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Well-posedness and blow-up solution for a modified two-component periodic Camassa–Holm system with peakons

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Abstract Considered herein is a modified two-component periodic Camassa–Holm system with peakons. The local well-posedness and low regularity result of solutions are established. The precise blow-up scenarios of strong solutions and several results of blow-up solutions with certain initial profiles are described in detail and the exact blow-up rate is also obtained.

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1 Introduction

It is well-known that the Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

has attracted much attention in the last decade because of its integrability and the existence of multi-peakon solitons. Eq. (1.1) models the propagation of unidirectional surface waves in irrotational flow over a flat bottom, where the fluid velocity $u(t, x)$ is

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taken at a certain depth at time t in the horizontal direction [6,21,33,34]. It is a water wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves described by the incompressible Euler equations. Indeed, Eq. (1.1) was found earlier by Fuchssteiner and Fokas [28] as a bi-Hamiltonian generalization of the KdV equation. It can be solved by the inverse scattering method for a large class of initial data [1,22] and posses an infinite number of conservation laws, so it is completely integrable. A remarkable fact for the Camassa–Holm equation is that it has an infinite number of non-smooth solitary wave solutions called peakons of the form

$$\varphi(t,x) = ce^{-|x-ct|}, \quad x \in \mathbb{R}, \quad t \geq 0, \quad c > 0 \tag{1.2}$$

and in the periodic case

$$\varphi(t,x) = c\frac{\cosh(x-ct-[x-ct]-1/2)}{\sinh(1/2)}, \quad x \in \mathbb{R}, \quad t \geq 0, \quad c > 0 \tag{1.3}$$

which interact like soliton for integrable systems and they are stable [2,23,40]. The peakons replicate a feature that is characteristic for the waves of great height-waves of the largest amplitude that are exact solutions of the governing equations for water waves [12,17,22,49]. From the geometric points of view, it is a reexpression of the geometric flow on the group of diffeomorphism of the line with the Riemannian structure induced by the H^1-right invariant metric [38]. This geometric interpretation leads to a proof that the Least Action Principle holds [19,37]. It is worth mentioning that recently it was pointed out by Lakshmanan [39] that the Camassa–Holm equation could be relevant to the modeling of tsunami waves (see also the discussion in Constantin and Johnson [20], and Segur [47]).

Another remarkable property of the Camassa–Holm equation is the presence of breaking waves (i.e. the solution remains bounded while its slope becomes unbounded in finite time [6,9–11,14,15,44,52]). It is noted that the KdV equation does not have wave breaking phenomena [36,48]. Wave breaking is one of the most intriguing long-standing problems of water wave theory [52]. As mentioned by Whitham [52], it is intriguing to know which mathematical models for shallow water waves exhibit both phenomena of soliton interaction and wave breaking. It is found that the Camassa–Holm equation could be first such a equation and has the potential to become the new master equation for shallow water wave theory, modeling the soliton interaction of peaked traveling waves, wave breaking, admitting solutions as permanent waves, and being integrable Hammiltonian systems.

The well-posedness and blow up phenomena for the Camassa–Holm equation have been studied extensively. Indeed, the local well-posedness of the periodic Camassa–Holm equation with initial data were proved in [9,13,15]. It has been known that there exist global strong solutions for certain class of initial data [9,13,15]. Existence and uniqueness results for classical solution of the periodic Camassa–Holm equation was established in [45]. The blow up phenomena of the periodic Camassa–Holm equation were investigated in a number of papers (see [9,13,15,16] and references therein).

After wave breaking the solutions can be continued as either global conservative or global dissipative solutions [3,4,29].

Note that Eq. (1.1) admits an infinite number of conservation laws. The first three conserved quantities for (1.1) are given by

$$F_1 = \int_Y m dx, \quad F_2 = \int_Y (u^2 + u_x^2) \mathrm{d}x, \quad F_3 = \int_Y (u^3 + u u_x^2) \mathrm{d}x, \tag{1.4}$$

where $m = u - u_{xx}$ and Y denotes either $\mathbb{R}$ or the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. These conservation laws play important role in the study of the well-posedness and blow up for both periodic and non-periodic cases.

Another important integrable equation admitting peakon solitons is the Degasperis–Procesi equation [24], it takes the form

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.5}$$

It is regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm shallow water equation, and it can also be obtained from the governing equations for water waves [21]. Wave breaking phenomena and global existence of solutions of the Degasperis-Procesi equation were investigated in [8,26,43], for example.

The Camassa–Holm equation also admits many integrable multi-component generalizations. The most popular one is

$$\begin{aligned} &m_t + um_x + 2mu_x + \sigma\rho\rho_x = 0, \\ &\rho_t + (\rho u)_x = 0, \end{aligned} \tag{1.6}$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$. It is reduced to Eq. (1.1) when $\rho = 0$. System (1.4) was derived in [18,46] and its mathematical properties, like the global existence and blow-up of strong solutions, have been studied further in many works, e.g. [7,18,25] and references therein. System (1.4) is integrable. However, it does not have the peakon solitons in the form of a superposition of multi-peakons.

Recently, the authors [27] introduced a new modified two-component Camassa–Holm system which has peakon solitons in the form of a superposition of multi-peakons. By parameterizing $\tilde{t} = -t$ for that modified two-componet Camassa–Holm system in [27], it then takes the following form.

$$\begin{aligned} &m_t + 2mu_x + m_x u + (mv)_x + nv_x = 0, \\ &n_t + 2nv_x + n_x v + (nu)_x + mu_x = 0, \end{aligned} \tag{1.7}$$

where $m = u - u_{xx}, n = v - v_{xx}$. System (1.7) can be rewritten as a Hamiltonian system,

$$\frac{\partial}{\partial t}\begin{pmatrix} m \\ n \end{pmatrix} = -\begin{pmatrix} \partial m + m\partial & \partial m + n\partial \\ \partial n + m\partial & \partial n + n\partial \end{pmatrix}\begin{pmatrix} \delta H/\delta m = u \\ \delta H/\delta n = v \end{pmatrix} \tag{1.8}$$

with the Hamiltonian

$$H = \frac{1}{2}\int (mG * m + nG * n)\mathrm{d}x$$

where $G * m = u, G * n = v$ and $G(x) = 1/2\mathrm{e}^{-|x|}$. Under the linear change of variables

$$\xi = m + n, \quad \eta = m - n,$$

system (1.8) is also equivalent to the following standard semidirect-product system

$$\frac{\partial}{\partial t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = -\begin{pmatrix}\partial\xi + \xi\partial & \eta\partial\\ \partial\eta & 0\end{pmatrix}\begin{pmatrix}\delta H/\delta\xi = \sigma\\ \delta H/\delta\eta = \varrho\end{pmatrix} \tag{1.9}$$

in which the Hamiltonian is

$$H = \frac{1}{2}\int (\xi G * \xi + \eta G * \eta)\mathrm{d}x$$

where $G * \xi = \sigma, G * \eta = \varrho$.

Note that System (1.9) was introduced by Holm et al. in [30], which is called the modified CH-2 system (MCH2), and is written in terms of velocity σ and locally averaged density ϱ (or depth, in the shallow water interpretation). From the geometric points of view, it is defined as geodesic motion on the corresponding semidirect product Lie group with respect to a certain metric and is given as a set of the Euler–Poincaré equations on the dual of the corresponding Lie algebra. In the general case, for a Lagrangian $L(\sigma, \varrho)$, the corresponding semidirect-product Euler–Poincaré equations [31] are written as

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\delta L}{\delta\sigma} &= -\pounds_\sigma\frac{\delta L}{\delta\sigma} - \frac{\delta L}{\delta\varrho}\nabla\varrho,\\ \frac{\partial}{\partial t}\frac{\delta L}{\delta\varrho} &= -\pounds_\sigma\frac{\delta L}{\delta\varrho},\end{aligned} \tag{1.10}$$

where $\pounds_\sigma(\delta L/\delta\sigma)$ is the Lie derivative of the one-form density $\xi = \delta L/\delta\sigma$ with respect to the vector field σ and $\pounds_\sigma(\delta L/\delta\varrho)$ is the corresponding Lie derivative of the scalar density $\delta L/\delta\varrho$.

By setting

$$\xi = \left(1 - \alpha_1^2\partial^2\right)\sigma, \quad \eta = \left(1 - \alpha_2^2\partial^2\right)\varrho,$$

in the following system in analogy to system (1.6)

$$\begin{aligned}&\xi_t + \sigma\xi_x + 2\xi\sigma_x = -g\eta\eta_x,\\ &\eta_t + (\eta\sigma)_x = 0,\end{aligned} \tag{1.11}$$

where $g > 0$ is the downward constant acceleration of gravity in applications to shallow water waves, the metric Lagrangian $L(\sigma, \varrho)$ in Hamilton's principle for the resulting system becomes into

$$
\begin{aligned}
L(\sigma,\varrho) &= \frac{1}{2}\|\sigma\|^2_{H^1(\mathbb{R})} + \frac{g}{2}\|\varrho-\varrho_0\|^2_{H^1(\mathbb{R})} \\
&= \frac{1}{2}\int \left(\sigma^2+\alpha_1^2\sigma_x^2\right)\mathrm{d}x + \frac{g}{2}\int\left[(\varrho-\varrho_0)^2+\alpha_2^2(\varrho-\varrho_0)_x^2\right]\mathrm{d}x,
\end{aligned}
\tag{1.12}
$$

where ϱ_0 is taken to be a constant. The Lax pair of the system (1.11) is established in [30] and the system is completely integrable.

MCH2 may be derived as semidirect-product Euler–Poincaré equations (1.10) from the following type of variational principle defined on the corresponding Lie algebra,

$$
\delta\int_{t_0}^{t_1} L(\sigma,\varrho)\mathrm{d}t = 0.
$$

The variational derivatives of this Lagrangian define the variables ξ and η as

$$
\begin{aligned}
\delta L/\delta\varrho &= g(1-\alpha_2^2\partial^2)(\varrho-\varrho_0) = g\eta, \\
\delta L/\delta\sigma &= (1-\alpha_1^2\partial^2)\sigma = \xi.
\end{aligned}
$$

Substituting these variational derivatives into the Euler–Poincaré equations (1.10) in one spatial dimension recovers the MCH2 equations (1.9) for the constants $g>0$, $\alpha_2^2>0$.

