac{1}{2}\overline{N}\left(r, \frac{1}{A}\right) + S(r, f). \tag{2.16}$$

It is clear that $F \not\equiv 0$ since $f$ is transcendental, and then by (2.14) we have $A \not\equiv 0$. Therefore, by applying Lemma 2.5 to (2.11), we obtain

$$m(r, A) = S(r, f). \tag{2.17}$$

Suppose that $z_0$ is a zero of $f$ with multiplicity $p\,(\geqslant k+1)$, then $F(z_0) = -1$ and $z_0$ must be a zero of $F' = ff^{(k+1)} + f'f^{(k)}$ with multiplicity at least $2p - (k+1) \geqslant p$. This together with (2.14) implies that $z_0$ must not be a pole of $A$, so the poles of $A$ only come from the zeros of $F$ and the zeros of $f$ with multiplicity $k$. Hence from (2.11), we have

$$N(r, A) \leqslant \overline{N}_{(k)}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{F}\right) = \overline{N}_{(k)}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.18}$$

Therefore, based in (2.17) and (2.18), we obtain consequently

$$T(r, A) = \overline{N}_{(k)}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f). \tag{2.19}$$

Moreover, (2.18) implies that $f = -\frac{1}{A}\frac{F'}{F}$, then by (2.11), (2.12), (2.19) and the first fundamental theorem, we have

$$m(r, f) \leqslant m\left(r, \frac{1}{A}\right) + S(r, f) = T(r, A) - N\left(r, \frac{1}{A}\right) + S(r, f) \leqslant \overline{N}_{(k)}\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{A}\right) + S(r, f). \tag{2.20}$$

By combining (2.16) with (2.20), we deduce that

$$T(r, f) \leqslant \frac{3}{2}\overline{N}_{(k)}\left(r, \frac{1}{f}\right) + S(r, f) \leqslant \frac{3}{2k}N\left(r, \frac{1}{f}\right) + S(r, f),$$

or equivalently

$$\left(1 - \frac{3}{2k}\right)T(r, f) = S(r, f). \tag{2.21}$$

Since $k \geqslant 2$, (2.21) implies that $T(r, f) = S(r, f)$. This contradicts the fact that $f$ is transcendental, and hence $ff^{(k)} - a$ has infinitely many zeros. □

## 3. Proof of Theorem 1.1

Without loss of generality, we may assume that $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Suppose, to the contrary, that $\mathscr{F}$ is not normal in $D$. Without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 = 0$. Then, by Lemma 2.1, there exist a sequence $\{z_j\}$ of complex numbers with $z_j \to 0$ $(j \to \infty)$; a sequence $\{f_j\}$ of $\mathcal{F}$; and a sequence $\{\rho_j\}$ of positive numbers with $\rho_j \to 0$ such that

$$g_j(\xi) = \rho_j^{-\frac{k}{2}} f_j(z_j + \rho_j \xi)$$

converges uniformly to a non-constant meromorphic function $g(\xi)$ in $C$ with respect to the spherical metric. Moreover, $g(\xi)$ is of order at most 2. By Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity $k+1$.

On every compact subset of $\mathbb{C}$ which contains no poles of $g$, we have uniformly

$$f_j(z_j + \rho_j \xi) f_j^{(k)}(z_j + \rho_j \xi) - a = g_j(\xi) g_j^{(k)}(\xi) - a \to g(\xi) g^{(k)}(\xi) - a \tag{3.1}$$

with respect to the spherical metric. If $gg^{(k)} \equiv a$, then $g$ has no zeros. Of course, $g$ also has no poles. Since $g$ is a non-constant meromorphic function of order at most 2, then there exist constants $c_i$ such that $(c_1, c_2) \neq (0, 0)$, and

$$g(\xi) = e^{c_0 + c_1\xi + c_2\xi^2}.$$

Obviously, this is contrary to the case $gg^{(k)} \equiv a$. Hence $gg^{(k)} \not\equiv a$.