Remark 1.1 One can easily check that in the periodic case Eq. (1.7) has the following peakons.

$$
\begin{aligned}
u(t,x) &= \frac{1}{\sinh(\frac{1}{2})}\Bigg(p_1(t)\cosh\left(x-q_1(t)-[x-q_1(t)]-\frac{1}{2}\right) \\
&\quad + p_2(t)\cosh\left(x-q_2(t)-[x-q_2(t)]-\frac{1}{2}\right)\Bigg), \\
v(t,x) &= \frac{1}{\sinh(\frac{1}{2})}\Bigg(p_1(t)\cosh\left(x-q_1(t)-[x-q_1(t)]-\frac{1}{2}\right) \\
&\quad + p_2(t)\cosh\left(x-q_2(t)-[x-q_2(t)]-\frac{1}{2}\right)\Bigg),
\end{aligned}
$$

where $p_1, p_2, q_1, q_2 \in W^{1,\infty}(\mathbb{R}^+)$, $q_1(t) < q_2(t)$ for any $t>0$ and $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ and the t-dependent functions satisfy the corresponding dynamical system in analogy to that in [27]. In particular, when $v=0$ (or $u=0$), the degenerated Eq. (1.7) has the same peakon solitons (1.3) as the Camassa–Holm equation. Moreover, when $u=v$, Eq. (1.7) is reduced to the scalar Camassa–Holm equation

$$
m_t + 4mu_x + 2um_x = 0. \tag{1.13}
$$

The above equation has the following peakon solitons

$$u(t,x)=c\frac{\cosh(x-2ct-[x-2ct]-1/2)}{\sinh(1/2)},\quad x\in\mathbb{R},\quad t\ge 0,\quad c>0.$$

Remark 1.2 System (1.7) can be extended to ones with n-components, which is given in the form

$$\begin{aligned}
&m_{1,t}+2m_1u_{1,x}+u_1m_{1,x}+\left(m_1\sum_{j\neq 1}^{n}u_j\right)_x+\sum_{j\neq 1}^{n}m_ju_{j,x}=0,\\
&m_{2,t}+2m_2u_{2,x}+u_2m_{2,x}+\left(m_2\sum_{j\neq 2}^{n}u_j\right)_x+\sum_{j\neq 2}^{n}m_ju_{j,x}=0,\\
&\quad\vdots\\
&m_{n,t}+2m_nu_{n,x}+u_nm_{n,x}+\left(m_n\sum_{j\neq n}^{n}u_j\right)_x+\sum_{j\neq n}^{n}m_ju_{j,x}=0,
\end{aligned}\tag{1.14}$$

where $m_i=u_i-u_{i,xx}$, $i=1,2,\ldots n$. A direct computation shows that system (1.14) admits multi-solitons and H^1 conservation laws.

In this paper, we shall study well-posedness and blow up of solutions to system (1.7) with the initial and periodic conditions

$$\begin{array}{lll}
u(0,x)=u_0(x), & & x\in\mathbb{R},\\
v(0,x)=v_0(x), & & x\in\mathbb{R},\\
u(t,x+1)=u(x), & t\ge 0, & x\in\mathbb{R},\\
v(t,x+1)=v(x), & t\ge 0, & x\in\mathbb{R}.
\end{array}\tag{1.15}$$

We firstly establish the local well-posedness of the periodic initial-value problem associated with Eq. (1.7) by applying Kato's semigroup approach to nonlinear hyperbolic evolution equations [35]. Then we prove the existence of lower regularity of solutions by regularizing this system and obtaining a solution of the equations as the limit of solutions to the regularized system. At last by means of energy estimates with a special technique of blow-up, we establish the precise blow-up scenarios and conditions for strong solutions as well as the exact blow-up rate for Eq. (1.7).

Throughout this paper, we identify all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$, i.e. $\mathbb{S}=\mathbb{R}/\mathbb{Z}$. It is easy to verify that this system has the following conserved quantities

$$
\begin{aligned}
E_1 &= \int_{\mathbb{S}} u \mathrm{d}x, \quad E_2 = \int_{\mathbb{S}} v \mathrm{d}x, \\
E_3 &= \int_{\mathbb{S}} m \,\mathrm{d}x, \quad E_4 = \int_{\mathbb{S}} n \mathrm{d}x, \\
E_5 &= \int_{\mathbb{S}} \left(u^2 + u_x^2 + v^2 + v_x^2\right) \mathrm{d}x.
\end{aligned} \tag{1.16}
$$

Let $f \in L^1(\mathbb{S})$. The Fourier transform of f is the complex sequence $Ff = \hat{f} = (\hat{f}(k))_{k\in\mathbb{Z}}$ defined by

$$
(Ff)(k) = \hat{f}(k) = c_k = \int_0^1 f(x) \exp(-2\pi ikx) \mathrm{d}x.
$$

The numbers $\hat{f}(k) = c_k$ are the Fourier coefficients of f and the series

$$
\sum_{k=-\infty}^{\infty} c_k \exp(2\pi ikx)
$$

is the Fourier series generated by f. Furthermore, the map $F : f \to \hat{f}$ is a continuous linear transformation from $L^1(\mathbb{S})$ into $l^\infty(\mathbb{Z})$.

Let $\Lambda = (1 - \partial_x^2)^{1/2}$. Then the operator Λ^{-2} acting on $L^2(\mathbb{S})$ can be expressed by its associated Green's function $G(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$, where $[x]$ stands for the integer part of x and $*$ the spacial convolution, as

$$
\Lambda^{-2} f(x) = G * f(x) = \frac{1}{2} \int_0^1 \frac{\cosh(x - y - [x - y] - \frac{1}{2})}{\sinh(\frac{1}{2})} f(y) \mathrm{d}y, \quad f \in L^2(\mathbb{S}).
$$

So the initial value problem (IVP) (1.7) and (1.15) is equivalent to the following IVP

$$
\begin{aligned}
& u_t + (u + v)\, u_x + \Lambda^{-2}(u v_x) \\
& \quad + \partial_x \Lambda^{-2} \left(u^2 + \frac{1}{2} u_x^2 + u_x v_x + \frac{1}{2} v^2 - \frac{1}{2} v_x^2\right) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\
& v_t + (u + v)\, v_x + \Lambda^{-2}(u_x v) \\
& \quad + \partial_x \Lambda^{-2} \left(v^2 + \frac{1}{2} v_x^2 + u_x v_x + \frac{1}{2} u^2 - \frac{1}{2} u_x^2\right) = 0, \quad t > 0, \quad x \in \mathbb{R}, \\
& u(0, x) = u_0(x), \qquad x \in \mathbb{R}, \\
& v(0, x) = v_0(x), \qquad x \in \mathbb{R}, \\
& u(t, x + 1) = u(t, x), \qquad t \ge 0, \quad x \in \mathbb{R}, \\
& v(t, x + 1) = v(t, x), \qquad t \ge 0, \quad x \in \mathbb{R}.
\end{aligned} \tag{1.17}
$$

The outline of this paper is as follows. In Sect. 2, based on Kato's theory, the local well-posedness for strong solutions of (1.17) is established and the proof of low regularity of local solution is carried out. Section 3 is devoted to investigating the phenomenon of blow-up and describes in detail the wave-breaking conditions as well as blow-up rate of solutions for (1.17).

Notation In the following, for given Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity. For any real s, we let $H^s = H^s(\mathbb{S})$ denote the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_{H^s} = \left(\sum_{k=-\infty}^{\infty} (1+|k|^2)^s |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty.$$

Given an unbounded operator A, we write $D(A)$ for the domain of the operator A. We let $[A, B]$ denote the commutator of linear operator A and B. For convenience, we let $(\cdot|\cdot)_{s\times r}$ and $(\cdot|\cdot)_s$ denote the inner products of $H^s \times H^r$, $s, r \in \mathbb{R}$ and H^s, $s \in \mathbb{R}$, respectively.

2 Well-posedness and low regularities of solutions

Let us firstly use Kato's theorem [35] to establish the local well-posedness for the Cauchy problem of (1.17).

Theorem 2.1 *Given* $z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in H^s \times H^s$, $s > 3/2$. *Then there exists a maximal* $T = T(z_0) > 0$, *and a unique solution* $z = \begin{pmatrix} u \\ v \end{pmatrix}$ *to the IVP* (1.17) *such that*

$$z = z(\cdot, z_0) \in C\left([0,T); H^s \times H^s\right) \cap C^1\left([0,T); H^{s-1} \times H^{s-1}\right).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$z_0 \to z(\cdot, z_0) : H^s \times H^s \to C\left([0,T); H^s \times H^s\right) \cap C^1\left([0,T); H^{s-1} \times H^{s-1}\right)$$

is continuous and the maximal time of existence $T > 0$ *can be chosen to be independent of* s.

Proof Consider the abstract quasi-linear evolution equation of the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} + A(z)z = f(z), \quad t > 0, \quad z(0) = z_0. \tag{2.1}$$

Let $z := \begin{pmatrix} u \\ v \end{pmatrix}$, $A(z) := \begin{pmatrix} (u+v)\partial_x & 0 \\ 0 & (u+v)\partial_x \end{pmatrix}$ and

$$f(z) := -\begin{pmatrix} \Lambda^{-2}(uv_x) + \partial_x\Lambda^{-2}(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2) \\ \Lambda^{-2}(u_x v) + \partial_x\Lambda^{-2}(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2) \end{pmatrix}.$$

We use the notations throughout the paper, $X = H^{s-1} \times H^{s-1}$, $Y = H^s \times H^s$ and $Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ with $\Lambda = (I - \partial_x^2)^{\frac{1}{2}}$. Obviously, Q is an isomorphism of $H^s \times H^s$ onto $H^{s-1} \times H^{s-1}$. In order to prove Theorem 2.1, we need to verify that $A(z)$ and $f(z)$ satisfy the corresponding conditions in view of Kato's Theorem.

For this purpose, we can prove the following lemmas analogous to Lemma 4.1–4.5 in [27]. So we omit the detailed proofs of the following lemmas.

Lemma 2.1 *The operator* $A(z) := \begin{pmatrix} (u+v)\partial_x & 0 \\ 0 & (u+v)\partial_x \end{pmatrix}$ *with* $z \in H^s \times H^s$, $s > 3/2$, *belongs to* $G(L^2 \times L^2, 1, \beta)$.

Lemma 2.2 *The operator* $A(z) := \begin{pmatrix} (u+v)\partial_x & 0 \\ 0 & (u+v)\partial_x \end{pmatrix}$ *with* $z \in H^s \times H^s$, $s > 3/2$, *belongs to* $G(H^{s-1} \times H^{s-1}, 1, \beta)$.

Lemma 2.3 *Let* $A(z) := \begin{pmatrix} (u+v)\partial_x & 0 \\ 0 & (u+v)\partial_x \end{pmatrix}$ *with* $z \in H^s \times H^s$, $s > 3/2$, *be given. Then* $A(z) \in L(H^s \times H^s, H^{s-1} \times H^{s-1})$ *and*

$$\|(A(y) - A(z))w\|_{H^{s-1}\times H^{s-1}} \le \mu_1 \|y - z\|_{H^{s-1}\times H^{s-1}} \|w\|_{H^s \times H^s}$$

holds for all $y, z, w \in H^s \times H^s$.

Lemma 2.4 *Setting* $B(z) = QA(z)Q^{-1} - A(z)$ *with* $z \in H^s \times H^s$, $s > 3/2$. *Then* $B(z) \in L(H^{s-1} \times H^{s-1})$ *and*

$$\|(B(y) - B(z))w\|_{H^{s-1}\times H^{s-1}} \le \mu_2 \|y - z\|_{H^s\times H^s} \|w\|_{H^{s-1} \times H^{s-1}}$$

holds for all $y, z \in H^s \times H^s$ *and* $w \in H^{s-1} \times H^{s-1}$.

Lemma 2.5 *Let* $z = \begin{pmatrix} u \\ v \end{pmatrix} \in H^s \times H^s$, $s > 3/2$ *and let*

$$f(z) := -\begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} := -\begin{pmatrix} \Lambda^{-2}(uv_x) + \partial_x\Lambda^{-2}(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2) \\ \Lambda^{-2}(u_x v) + \partial_x\Lambda^{-2}(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2) \end{pmatrix}.$$

Then f *is bounded on any bounded sets in* $H^s \times H^s$, *and it satisfies*

(a) $\|f(y) - f(z)\|_{H^s\times H^s} \le \mu_3 \|y - z\|_{H^s\times H^s}$, $y, z \in H^s \times H^s$,
(b) $\|f(y) - f(z)\|_{H^{s-1}\times H^{s-1}} \le \mu_4 \|y - z\|_{H^{s-1}\times H^{s-1}}$, $y, z \in H^s \times H^s$.

At last, applying Kato's theory for abstract quasi-linear evolution equation of hyperbolic type (2.1) we can obtain the local well-posedness of (1.17). □

In the following, we are concerned with the Cauchy problem for a regularized version of the periodic coupled Camassa–Holm equation (1.17). The idea of proving the existence of the local weak solution is inspired by Li and Olver in [41]. Firstly, we present some preliminary lemmas.

Lemma 2.6 [32] *Let $s, t \in \mathbb{R}$, $s \geq r$. Then $H^s \hookrightarrow H^r$, that is, H^s is continuously and densely embedded in H^r and*

$$\|f\|_{H^r} \leq \|f\|_{H^s}, \quad \forall f \in H^s.$$

Lemma 2.7 [32] *If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\|fg\|_{H^r} \leq c\,(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty})\,,$$

where c is a constant depending only on r.

Lemma 2.8 [32] *If $r > 0$, then*

$$\|[\Lambda^r, f]g\|_{L^2} \leq c\left(\|\partial_x f\|_{L^\infty}\|\Lambda^{r-1}g\|_{L^2} + \|\Lambda^r f\|_{L^2}\|g\|_{L^\infty}\right),$$

where c is a constant depending only on r.

Consider the initial value problem in the following.

$$\begin{cases} u_t - u_{xxt} + \varepsilon u_{xxxxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + uv_x + u_x v \\ \qquad\qquad - u_{xx}v_x - vu_{xxx} + vv_x - v_x v_{xx} = 0, \\ v_t - v_{xxt} + \varepsilon v_{xxxxt} + 3vv_x - 2v_x v_{xx} - vv_{xxx} + uv_x + u_x v \\ \qquad\qquad - u_x v_{xx} - uv_{xxx} + uu_x - u_x u_{xx} = 0, \\ u(0,x) = u_0(x) \in H^s, \qquad s \geq 1, \quad x \in \mathbb{R}, \\ v(0,x) = v_0(x) \in H^s, \qquad s \geq 1, \quad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}, \\ v(t,x+1) = v(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Here ε is a constant, and $0 < \varepsilon < \frac{1}{4}$. One can easily check that when $\varepsilon = 0$, Eq.(2.2) is equivalent to the IVP (1.17).

Lemma 2.9 *For any* $0 < \varepsilon < \frac{1}{4}$ *and any* s*, the integral operator*

$$\mathcal{D} = \left(1 - \partial_x^2 + \varepsilon \partial_x^4\right)^{-1} : H^s \to H^{s+4} \tag{2.3}$$

defines a bounded linear operator on the indicated Sobolev spaces.

To prove the existence of a solution to the problem (2.2), we apply the operator (2.3) to both sides of (2.2) and then integrate the resulting equations with regard to t. This leads to the following equations

$$\begin{cases} u(t,x) = u_0(x) - \displaystyle\int_0^t \mathcal{D}\Big[\frac{3}{2}(u^2)_x + \frac{1}{2}(u_x^2)_x - \frac{1}{2}(u^2)_{xxx} + (uv)_x \\ \qquad -(u_x v)_{xx} + u_x v_x + \frac{1}{2}(v^2)_x - \frac{1}{2}(v_x^2)_x\Big]\mathrm{d}\tau, \\ v(t,x) = v_0(x) - \displaystyle\int_0^t \mathcal{D}\Big[\frac{3}{2}(v^2)_x + \frac{1}{2}(v_x^2)_x - \frac{1}{2}(v^2)_{xxx} + (uv)_x \\ \qquad -(u v_x)_{xx} + u_x v_x + \frac{1}{2}(u^2)_x - \frac{1}{2}(u_x^2)_x\Big]\mathrm{d}\tau. \end{cases}$$

A standard application of the contraction mapping theorem leads to the following existence result.

Theorem 2.2 *For each initial data* $u_0, v_0 \in H^s$ *with* $s \geq 1$*, there is a* $T > 0$ *depending only on the norm of* u_0 *and* v_0 *in* H^s *such that there corresponds a unique solution* $(u(t,x), v(t,x)) \in C([0,T]; H^s) \times C([0,T]; H^s)$ *of Eq.* (2.2) *in the sense of distribution. If* $s \geq 2$*, the solution* $(u,v) \in C^\infty([0,\infty); H^s) \times C^\infty([0,\infty); H^s)$ *exists for all time. In particular, when* $s \geq 4$*, the corresponding solution is a classical globally defined solution of* (2.2).

The global existence result follows from the conservation law

$$\int_0^1 \left(u^2 + v^2 + u_x^2 + v_x^2 + \varepsilon u_{xx}^2 + \varepsilon v_{xx}^2\right)\mathrm{d}x$$
$$= \int_0^1 \left(u_0^2 + v_0^2 + u_{0x}^2 + v_{0x}^2 + \varepsilon u_{0xx}^2 + \varepsilon v_{0xx}^2\right)\mathrm{d}x,$$

admitted by (2.2) in its integral form.

Now we study norms of solutions of (2.2) using energy estimates.

Theorem 2.3 *Suppose that for some* $s \geq 4$*, the functions* $u(t,x)$*,* $v(t,x)$ *are the solution of Eq.* (2.2) *corresponding to the initial data* $u_0, v_0 \in H^s$*. Then the following inequalities hold.*

$$\|u\|_{H^1}^2, \|v\|_{H^1}^2 \le \int_0^1 \left(u^2+v^2+u_x^2+v_x^2+\varepsilon u_{xx}^2+\varepsilon v_{xx}^2\right) \mathrm{d}x$$
$$= \int_0^1 \left(u_0^2+v_0^2+u_{0x}^2+v_{0x}^2+\varepsilon u_{0xx}^2+\varepsilon v_{0xx}^2\right) \mathrm{d}x. \tag{2.4}$$

For any real number $q \in (1, s]$, there exists a constant c depending only on q such that

$$\begin{aligned}
&\int_0^1 [(1-\varepsilon)(\Lambda^q u)^2 + (1-\varepsilon)(\Lambda^q v)^2 + \varepsilon(\Lambda^q u_x)^2 \\
&\quad + \varepsilon(\Lambda^q v_x)^2 + \varepsilon(\Lambda^{q-1} u)^2 + \varepsilon(\Lambda^{q-1} v)^2]\mathrm{d}x \\
&\le \int_0^1 \Big[(1-\varepsilon)(\Lambda^q u_0)^2 + (1-\varepsilon)(\Lambda^q v_0)^2 + \varepsilon(\Lambda^q u_{0x})^2 \\
&\quad + \varepsilon(\Lambda^q v_{0x})^2 + \varepsilon(\Lambda^{q-1} u_0)^2 + \varepsilon(\Lambda^{q-1} v_0)^2\Big] \mathrm{d}x \\
&\quad + c\int_0^t (\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \left(\|u\|_{H^q}^2 + \|v\|_{H^q}^2\right) \mathrm{d}\tau.
\end{aligned} \tag{2.5}$$

For any $q \in [0, s-1]$, there is a constant c such that

$$\begin{aligned}
&(1-2\varepsilon)(\|u_t\|_{H^q} + \|v_t\|_{H^q}) \\
&\quad \le c(\|u\|_{H^1} + \|v\|_{H^1} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}}).
\end{aligned} \tag{2.6}$$

Proof Multiplying both sides of the first and second equation of (2.2) by u and v respectively, integrating with respect to x and summing up leads to the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(u^2+v^2+u_x^2+v_x^2+\varepsilon u_{xx}^2+\varepsilon v_{xx}^2\right) \mathrm{d}x = 0,$$

which implies inequality (2.4).