By Lemma 2.3, the function $gg^{(k)} - a$ has at least two distinct zeros. Let $\xi_0$ and $\xi_0^*$ be two distinct zeros of $gg^{(k)} - a$. We choose a positive number $\delta$ small enough such that $D_1 \cap D_2 = \emptyset$ and such that $gg^{(k)} - a$ has no other zeros in $D_1 \cup D_2$ except for $\xi_0$ and $\xi_0^*$, where

$$D_1 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0| < \delta\}, \qquad D_2 = \{\xi \in \mathbb{C} \mid |\xi - \xi_0^*| < \delta\}.$$

By (3.1) and Hurwitz's theorem, for sufficiently large $j$ there exist points $\xi_j \in D_1$, $\xi_j^* \in D_2$ such that

$$f_j(z_j + \rho_j\xi_j) f_j^{(k)}(z_j + \rho_j\xi_j) - a = 0,$$
$$f_j(z_j + \rho_j\xi_j^*) f_j^{(k)}(z_j + \rho_j\xi_j^*) - a = 0.$$

Since, by the assumption in Theorem 1.1, $f_1 f_1^{(k)}$ and $f_j f_j^{(k)}$ share $a$ IM for each $j$, it follows that

$$f_1(z_j + \rho_j\xi_j) f_1^{(k)}(z_j + \rho_j\xi_j) - a = 0,$$
$$f_1(z_j + \rho_j\xi_j^*) f_1^{(k)}(z_j + \rho_j\xi_j^*) - a = 0.$$

By letting $j \to \infty$, and noting $z_j + \rho_j\xi_j \to 0$, $z_j + \rho_j\xi_j^* \to 0$, we obtain

$$f_1(0) f_1^{(k)}(0) - a = 0.$$

Since the zeros of $f_1 f_1^{(k)} - a$ has no accumulation points, in fact we have

$$z_j + \rho_j\xi_j = 0, \; z_j + \rho_j\xi_j^* = 0,$$

or equivalently

$$\xi_j = -\frac{z_j}{\rho_j}, \qquad \xi_j^* = -\frac{z_j}{\rho_j}.$$

This contradicts with the facts that $\xi_j \in D_1$, $\xi_j^* \in D_2$, $D_1 \cap D_2 = \emptyset$. Theorem 1.1 is proved completely.

## 4. Proof of Theorem 1.6

By using the notations in the proof of Theorem 1.1, and now noting that, by Hurwitz's theorem, the zeros of $g(\xi)$ have at least multiplicity $k$ and $g^{(k)}(\xi)$ has no simple zeros, so the function $gg^{(k)} - a$ has at least one zero $\xi_0$ based on Lemmas 2.4 and 2.6. Thus we have

$$\left|g_j^{(k)}(\xi_j)\right| = \rho_j^{\frac{k}{2}} \left|f_j^{(k)}(z_j + \rho_j\xi_j)\right| \leqslant A\rho_j^{\frac{k}{2}}.$$

Since Hurwitz's theorem implies $\xi_j \to \xi_0$ as $j \to \infty$, we obtain consequently

$$g^{(k)}(\xi_0) = \lim_{j\to\infty} g_j^{(k)}(\xi_j) = 0.$$

This contradicts $g(\xi_0) g^{(k)}(\xi_0) = a \neq 0$. Theorem 1.6 is proved.

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## References

[1] A. Alotaibi, On the zeros of $af(f^{(k)}) - 1$ for $n \geqslant 2$, Comput. Methods Funct. Theory 4 (1) (2004) 227–235.
[2] W. Bergweiler, Bloch's principle, Comput. Methods Funct. Theory 6 (2006) 77–108.
[3] W. Bergweiler, A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoam. 11 (1995) 355–373.
[4] H.H. Chen, M.L. Fang, On the value distribution of $f^n f'$, Sci. China Ser. A 38 (1995) 789–798.
[5] H.H. Chen, Y.X. Gu, An improvement of Marty's criterion and its applications, Sci. China Ser. A 36 (1993) 674–681.
[6] J. Clunie, On integral and meromorphic functions, J. Lond. Math. Soc. 37 (1962) 17–27.
[7] J. Clunie, On a result of Hayman, J. Lond. Math. Soc. 42 (1967) 389–392.