In order to prove (2.5) and (2.6), we use the following equivalent form of (2.2):

$$\begin{cases} (1-\varepsilon)u_t - \varepsilon u_{xxt} + (u+v)u_x + \Lambda^{-2}(uv_x + \varepsilon u_t) \\ \quad + \partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right) = 0, \\ (1-\varepsilon)v_t - \varepsilon v_{xxt} + (u+v)v_x + \Lambda^{-2}(u_x v + \varepsilon v_t) \\ \quad + \partial_x \Lambda^{-2}\left(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right) = 0, \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \\ v(0,x) = v_0(x), \qquad x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}, \\ v(t,x+1) = v(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}. \end{cases} \tag{2.7}$$

For any $q \in (1, s]$, applying $(\Lambda^q u)\Lambda^q$ and $(\Lambda^q v)\Lambda^q$ to both sides of the first and second equation of (2.7) respectively and integrating with regard to x again, one obtains

$$\begin{aligned} &(1-\varepsilon)(\Lambda^q u, \Lambda^q u_t) - \varepsilon(\Lambda^q u, \Lambda^q u_{xxt}) + \varepsilon(\Lambda^{-2}u_t, u)_q \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left[(1-\varepsilon)(\Lambda^q u)^2 + \varepsilon(\Lambda^q u_x)^2 + \varepsilon(\Lambda^{q-1}u)^2\right]\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} |((u+v)u_x, u)_q| &= |(\Lambda^q (u+v)u_x, \Lambda^q u)_0| \\ &= |([\Lambda^s, (u+v)]u_x, \Lambda^q u)_0 + ((u+v)\Lambda^q u_x, \Lambda^q u)_0| \\ &\leq \|[\Lambda^q, u+v]u_x\|_{L^2}\|\Lambda^q u\|_{L^2} + \frac{1}{2}|((u_x+v_x)\Lambda^q u, \Lambda^q u)_0| \\ &\leq c(\|u_x+v_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u+v\|_{H^q})\|u\|_{H^q} \\ &\quad + \frac{1}{2}\|u_x+v_x\|_{L^\infty}\|u\|_{H^q}^2 \\ &\leq c(\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^q}^2 + \|v\|_{H^q}^2). \end{aligned} \tag{2.8}$$

In the above inequality, we have used Lemma 2.8 with $r = q$.

$$\begin{aligned} (\Lambda^{-2}uv_x, u)_q = (uv_x, u)_{q-1} &\leq \|uv_x\|_{H^{q-1}}\|u\|_{H^{q-1}} \\ &\leq c(\|u\|_{L^\infty}\|v_x\|_{H^{q-1}} + \|v_x\|_{L^\infty}\|u\|_{H^{q-1}})\|u\|_{H^{q-1}} \\ &\leq c(\|u\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^q}^2 + \|v\|_{H^q}^2), \end{aligned}$$

where we used Lemma 2.7 with $r = q - 1 > 0$. For the last term,

$$\begin{aligned}
&\left(\partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right), u\right)_q \\
&\le \left\| u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 \right\|_{H^{q-1}} \|u\|_{H^q} \\
&\le c\left(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}\right)\left(\|u\|_{H^q}^2 + \|v\|_{H^q}^2\right),
\end{aligned}$$

where we used Lemma 2.7 with $r = q - 1 > 0$. For the second equation we have the similar estimates. Summing the results up we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \Big[(1-\varepsilon)(\Lambda^q u)^2 + (1-\varepsilon)(\Lambda^q v)^2 + \varepsilon(\Lambda^q u_x)^2 + \varepsilon(\Lambda^q v_x)^2 \\
&\quad + \varepsilon(\Lambda^{q-1} u)^2 + \varepsilon(\Lambda^{q-1} v)^2\Big]\mathrm{d}x \\
&\le c\left(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}\right)\left(\|u\|_{H^q}^2 + \|v\|_{H^q}^2\right).
\end{aligned}$$

Integrating with respect to t on both sides of the above inequality and considering $0 < \varepsilon < \frac{1}{4}$ is a constant leads to inequality (2.5).

To estimate the norm of u_t and v_t, for some $q \in [0, s-1]$ applying $(\Lambda^q u_t)\Lambda^q$ and $(\Lambda^q v_t)\Lambda^q$ to both sides of the first and second equation of (2.7) respectively and integrating with regard to x again, one obtains

$$\begin{aligned}
&((1-\varepsilon)u_t - \varepsilon u_{xxt}, u_t)_q \\
&= (1-\varepsilon)\int_0^1 (\Lambda^q u_t)^2 \mathrm{d}x + \varepsilon \int_0^1 (\Lambda^q u_{xt})^2 \mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
&((u+v)u_x, u_t)_q \\
&\le c\|[\Lambda^q, u+v]u_x\|_{L^2}\|\Lambda^q u_t\|_{L^2} + ((u+v)\Lambda^q u_x, \Lambda^q u_t) \\
&\le c(\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}})\|u_t\|_{H^q} \\
&\quad + (\|u\|_{L^\infty} + \|v\|_{L^\infty})\|u\|_{H^{q+1}}\|u_t\|_{H^q} \\
&\le c(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}})\|u_t\|_{H^q},
\end{aligned}$$

$$\begin{aligned}
&\left(\Lambda^{-2}(uv_x + \varepsilon u_t)\right., u_t)_q \\
&\le \varepsilon\|u_t\|_{H^q}^2 + c\|u_t\|_{H^q}\|\Lambda^{-2}(uv_x)\|_{H^q} \\
&\le \varepsilon\|u_t\|_{H^q}^2 + c\|u_t\|_{H^q}\|uv_x\|_{H^q} \\
&\le \varepsilon\|u_t\|_{H^q}^2 + c\|u_t\|_{H^q}(\|u\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}}),
\end{aligned}$$

where we applied Lemmas 2.6 and 2.7 with $r = q > 0$. For the last term,

$$
\begin{aligned}
&\left| \int_0^1 \Lambda^q u_t \partial_x \Lambda^{q-2} \left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 \right) \mathrm{d}x \right| \\
&\le \|u_t\|_{H^q} \|u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\|_{H^{q-1}} \\
&\le \|u_t\|_{H^q} \|u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\|_{H^q} \\
&\le c(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}})\|u_t\|_{H^q},
\end{aligned}
$$

where we have used Lemmas 2.6 and 2.7 with $r = q > 0$. Since $\|u\|_{L^\infty} \le c\|u\|_{H^1}$, $\|v\|_{L^\infty} \le c\|v\|_{H^1}$, with $c = \sqrt{1/2\coth(1/2)}$, summing the above inequalities up yields the inequality

$$
\begin{aligned}
(1-\varepsilon)\|u_t\|_{H^q}^2 &\le (1-\varepsilon)\|u_t\|_{H^q}^2 + \varepsilon\|u_{xt}\|_{H^q}^2 \\
&\le \varepsilon\|u_t\|_{H^q}^2 + c(\|u\|_{H^1} + \|v\|_{H^1} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \\
&\quad \times(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}})\|u_t\|_{H^q},
\end{aligned}
$$

and so

$$
(1-2\varepsilon)\|u_t\|_{H^q} \le c(\|u\|_{H^1} + \|v\|_{H^1} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^{q+1}} + \|v\|_{H^{q+1}}).
$$

For the second equation, we can obtain the above similar inequality. Summing the two inequalities up, it yields inequality (2.6). This completes the proof of the theorem. □

For any fixed real number s with $s \ge 1$, suppose that the function u_0, v_0 are in H^s, and let $u_{\varepsilon 0}$, $v_{\varepsilon 0}$ have the following Fourier coefficients

$$
\hat{u}_{\varepsilon 0}(k) = \varphi(\varepsilon^{-\frac{1}{4}}k)\hat{u}_0(k), \quad \hat{v}_{\varepsilon 0}(k) = \varphi(\varepsilon^{-\frac{1}{4}}k)\hat{v}_0(k),
$$

where $\hat{u}_0(k)$, $\hat{v}_0(k)$ denote the Fourier coefficients of u_0, v_0, respectively, and φ is an even C^∞ function, with $0 \le \varphi \le 1$ everywhere and $\varphi(0) = 1$, such that $\psi(k) = 1 - \varphi(k)$ has a zero of infinite order at 0 and such that φ tends exponentially to 0 at $\pm\infty$. Then it follows from Theorem 2.2 that for each ε with $0 < \varepsilon < \frac{1}{4}$ the Cauchy problem

$$\begin{cases} u_t - u_{xxt} + \varepsilon u_{xxxxxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + uv_x + u_x v \\ \qquad\qquad - u_{xx}v_x - vu_{xxx} + vv_x - v_x v_{xx} = 0, \\ v_t - v_{xxt} + \varepsilon v_{xxxxxt} + 3vv_x - 2v_x v_{xx} - vv_{xxx} + uv_x + u_x v \\ \qquad\qquad - u_x v_{xx} - uv_{xxx} + uu_x - u_x u_{xx} = 0, \\ u(0,x) = u_{\varepsilon 0}(x), \qquad x \in \mathbb{R}, \\ v(0,x) = v_{\varepsilon 0}(x), \qquad x \in \mathbb{R}, \\ u(t, x+1) = u(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}, \\ v(t, x+1) = v(t,x), \qquad t \geq 0, \quad x \in \mathbb{R}. \end{cases} \tag{2.9}$$

has a unique solution $u_\varepsilon(t,x), v_\varepsilon(t,x) \in C^\infty([0,\infty); H^\infty)$. We first demonstrate the properties of the initial data $u_{\varepsilon 0},\ v_{\varepsilon 0}$ in the following lemma. The proof is similar to that of Lemma 5 in [5].

Lemma 2.10 *Under the above assumptions, the following estimates hold for any ε with $0 < \varepsilon < \frac{1}{4}$:*

$$\begin{aligned} &\|u_{\varepsilon 0}\|_{H^q}, \|v_{\varepsilon 0}\|_{H^q} \leq c, && \text{if } q \leq s, \\ &\|u_{\varepsilon 0}\|_{H^q}, \|v_{\varepsilon 0}\|_{H^q} \leq c\varepsilon^{(s-q)/4}, && \text{if } q > s, \end{aligned} \tag{2.10}$$

here c is a constant independent of ε.

Theorem 2.4 *Suppose that $u_0(x), v_0(x)$ are functions in the Sobolev space H^s for some $s \in [1, 3/2]$ such that $\|u_{0x}\|_{L^\infty} < \infty$, $\|v_{0x}\|_{L^\infty} < \infty$. Let $u_{\varepsilon 0},\ v_{\varepsilon 0}$ be defined the same as above. Then there are constants $T > 0$ and $c > 0$ independent of ε such that the corresponding solution $u_\varepsilon, v_\varepsilon$ of (2.9) satisfy the inequalities $\|u_{\varepsilon x}\|_{L^\infty} \leq c$, $\|v_{\varepsilon x}\|_{L^\infty} \leq c$ for any $t \in [0, T)$.*

Proof We start from Eq. (2.7) with $u = u_\varepsilon$, $v = v_\varepsilon$. Differentiating with respect to x on both sides of the first equation of (2.7) and noticing that $\partial_x^2 \Lambda^{-2} = \Lambda^{-2} - I$, we obtain

$$\begin{aligned} &(1-\varepsilon)u_{xt} - \varepsilon u_{xxxt} \\ &\quad = -(u_x + v_x)u_x - (u+v)u_{xx} + u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 \\ &\qquad - \partial_x \Lambda^{-2}(uv_x + \varepsilon u_t) - \Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right) \\ &\quad = -(u+v)u_{xx} + u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 \\ &\qquad - \partial_x \Lambda^{-2}(uv_x + \varepsilon u_t) - \Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right). \end{aligned}$$

Let $n > 0$ be an integer. Then multiplying the above equation by $(u_x)^{2n+1}$ to integrate with respect to x yields the equality

$$\begin{aligned}
&\frac{1-\varepsilon}{2n+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_x)^{2n+2}\mathrm{d}x - \varepsilon\int_0^1 (u_x)^{2n+1}u_{xxxt}\mathrm{d}x - \frac{1}{2n+2}\int_0^1 (u_x+v_x)(u_x)^{2n+2}\mathrm{d}x\\
&= -\frac{1}{2}\int_0^1 (u_x)^{2n+3}\mathrm{d}x - \int_0^1 (u_x)^{2n+1}\left(-u^2-\frac{1}{2}v^2+\frac{1}{2}v_x^2\right)\mathrm{d}x\\
&\quad -\int_0^1 (u_x)^{2n+1}\left[\partial_x\Lambda^{-2}(uv_x-\varepsilon u_t)+\Lambda^{-2}\left(u^2+\frac{1}{2}u_x^2+u_xv_x+\frac{1}{2}v^2-\frac{1}{2}v_x^2\right)\right]\mathrm{d}x,
\end{aligned}$$

where we have used

$$\int_0^1 (u+v)u_{xx}(u_x)^{2n+1}\mathrm{d}x = -\frac{1}{2n+2}\int_0^1 (u_x+v_x)(u_x)^{2n+2}\mathrm{d}x.$$

It follows from Hölder's inequality that

$$\begin{aligned}
&\frac{1-\varepsilon}{2n+2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_x)^{2n+2}\mathrm{d}x\\
&\le \left(\int_0^1 (u_x)^{2n+2}\mathrm{d}x\right)^{\frac{2n+1}{2n+2}}\left[\varepsilon\left(\int_0^1 |u_{xxxt}|^{2n+2}\mathrm{d}x\right)^{\frac{1}{2n+2}} + \left(\int_0^1 |u|^{4n+4}\mathrm{d}x\right)^{\frac{1}{2n+2}}\right.\\
&\quad \left.+\frac{1}{2}\left(\int_0^1 (|v_x|^{4n+4}+|v|^{4n+4})\mathrm{d}x\right)^{\frac{1}{2n+2}} + \left(\int_0^1 |g|^{2n+2}\mathrm{d}x\right)^{\frac{1}{2n+2}}\right]\\
&\quad +\left(\frac{n}{2n+2}\|u_x\|_{L^\infty}+\frac{1}{2n+2}\|v_x\|_{L^\infty}\right)\int_0^1 |u_x|^{2n+2}\mathrm{d}x,
\end{aligned}$$

where $g = \partial_x\Lambda^{-2}(uv_x-\varepsilon u_t)+\Lambda^{-2}(u^2+\frac{1}{2}u_x^2+u_xv_x+\frac{1}{2}v^2-\frac{1}{2}v_x^2)$. Then multiplying both sides of the above inequality by $(\int_0^1 (u_x)^{2n+2}\mathrm{d}x)^{-\frac{2n+1}{2n+2}}$ one finds

$$
\begin{aligned}
&(1-\varepsilon)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 (u_x)^{2n+2}\mathrm{d}x\right)^{\frac{1}{2n+2}}\\
&\leq \varepsilon\left(\int_0^1 |u_{xxxt}|^{2n+2}\mathrm{d}x\right)^{\frac{1}{2n+2}} + \left(\int_0^1 |u|^{4n+4}\mathrm{d}x\right)^{\frac{1}{2n+2}}\\
&\quad +\frac{1}{2}\left(\int_0^1 (|v_x|^{4n+4}+|v|^{4n+4})\mathrm{d}x\right)^{\frac{1}{2n+2}} + \left(\int_0^1 |g|^{2n+2}\mathrm{d}x\right)^{\frac{1}{2n+2}}\\
&\quad +\left(\frac{n}{2n+2}\|u_x\|_{L^\infty}+\frac{1}{2n+2}\|v_x\|_{L^\infty}\right)\left(\int_0^1 |u_x|^{2n+2}\mathrm{d}x\right)^{\frac{1}{2n+2}}.
\end{aligned}
$$

Because $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p\to\infty$ for any $f\in L^\infty\cap L^2$, integration with respect to t and taking the limit as $n\to\infty$ on both sides of the above inequality leads to the estimate

$$
\begin{aligned}
&(1-\varepsilon)\|u_x\|_{L^\infty}\\
&\leq (1-\varepsilon)\|u_{0x}\|_{L^\infty} + \int_0^t \Big[\varepsilon\|u_{xxxt}\|_{L^\infty} + c\left(\|u^2\|_{L^\infty}+\|v_x^2\|_{L^\infty}+\|v^2\|_{L^\infty}+\|g\|_{L^\infty}\right)\\
&\quad +\frac{1}{2}\|u_x\|^2_{L^\infty}\Big]\mathrm{d}\tau.
\end{aligned}
$$

Because

$$
\|g\|_{L^\infty} \leq \tilde{c}\left(\|u_t\|_{L^2}+\|u\|^2_{L^2}+\|u_x\|^2_{L^2}+\|v\|^2_{L^2}+\|v_x\|^2_{L^2}\right),
$$

for some $\tilde{c}$ depending only on Λ^{-2}, it follows from (2.4), (2.6) and (2.10) that

$$
\|g\|_{L^\infty} \leq c(1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty}),
$$

where c is a constant independent of ε when ε is sufficiently small. Moreover, for any fixed $r\in(1/2,1)$, there is a constant c_r such that

$$
\|u_{xxxt}\|_{L^\infty} \leq c_r\|u_{xxxt}\|_{H^r} \leq c_r\|u_t\|_{H^{r+3}},
$$

which combined with (2.4), (2.6) yields

$$
\|u_{xxxt}\|_{L^\infty} \leq c(1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty})(\|u\|_{H^{r+4}}+\|v\|_{H^{r+4}}).
$$

Applying Gronwall's inequality to (2.5) with $q = r + 4$ and $u = u_\varepsilon$ and (2.10) one has

$$
\begin{aligned}
&\|u\|_{H^{r+4}} + \|v\|_{H^{r+4}} \\
&\leq \sqrt{2}\left(\|u\|^2_{H^{r+4}} + \|v\|^2_{H^{r+4}}\right)^{\frac{1}{2}} \\
&\leq \sqrt{\frac{2}{1-\varepsilon}}\left\{\left[(1-\varepsilon)(\|u_0\|^2_{H^{r+4}} + \|v_0\|^2_{H^{r+4}}) + \varepsilon(\|u_{0x}\|^2_{H^{r+4}} + \|v_{0x}\|^2_{H^{r+4}}\right.\right. \\
&\quad\left.\left. + \|u_0\|^2_{H^{r+3}} + \|v_0\|^2_{H^{r+3}})\right]\exp\left[c\int_0^t (1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})\mathrm{d}\tau\right]\right\}^{\frac{1}{2}} \\
&\leq c\varepsilon^{(s-r-4)/4}\exp\left[c\int_0^t (1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})\mathrm{d}\tau\right].
\end{aligned}
$$

And so

$$
\begin{aligned}
\|u_{xxxt}\|_{L^\infty} \leq c\varepsilon^{(s-r-4)/4}&(1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \\
&\times\exp\left[c\int_0^t (1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})\mathrm{d}\tau\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_x\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty} + \frac{c}{1-\varepsilon}\int_0^t &\Big\{\varepsilon^{(s-r)/4}(1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \\
&\times\exp\left[c\int_0^t (1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})\mathrm{d}\tau\right] \\
&+1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty} + \|u_x\|^2_{L^\infty} + \|v_x\|^2_{L^\infty}\Big\}\,\mathrm{d}\tau.
\end{aligned}
$$

For the second equation of Eq. (2.7), we can obtain the similar result. Summing the results up, we get

$$
\begin{aligned}
\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty} &\leq \|u_{0x}\|_{L^\infty} + \|v_{0x}\|_{L^\infty} \\
&+\frac{c}{1-\varepsilon}\int_0^t \Big\{\varepsilon^{(s-r)/4}(1 + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})
\end{aligned}
$$

$$\times \exp\left[c\int_0^t (1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty})\mathrm{d}\tau\right]$$
$$+1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty}+(\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty})^2\Big\}\,\mathrm{d}\tau.$$

Therefore, as $0<\varepsilon<1/4$, one obtains the inequality

$$\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty}\le \|u_{0x}\|_{L^\infty}+\|v_{0x}\|_{L^\infty}$$
$$+\frac{4c}{3}\int_0^t\left\{\varepsilon^{(s-r)/4}(1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty})\exp\left[c\int_0^t(1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty})\mathrm{d}\tau\right]\right.$$
$$\left.+1+\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty}+(\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty})^2\right\}\mathrm{d}\tau.$$

It follows from the contraction mapping theorem that there is a $T>0$ such that the equation

$$f(t)=\|u_{0x}\|_{L^\infty}+\|v_{0x}\|_{L^\infty}$$
$$+\frac{4c}{3}\int_0^t\left[1+f(\tau)+f^2(\tau)+(1+f(\tau))\exp\left(c\int_0^\tau(1+f(s))\mathrm{d}s\right)\right]\mathrm{d}\tau$$

has a unique solution $f(t)\in C[0,T]$. Theorem II in [51]: Sect. I.1 shows that $\|u_x\|_{L^\infty}+\|v_x\|_{L^\infty}\le f(t)$ for any $t\in[0,T]$ which implies the conclusion of Theorem 2.4. □

As a direct result of Theorem 2.4, one may estimate norms of $u=u_\varepsilon$, $v=v_\varepsilon$ by using (2.5), (2.6), (2.10) and Gronwall's inequality to show that there is a constant $c>0$ such that the inequalities

$$\|u_\varepsilon\|_{H^q}+\|v_\varepsilon\|_{H^q}=\|u\|_{H^q}+\|v\|_{H^q}\le c\exp\left(c\int_0^t(1+f(\tau))\mathrm{d}\tau\right),$$

and

$$\|u_{\varepsilon t}\|_{H^r}+\|v_{\varepsilon t}\|_{H^r}=\|u_t\|_{H^r}+\|v_t\|_{H^r}\le c(1+f(t))\exp\left(c\int_0^t(1+f(\tau))\mathrm{d}\tau\right),$$

hold for any $q\in(0,s]$, $r\in(0,s-1]$ and any $t\in[0,T]$. Then it follows from Aubin's compactness theorem [42], that there is a subsequence of $\{u_\varepsilon\}$, $\{v_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}, \{v_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}, \{v_{\varepsilon_n}\}, \{u_{\varepsilon_n t}\}, \{v_{\varepsilon_n t}\}$ are weakly convergent to $u(t,x), v(t,x)$ and u_t, v_t in $L^2([0,T]; H^s)$ and $L^2([0,T]; H^{s-1})$, respectively. Moreover, for any real $0 < \delta < 1/2$, $\{u_{\varepsilon n}\}, \{v_{\varepsilon n}\}$ are convergent to u, v strongly in $L^2([0,T]; H^q(\delta, 1-\delta))$ for any $q \in [0,s)$ and $\{u_{\varepsilon nt}\}, \{v_{\varepsilon nt}\}$ converge to u_t, v_t strongly in $L^2([0,T]; H^r(\delta, 1-\delta))$ for any $r \in [0, s-1)$. Therefore, one obtains the existence of a weak solution to the Cauchy problem (1.7) as follows:

Theorem 2.5 *Let $u_0(x), v_0(x)$ be the functions in H^s for some $s \in [1, 3/2]$, satisfying $\|u_{0x}\|_{L^\infty} < \infty$, $\|v_{0x}\|_{L^\infty} < \infty$. Then there is a $T > 0$ such that the Cauchy problem* (1.17) *with the initial data u_0, v_0 has a solution*

$$(u(t,x), v(t,x)) \in L^2([0,T]; H^s) \times L^2([0,T]; H^s)$$

in the sense of distribution, and $u_x, v_x \in L^\infty([0,T] \times \mathbb{S})$.