[8] D. Drasin, Normal families and Nevanlinna theory, Acta Math. 122 (1969) 231–263.
[9] Y.X. Gu, On normal families of meromorphic functions, Sci. China Ser. A 4 (1978) 373–384.
[10] W. Hayman, Picard value of meromorphic functions and their derivatives, Ann. of Math. 70 (1959) 9–42.
[11] W. Hayman, Research Problems in Function Theory, Athlone Press, University of London, 1967.
[12] W. Hennekemper, Über die Wertverteilung von $(f^{k+1})^{(k)}$, Math. Z. 177 (1981) 375–380.
[13] P.C. Hu, D.W. Meng, Normal criteria of meromorphic functions with multiple zeros, J. Math. Anal. Appl. 357 (2009) 323–329.
[14] Yung-hsing Ku, Sur les familles normales de fonctions méromorphes, Sci. Sin. 21 (1978) 431–445.
[15] Song-ying Li, Hui-chun Xie, On normal families of meromorphic functions, Acta Math. Sinica 29 (1986) 468–476.
[16] Y.T. Li, Y.X. Gu, On normal families of meromorphic functions, J. Math. Anal. Appl. 354 (2009) 421–425.
[17] E. Mues, Über ein problem von Hayman, Math. Z. 164 (1979) 239–259.
[18] I.B. Oshkin, On a test for the normality of families of holomorphic functions, Uspekhi Mat. Nauk 37 (2) (1982) 221–222; Russian Math. Surveys 37 (2) (1982) 237–238.
[19] X.C. Pang, Normality conditions for differential polynomials, Kexue Tongbao 33 (1988) 1690–1693 (in Chinese).
[20] X.C. Pang, Bloch principle and normality criterion, Sci. Sin. Ser. A 11 (1988) 1153–1159; Sci. China Ser. A 32 (1989) 782–791.
[21] X.C. Pang, On normal criterion of meromorphic functions, Sci. China Ser. A 33 (1990) 521–527.
[22] X.C. Pang, L. Zalcman, On theorems of Hayman and Clunie, New Zealand J. Math. 28 (1999) 71–75.
[23] J. Schiff, Normal Families, Springer-Verlag, Berlin, 1993.
[24] W. Schwick, Normal criteria for families of meromorphic function, J. Anal. Math. 52 (1989) 241–289.
[25] J.P. Wang, A fundamental inequality of theory of meromorphic function and its applications, Acta Math. Sinica (Chin. Ser.) 49 (2) (2006) 443–450.
[26] Y.F. Wang, M.L. Fang, Picard values and normal families of meromorphic functions with multiple zeros, Acta Math. Sinica (Chin. Ser.) 41 (1998) 743–748.
[27] Guo-fen Xue, Xue-cheng Pang, A criterion for normality of a family of meromorphic functions, J. East China Norm. Univ. Natur. Sci. Ed. 2 (1988) 15–22 (in Chinese).
[28] C.C. Yang, P.C. Hu, On the value distribution of $ff^{(k)}$, Kodai Math. J. 19 (1996) 157–167.
[29] C.C. Yang, L. Yang, Y.F. Wang, On the zeros of $f(f^{(k)})^n - a$, Chinese Sci. Bull. 38 (1993) 2125–2128.
[30] C.C. Yang, H.X. Yi, Uniquess Theory of Meromorphic Functions, Science Press, Kluwer Academic, Beijing, New York, 2003.
[31] L. Yang, C.C. Yang, Angular distribution of values of $ff'$, Sci. China Ser. A 37 (1994) 284–294.
[32] L. Yang, G. Zhuang, Recherches sur la normalité des familes de fonctions analytiques à des valeurs multiples, I. Un nouveau critère et quelques applications, Sci. Sin. Ser. A 14 (1965);
L. Yang, G. Zhuang, Recherches sur la normalité des familes de fonctions analytiques à des valeurs multiples, II. Géneralisations, Sci. Sin. Ser. A 15 (1966) 433–453.
[33] L. Zalcman, A heuristic principle in complex function theory, Amer. Math. Monthly 82 (1975) 813–817.
[34] L. Zalcman, Normal families: New perspectives, Bull. Amer. Math. Soc. 35 (1998) 215–230.
[35] Q.C. Zhang, Normal families of meromorphic functions concerning shared values, J. Math. Anal. Appl. 338 (2008) 545–551.
[36] Q.C. Zhang, Some normality criteria of meromorphic functions, Complex Var. Elliptic Equ. 53 (8) (2008) 791–795.
[37] Z.F. Zhang, G.D. Song, On the zeros of $f(f^{(k)})^n - a(z)$, Chinese Ann. Math. Ser. A 19 (2) (1998) 275–282.