Proof It follows from Theorem 2.4 that $\{u_{\varepsilon_n x}\}, \{v_{\varepsilon_n x}\}$ is bounded in the space L^∞. Hence, the sequences $\{u^2_{\varepsilon_n}\}, \{v^2_{\varepsilon_n}\}$, $\{u^2_{\varepsilon_n x}\}, \{v^2_{\varepsilon_n x}\}$, $\{u_{\varepsilon_n} v_{\varepsilon_n}\}, \{u_{\varepsilon_n x} v_{\varepsilon_n x}\}$, $\{u_{\varepsilon_n x} v_{\varepsilon_n}\}$, $\{u_{\varepsilon_n} v_{\varepsilon_n x}\}$ are also weakly convergent to $u^2, v^2, u_x^2, v_x^2, uv, u_x v_x, u_x v, u v_x$ in $L^2([0,T]; H^r(\delta, 1-\delta))$ for any $r \in [0, s-1]$ and $0 < \delta < 1/2$, respectively. Therefore, u, v satisfy the equations

$$\int_0^T \int_0^1 u(f_t - f_{xxt}) \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_0^1 \left[u_x v_x f - \left(\frac{3}{2}u^2 + \frac{1}{2}u_x^2 + uv + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 \right) f_x - u_x v f_{xx} + \frac{1}{2}u^2 f_{xxx} \right] \mathrm{d}x\mathrm{d}t,$$

and

$$\int_0^T \int_0^1 v(f_t - f_{xxt}) \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_0^1 \left[u_x v_x f - \left(\frac{3}{2}v^2 + \frac{1}{2}v_x^2 + uv + \frac{1}{2}u^2 - \frac{1}{2}u_x^2 \right) f_x - u v_x f_{xx} + \frac{1}{2}v^2 f_{xxx} \right] \mathrm{d}x\mathrm{d}t,$$

with $u(0,x) = u_0(x), v(0,x) = v_0(x)$ and any $f \in C_0^\infty$. Moreover, since $X = L^1([0,T] \times \mathbb{S})$ is a separable Banach space and $\{u_{\varepsilon_n x}\}, \{v_{\varepsilon_n x}\}$ are bounded sequences in the dual space $X^* = L^\infty([0,T] \times \mathbb{S})$ of X, there are two subsequences of $\{u_{\varepsilon_n x}\}, \{v_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}, \{v_{\varepsilon_n x}\}$, weak star convergent to two functions U, V in $L^\infty([0,T] \times \mathbb{S})$, respectively. Because $\{u_{\varepsilon_n x}\}, \{v_{\varepsilon_n x}\}$ are also weakly convergent to u_x, v_x in $L^2([0,T] \times \mathbb{S})$, respectively, it follows that $u_x = U, v_x = V$ almost everywhere. Hence, $u_x, v_x \in L^\infty([0,T] \times \mathbb{S})$. □

3 The wave breaking phenomena and blow-up rate

In this section, we establish the precise blow-up scenario, wave breaking phenomena and blow-up rate for solutions to the IVP (1.17).

Theorem 3.1 *Let $z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in H^s \times H^s$, $s > 3/2$ be given and assume that T is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ v \end{pmatrix}$ to the IVP (1.17) with the initial data z_0. If there exists $M > 0$ such that*

$$\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty} \leq M, \quad t \in [0, T),$$

then the $H^s \times H^s$-norm of $z(\cdot, t)$ does not blow up on $[0, T)$.

Proof Let $z = \begin{pmatrix} u \\ v \end{pmatrix}$ be the solution to (1.17) with initial data $z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in H^s \times H^s$, $s > 3/2$ and let T be the maximal existence time of the solution $z = \begin{pmatrix} u \\ v \end{pmatrix}$, which is guaranteed by Theorem 2.1. Throughout this proof, $c > 0$ stands for a generic constant depending only on s.

Applying the operator Λ^s to the first and second equation in (1.17), multiplying by $\Lambda^s u$ and $\Lambda^s v$, respectively, and integrating over $\mathbb{S}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^s}^2 + 2((u+v)u_x, u)_s + 2(u, f_1(z))_s = 0, \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{H^s}^2 + 2((u+v)v_x, v)_s + 2(v, f_2(z))_s = 0, \tag{3.2}$$

where

$$f_1(z) = \Lambda^{-2}(uv_x) + \partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2\right),$$

$$f_2(z) = \Lambda^{-2}(u_x v) + \partial_x \Lambda^{-2}\left(v^2 + \frac{1}{2}v_x^2 + u_x v_x + \frac{1}{2}u^2 - \frac{1}{2}u_x^2\right).$$

Let us estimate the right hand side of (3.1) and (3.2). From (2.8), it follows that

$$|((u+v)u_x, u)_s| \leq c(\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).$$

Similarly,

$$|((u+v)v_x, v)_s| \leq c(\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).$$

For the other two terms, we have

$$
\begin{aligned}
&|(u, f_1(z))_s| \\
&= \left| \left(\Lambda^{-2}(uv_x) + \partial_x \Lambda^{-2}\left(u^2 + \frac{1}{2}u_x^2 + u_x v_x + \frac{1}{2}v^2 - \frac{1}{2}v_x^2 \right), u \right)_s \right| \\
&\le c\|u\|_{H^s} \left(\|u^2\|_{H^{s-1}} + \|u_x^2\|_{H^{s-1}} + \|u_x v_x\|_{H^{s-1}} + \|v^2\|_{H^{s-1}} + \|v_x^2\|_{H^{s-1}} \right) \\
&\quad + |(\Lambda^{s-1}(uv_x), \Lambda^{s-1}u)_0| \\
&\le c\|u\|_{H^s}(\|u\|_{L^\infty}\|u\|_{H^{s-1}} + \|u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + \|v_x\|_{L^\infty}\|u_x\|_{H^{s-1}} \\
&\quad + \|v\|_{L^\infty}\|v\|_{H^{s-1}} + \|v_x\|_{L^\infty}\|v_x\|_{H^{s-1}}) \\
&\quad + \|[\Lambda^{s-1}, u]v_x\|_{L^2}\|\Lambda^{s-1}u\|_{L^2} + |(u\Lambda^{s-1}v_x, \Lambda^{s-1}u)_0| \\
&\le c\|u\|_{H^s}(\|u\|_{L^\infty}\|u\|_{H^{s-1}} + \|u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + \|v_x\|_{L^\infty}\|u_x\|_{H^{s-1}} \\
&\quad + \|v\|_{L^\infty}\|v\|_{H^{s-1}} + \|v_x\|_{L^\infty}\|v_x\|_{H^{s-1}}) \\
&\quad + c\|u\|_{H^{s-1}}(\|u_x\|_{L^\infty}\|v\|_{H^{s-1}} + \|v_x\|_{L^\infty}\|u\|_{H^{s-1}}) + \|u\|_{L^\infty}\|u\|_{H^{s-1}}\|v\|_{H^s} \\
&\le c\left(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}\right)\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2 \right),
\end{aligned}
$$

where we used Lemmas 2.7 and 2.8 with $r = s - 1$. Similarly

$$
|(v, f_2(z))_s| \le c(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).
$$

Therefore

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2 \right) \le c(\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^s}^2 + \|v\|_{H^s}^2).
$$

An application of Gronwall's inequality and the assumption of the theorem yields

$$
\|u\|_{H^s}^2 + \|v\|_{H^s}^2 \le \exp(cMt)\left(\|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 \right).
$$

This completes the proof of the theorem. □

In the sequel, we shall use the conserved quantity H^1-norm of the solution (u, v) of Eq. (1.7) to show that some of its solutions exist only in finite time.

Lemma 3.1 *Let $u_0(x), v_0(x) \in H^s$, $s > \frac{3}{2}$, and $(u(x,t), v(x,t))$ be the solution of the IVP (1.17) with life-span T. Then T is finite if and only if*

$$
\liminf_{t \uparrow T}\{\inf_{x\in\mathbb{R}}[u_x(x,t)]\} = -\infty,
$$

or

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{R}}[v_x(x,t)]\} = -\infty.$$

Proof By the definition of m and n, we have

$$\|m\|_{L^2}^2 = \int_{\mathbb{S}} (u-u_{xx})^2 \mathrm{d}x = \int_{\mathbb{S}} \left(u^2 + 2u_x^2 + u_{xx}^2\right) \mathrm{d}x,$$

and

$$\|n\|_{L^2}^2 = \int_{\mathbb{S}} (v-v_{xx})^2 \mathrm{d}x = \int_{\mathbb{S}} \left(v^2 + 2v_x^2 + v_{xx}^2\right) \mathrm{d}x.$$

Hence,

$$\|u\|_{H^2}^2 \le \|m\|_{L^2}^2 \le 2\|u\|_{H^2}^2, \tag{3.3}$$

$$\|v\|_{H^2}^2 \le \|n\|_{L^2}^2 \le 2\|v\|_{H^2}^2. \tag{3.4}$$

Applying Theorem 2.1 and a simple density argument, it suffices to consider the case $s \ge 2$ and show that the H^2-norm does not blow up in finite time.

Multiplying the first equation by m and the second one by n in (1.17), after integration by parts and adding up the results, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}} (m^2+n^2)\mathrm{d}x \\
&= -\int_{\mathbb{S}} [(2u_x+v_x)m^2 + (2v_x+u_x)n^2 + (mm_x+nn_x)(u+v) \\
&\qquad +mn(u_x+v_x)]\mathrm{d}x \\
&= -\int_{\mathbb{S}} \Bigg[\left(2u_x+v_x-\frac{1}{2}u_x-\frac{1}{2}v_x\right)m^2 + \left(2v_x+u_x-\frac{1}{2}u_x-\frac{1}{2}v_x\right)n^2 \\
&\qquad +mn(u_x+v_x)\Bigg]\mathrm{d}x \\
&= -\int_{\mathbb{S}} \left[\left(\frac{3}{2}u_x+\frac{1}{2}v_x\right)m^2 + \left(\frac{1}{2}u_x+\frac{3}{2}v_x\right)n^2 + mn(u_x+v_x)\right]\mathrm{d}x \\
&= -\int_{\mathbb{S}} \left[\frac{1}{2}(u_x+v_x)(m+n)^2 + m^2u_x + n^2v_x\right]\mathrm{d}x.
\end{aligned}$$

If there exist two constants $M > 0$ and $N > 0$ such that $u_x \geq -M$, $v_x \geq -N$, one finds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}(m^2+n^2)\mathrm{d}x &\leq (M+N)\int_{\mathbb{S}}\left[(m+n)^2+m^2+n^2\right]\mathrm{d}x\\ &\leq 3(M+N)\int_{\mathbb{S}}\left(m^2+n^2\right)\mathrm{d}x,\end{aligned}$$

By Gronwall's inequality, we get

$$\begin{aligned}\|u\|_{H^2}^2+\|v\|_{H^2}^2 &\leq \int_{\mathbb{S}}\left(m^2+n^2\right)\mathrm{d}x \leq \exp[3T(M+N)]\int_{\mathbb{S}}\left(m_0^2+n_0^2\right)\mathrm{d}x\\ &\leq 2\exp[3T(M+N)]\left(\|u_0\|_{H^2}^2+\|v_0\|_{H^2}^2\right),\end{aligned}$$

The above inequality, Sobolev's embedding theorem and Theorem 3.1 ensure that the solution (u, v) does not blow up in finite time.

On the other hand, if

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{R}}[u_x(x,t)]\} = -\infty,$$

or

$$\liminf_{t\uparrow T}\{\inf_{x\in\mathbb{R}}[v_x(x,t)]\} = -\infty,$$

then the solution will blow up in finite time. □

To establish the following result in view, we follow the general approach by Li and Olver [41] to deal with blow-up solutions for the Camassa–Holm equation, but using the more detail analysis of solutions.

Theorem 3.2 *Let $s \in [2, \infty)$ be any real number. If the initial data u_0, v_0 of the IVP* (1.17) *satisfies the conditions*

$$u_0, v_0 \in H^s, \quad \int_{\mathbb{S}}(u_{0x}+v_{0x})^3\mathrm{d}x < 0$$

and

$$\left(\int_{\mathbb{S}}(u_{0x}+v_{0x})^3\mathrm{d}x\right)^2 > 60\coth(1/2)\left(\|u_0\|_{H^1}^2+\|v_0\|_{H^1}^2\right)^3,$$

then there exists a T^* such that the corresponding solution $(u, v) \in C([0, T^*); H^s) \times C([0, T^*); H^s)$ ceases to exist in H^s at the time T^* in the sense that

$$\limsup_{t\to T^*} (\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) = \infty, \quad \lim_{t\to T^*} (\|u\|_{H^q} + \|v\|_{H^q}) = \infty$$

for any $q \in (3/2, s]$.

Proof From Theorem 2.1, we know that there exists a $T_0 > 0$ such that the IVP (1.17) has a unique solution $(u, v) \in C([0, T_0); H^s) \times C([0, T_0); H^s)$. Adding up the first and second equation of (1.17), one finds that

$$(u+v)_t + (u+v)(u_x+v_x) = -\partial_x \Lambda^{-2}\left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_xv_x\right).$$

Applying ∂_x to both sides of the above equation, it follows that

$$\begin{aligned}&(u_x+v_x)_t + u_x^2 + v_x^2 + (u+v)(u_{xx}+v_{xx})\\ &\quad = \frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - \Lambda^{-2}\left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_xv_x\right).\end{aligned}$$

Multiplying it by $(u_x + v_x)^2$ and integrating with respect to x, one obtains

$$\begin{aligned}&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}(u_x+v_x)^3\mathrm{d}x + \int_{\mathbb{S}}(u_x^2+v_x^2)(u_x+v_x)^2\mathrm{d}x - \frac{1}{3}\int_{\mathbb{S}}(u_x+v_x)^4\mathrm{d}x\\ &\quad = \int_{\mathbb{S}}(u_x+v_x)^2\left[\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - \Lambda^{-2}\left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_xv_x\right)\right]\mathrm{d}x.\end{aligned} \tag{3.5}$$

Since

$$\int_{\mathbb{S}}(u_x^2+v_x^2)(u_x+v_x)^2\mathrm{d}x \geq \frac{1}{2}\int_{\mathbb{S}}(u_x+v_x)^4\mathrm{d}x,$$

it turns out from (3.5) that

$$\begin{aligned}&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}}(u_x+v_x)^3\mathrm{d}x + \frac{1}{6}\int_{\mathbb{S}}(u_x+v_x)^4\mathrm{d}x\\ &\quad \leq \int_{\mathbb{S}}(u_x+v_x)^2\left[\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - \Lambda^{-2}\left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_xv_x\right)\right]\mathrm{d}x.\end{aligned} \tag{3.6}$$

Also

$$\left|\int_{\mathbb{S}} (u_x + v_x)^3 \mathrm{d}x\right| \leq \left(\int_{\mathbb{S}} (u_x + v_x)^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}} (u_x + v_x)^4 \mathrm{d}x\right)^{\frac{1}{2}},$$

it follows that

$$\int_{\mathbb{S}} (u_x + v_x)^4 \mathrm{d}x \geq \frac{1}{2(\|u\|_{H^1}^2 + \|v\|_{H^1}^2)} \left(\int_{\mathbb{S}} (u_x + v_x)^3 \mathrm{d}x\right)^2. \tag{3.7}$$

Using

$$\begin{aligned}|\Lambda^{-2} f(x)| &= \left|\frac{1}{2\sinh(1/2)} \int_{\mathbb{S}} \cosh\left(x - y - |x - y| - \frac{1}{2}\right) f(y)\mathrm{d}y\right| \\ &\leq \frac{\cosh(1/2)}{2\sinh(1/2)} \|f\|_{L^1},\end{aligned}$$

and $|f|_{L^\infty} \leq c\|f\|_{H^1}$, with $c = \sqrt{1/2\coth(1/2)} = \sqrt{\frac{\cosh(1/2)}{2\sinh(1/2)}}$ (see [50]), we have

$$\begin{aligned}&\left|\int_{\mathbb{S}} 3(u_x + v_x)^2 \left[\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - \Lambda^{-2}\left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_x v_x\right)\right] \mathrm{d}x\right| \\ &\leq 6(\|u\|_{H^1}^2 + \|v\|_{H^1}^2) \Big(2\|u^2\|_{L^\infty} + 2\|v^2\|_{L^\infty} \\ &\quad + \|\Lambda^{-2}\left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_x v_x\right)\|_{L^\infty}\Big) \\ &\leq 6\left(2c^2 + \frac{3\cosh(\frac{1}{2})}{2\sinh(\frac{1}{2})}\right)(\|u\|_{H^1}^2 + \|v\|_{H^1}^2)(\|u\|_{H^1}^2 + \|v\|_{H^1}^2) \\ &= 15\coth(1/2)(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)^2. \end{aligned} \tag{3.8}$$

Moreover, using (3.7), (3.8) and the conserved quantity E_5 to (3.6) lead to the estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}} (u_x + v_x)^3 \mathrm{d}x &\leq -\frac{1}{4(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)} \left(\int_{\mathbb{S}} (u_x + v_x)^3 \mathrm{d}x\right)^2 \\ &\quad + 15\coth(1/2)(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)^2.\end{aligned}$$

It is well-known that if

$$\int_{\mathbb{S}} (u_{0x} + v_{0x})^3 \mathrm{d}x < 0, \quad \text{and}$$

$$\left(\int_{\mathbb{S}} (u_{0x} + v_{0x})^3 \mathrm{d}x \right)^2 > 60 \coth(1/2) \left(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2 \right)^3,$$

then there exists a T_0 such that the solution $\int_{\mathbb{S}} (u_{0x} + v_{0x})^3 \mathrm{d}x$ of this Riccati type equation will go to $-\infty$ when t goes to T_0. However

$$\begin{aligned} \left| \int_{\mathbb{S}} (u_x + v_x)^3 \mathrm{d}x \right| &\leq c_q (\|u\|_{H^1}^2 + \|v\|_{H^1}^2)(\|u\|_{H^q} + \|v\|_{H^q}) \\ &= c_q (\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)(\|u\|_{H^q} + \|v\|_{H^q}) \end{aligned}$$

shows that $\lim_{t \to T_0} (\|u\|_{H^q} + \|v\|_{H^q}) = \infty$, where q is any real number with $q \in (\frac{3}{2}, \infty)$ and c_q is a constant independent of u and v.

To verify $\limsup_{t \to T_0} (\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) = \infty$, one may use (2.5), (2.10) and the fact that u, v as the limit of (2.2) satisfy the inequality

$$\begin{aligned} \|u\|_{H^q}^2 + \|v\|_{H^q}^2 &\leq \|u_0\|_{H^q}^2 + \|v_0\|_{H^q}^2 \\ &\quad + c \int_0^t (\|u\|_{L^\infty} + \|v\|_{L^\infty} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty})(\|u\|_{H^q}^2 + \|v\|_{H^q}^2) \mathrm{d}\tau \end{aligned}$$

for any $q \in (\frac{3}{2}, s]$. It follows from Gronwall's inequality that

$$\begin{aligned} &\|u\|_{H^q}^2 + \|v\|_{H^q}^2 \\ &\quad \leq \left(\|u_0\|_{H^q}^2 + \|v_0\|_{H^q}^2 \right) \exp \left[C \int_0^t (\|u_0\|_{H^1} + \|v_0\|_{H^1} + \|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) \mathrm{d}\tau \right]. \end{aligned}$$

Therefore, if $\limsup_{t \to T_0} (\|u_x\|_{L^\infty} + \|v_x\|_{L^\infty}) < \infty$, then it leads to the boundedness of $\|u\|_{H^q}^2 + \|v\|_{H^q}^2$, which contradicts $\lim_{t \to T_0} (\|u\|_{H^q} + \|v\|_{H^q}) = \infty$. Hence $T_0 = T^*$ is the finite time for u, v, u_x, v_x to cease existing in L^∞ and H^q for any $q \in (\frac{3}{2}, s]$, respectively. This completes the proof of the theorem. □

It is easy to observe that if $(u(t, x), v(t, x))$ is a solution to (1.17) system with initial data $u_0(x)$, $v_0(x)$, then $-u(t, -x)$, $-v(t, -x)$ is also solutions to the system with initial data $-u_0(-x)$, $-v_0(-x)$. Hence, due to the uniqueness of the solution, the solution to the system is odd as long as the initial data $u_0(x)$, $v_0(x)$ are odd.

Theorem 3.3 *Suppose that $u_0(x), v_0(x) \in H^s, \quad s > 3/2$, and let T be the maximal existence time of the solution to the initial value problem (1.17) with the initial data u_0, v_0. If u_0, v_0 are odd and*

$$u_0'(0) + v_0'(0) < -\sqrt{\coth(1/2)(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)},$$

then T is finite and $u_x(t,0) + v_x(t,0)$ tends to negative infinity as t goes to T.

Proof Let $w = u + v$, since $w_0 = u_0 + v_0$ is odd, w is also odd. It follows that

$$w_t + ww_x + \partial_x G * \left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_x v_x\right) = 0.$$

Then

$$\begin{aligned} & w_{xt} + ww_{xx} + (u_x + v_x)^2 \\ & = \frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_x v_x - G * \left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_x v_x\right). \end{aligned}$$

By continuity with respect to x of u, v, u_{xx}, v_{xx}, we have

$$u(t,0) = v(t,0) = u_{xx}(t,0) = v_{xx}(t,0) = 0.$$

So

$$w_{tx}(t,0) + u_x^2(t,0) + v_x^2(t,0) = -G * \left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_x v_x\right)(t,0).$$

However, $u_x^2 + v_x^2 \geq \frac{(u_x+v_x)^2}{2}$ and $G * (\frac{3}{2}u^2 + uv + \frac{3}{2}v^2) \geq 0$, so

$$w_{tx}(t,0) + \frac{1}{2}w_x^2(t,0) \leq -G * (2u_x v_x)(t,0),$$

i.e.

$$w_{tx}(t,0) \leq -\frac{1}{2}w_x^2(t,0) - G * (2u_x v_x)(t,0).$$

By Young's inequality,

$$\begin{aligned} \|G * (2u_x v_x)(t,0)\|_{L^\infty} & \leq 2\|G\|_{\text{Ł}^\infty}\|u_x v_x\|_{L^1} \leq \|G\|_{\text{Ł}^\infty}\left(\|u_x^2\|_{L^1} + \|v_x^2\|_{L^1}\right) \\ & \leq \frac{1}{2}\coth\left(\frac{1}{2}\right)\left(\|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2\right) \\ & \leq \frac{1}{2}\coth\left(\frac{1}{2}\right)\left(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2\right). \end{aligned}$$

Therefore, we obtain

$$w_{tx}(t,0) \leq -\frac{1}{2}w_x^2(t,0) + K^2,$$

where $K = \frac{\sqrt{2\coth(1/2)}}{2}(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)^{\frac{1}{2}}$. Note that if $w_x(0,0) \leq -\sqrt{2}K$, then $w_x(t,0) \leq -\sqrt{2}K$ for all $t \in [0,T)$.

From the above inequality we obtain

$$\frac{w_x(0,0)+\sqrt{2}K}{w_x(0,0)-\sqrt{2}K}\mathrm{e}^{\sqrt{2}Kt} - 1 \leq \frac{2\sqrt{2}K}{w_x(t,0)-\sqrt{2}K} \leq 0.$$

Due to $0 < \frac{w_x(0,0)+\sqrt{2}K}{w_x(0,0)-\sqrt{2}K} < 1$, there exists $0 < T < \frac{1}{\sqrt{2}K}\ln(\frac{w_x(0,0)-\sqrt{2}K}{w_x(0,0)+\sqrt{2}K})$, such that $\lim_{t\uparrow T} w_x(t,0) = -\infty$. This completes the proof of the theorem. □

Lemma 3.2 [14] *Let $T > 0$ and $w \in C^1([0,T); H^2)$. Then for every $t \in [0,T)$, there exists at least one point $\xi(t) \in \mathbb{S}$ with $l(t) := \inf_{x\in\mathbb{S}}[w_x(t,x)] = w_x(t,\xi(t))$. The function $l(t)$ is absolutely continuous on (0,T) with*

$$\frac{\mathrm{d}l}{\mathrm{d}t} = w_{xt}(t,\xi(t)), \quad a.e. \quad on \quad (0,T).$$

Theorem 3.4 *Let $u_0(x), v_0(x) \in H^s, s > 3/2$, and let T be the maximal existence time of the solution to the initial value problem (1.17) with the initial data u_0, v_0. If there exists some $x_0 \in \mathbb{S}$ such that*

$$u_0'(x_0) + v_0'(x_0) < -\sqrt{3\coth(1/2)(\|u_0\|_{H^1}^2 + \|v_0\|_{H^1}^2)},$$

then T is finite and $\lim_{t\uparrow T}\inf_{x\in\mathbb{S}}[u_x(t,x) + v_x(t,x)] = -\infty$.

Proof By Theorem 2.1 and a simple density argument, we only need to consider the case $s = 2$. Let $w = u + v$. Then we have

$$w_{xt} + ww_{xx} + u_x^2 + v_x^2 = \frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - G * \left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_xv_x\right).$$

Define $l(t) = w_x(t,\xi(t)) = \inf_{x\in\mathbb{S}}[w_x(t,x)]$. Since we deal with a minimum, $w_{xx}(t,\xi(t)) = 0$ for all $t \in [0,T)$. Thus, we obtain a.e. on $[0,T)$,

$$l'(t) + u_x^2 + v_x^2 = \frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - G * \left(\frac{3}{2}u^2 + uv + \frac{3}{2}v^2 + 2u_xv_x\right).$$

However, $u_x^2 + v_x^2 \geq \frac{(u_x+v_x)^2}{2}$ and $G * (\frac{3}{2}u^2 + uv + \frac{3}{2}v^2) \geq 0$, so

$$l'(t) \leq -\frac{1}{2}l^2(t) + \frac{3}{2}u^2 + uv + \frac{3}{2}v^2 - G * (2u_xv_x).$$

As above,

$$\|G*(2u_xv_x)\|_{L^\infty} \leq \frac{1}{2}\coth\left(\frac{1}{2}\right)\left(\|u_0\|^2_{H^1}+\|v_0\|^2_{H^1}\right),$$

since

$$\begin{aligned}\left\|\frac{3}{2}u^2+uv+\frac{3}{2}v^2\right\|_{L^\infty} &\leq 2\|u^2+v^2\|_{L^\infty} \leq \coth(1/2)\left(\|u\|^2_{H^1}+\|v\|^2_{H^1}\right)\\ &= \coth(1/2)\left(\|u_0\|^2_{H^1}+\|v_0\|^2_{H^1}\right).\end{aligned}$$

So

$$l'(t) \leq -\frac{1}{2}l^2(t)+M^2,$$

where $M=\sqrt{\frac{3}{2}\coth(1/2))(\|u_0\|^2_{H^1}+\|v_0\|^2_{H^1})}$. Using the above inequality and following the lines of the proof of Theorem 3.3, we see that if

$$l(0) < -\sqrt{3\coth\left(\frac{1}{2}\right)\left(\|u_0\|^2_{H^1}+\|v_0\|^2_{H^1}\right)},$$

then T is finite and $\lim_{t\uparrow T} l(t)=-\infty$. This completes the proof of the theorem. □

Our attention is now turned to the question of the blow-up rate of the slope to a breaking wave for (1.17).

Theorem 3.5 *If $T<\infty$ is the blow up time of the solution (u,v) of the initial value problem (1.17) with initial data $u_0, v_0 \in H^s$, $s>3/2$, then we have*

$$\lim_{t\uparrow T}\left\{\inf_{x\in\mathbb{S}}[(u_x(t,x)+v_x(t,x))(T-t)]\right\}=-2$$

while the solution remains uniformly bounded.

Proof The uniform boundedness of the solution can be easily obtained by the conserved quantity in H^1. By Lemma 3.2, we can see that the function

$$l(t)=\inf_{x\in\mathbb{S}}[u_x(t,x)+v_x(t,x)]$$

is locally Lipschitz with $l(t)<0$, $t\in[0,T)$.

From the above theorem, we get

$$l'(t) \leq -\frac{1}{2}l^2(t)+N, \quad t\in[0,T), \tag{3.9}$$

where $N=3/2\coth(1/2)(\|u_0\|^2_{H^1}+\|v_0\|^2_{H^1})$.

Now fix any $\varepsilon \in (0, \frac{1}{2})$. From Lemma 3.1, there exists $t_0 \in (0, T)$ such that $l(t_0) < -\sqrt{2N + \frac{N}{\varepsilon}}$. Notice that $l(t)$ is locally Lipschitz so that it is absolutely continuous on $[0, T)$. It is then from the above inequality that $l(t)$ is decreasing on $[t_0, T)$ and satisfies that

$$l(t) < -\sqrt{2N + \frac{N}{\varepsilon}} < -\sqrt{\frac{N}{\varepsilon}}, \quad t \in [t_0, T).$$

Since $l(t)$ is decreasing on $[t_0, T)$, it follows that

$$\lim_{t \uparrow T} l(t) = -\infty.$$

From (3.9), we obtain

$$\frac{1}{2} - \varepsilon \leq \frac{\mathrm{d}}{\mathrm{d}t}(l(t)^{-1}) = -\frac{l'(t)}{l^2(t)} \leq \frac{1}{2} + \varepsilon.$$

Integrating the above equation on (t, T) with $t \in (t_0, T)$ and noticing that $\lim_{t \uparrow T} l(t) = -\infty$, we get

$$\left(\frac{1}{2} - \varepsilon\right)(T - t) \leq -\frac{1}{l(t)} \leq \left(\frac{1}{2} + \varepsilon\right)(T - t).$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, in view of the definition of $l(t)$, the above inequality implies the desired result of the theorem. □

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