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Multiple solutions for nonhomogeneous p-Laplacian equations with nonlinear boundary conditions on $\mathbb{R}^N_+$

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Abstract

In this paper, we study the existence of multiple solutions for the nonlinear boundary value problem

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = h(x), & x \in \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u, & x \in \partial\mathbb{R}^N_+, \end{cases} \tag{0.1}$$

where $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^+\}$ is an upper half space in $\mathbb{R}^N$ and $1 < p < N, \lambda > 0$ and $1 < q < p < r < p_* = \frac{p(N-1)}{N-p}$, ν denotes the unit outward normal to boundary $\partial\mathbb{R}^N_+$. The functions $V(x), h(x), h_1(x)$ and $h_2(x)$ satisfy some suitable conditions. Using the Mountain Pass Theorem and Ekeland's variational principle, we prove that there exist $\lambda_0, m_0 > 0$ such that problem (0.1) admits at least two solutions provided $\lambda \in (0, \lambda_0)$ and $\|h\|_{p'} \le m_0 < c_1\lambda^{(p-1)/(r-q)}$, where the constant $c_1 > 0$ is independent of $\lambda > 0$. On the other hand, if $h_2 = 0$, we prove that the problem (0.1) admits at least one solution for any $\lambda > 0$ and $h \in L^{p'}(\mathbb{R}^N_+)$.

Keywords: p-Laplacian equation; Mountain Pass Theorem; Nonlinear boundary condition; Ekeland's variational principle; Multiple solutions.

AMS Subject Classifications: 35J20; 35J66; 35J92.

1 Introduction

Recently, by Nehari manifold and fibering maps method, T.F.Wu in [18] studied the existence of multiple solutions for the nonlinear boundary value problem

$$\begin{cases} -\Delta u + u = 0, & x \in \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u, & x \in \partial\mathbb{R}^N_+ \end{cases} \tag{1.1}$$

where $1 < q < 2 < r < 2_*(2_* = \frac{2(N-1)}{N-2}$ if $N > 2, 2_* = \infty$ if $N = 2)$, $\mathbb{R}^N_+$ is an upper half space in $\mathbb{R}^N$ and $\lambda > 0$. The functions h_1 and h_2 satisfy the following conditions:

(D_1) $h_1 \in L^{\frac{r}{r-q}}(\partial\mathbb{R}^N_+) \setminus \{0\}$ with $(h_1)_\pm(x) = \max\{\pm h_1(x)\} \not\equiv 0$;

(D_2) $h_2 \in C(\partial\mathbb{R}^N_+)$ and there is a positive number $r_0 < r$ such that

$$h_2(x) \ge 1 + c_0\exp(-r_0|x|) \quad \text{for some } c_0 < 1 \text{ and for all } x \in \partial\mathbb{R}^N_+$$

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and $h_2(x) \to 1$ as$|x| \to \infty$.

In this paper, motivated by [18], we study the existence of multiple solutions for the nonlinear boundary value problem

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u) + V(x)|u|^{p-2}u = h(x), & x \in \mathbb{R}^N_+, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda h_1(x)|u|^{q-2}u + h_2(x)|u|^{r-2}u, & x \in \partial\mathbb{R}^N_+, \end{cases} \tag{1.2}$$

where $1 < p < N, \lambda > 0$ and ν denotes the unit outward normal to the boundary $\partial\mathbb{R}^N_+$. The parameters r, q satisfy $1 < q < p < r < p_* = \frac{p(N-1)}{N-p}$, where p_* is the critical exponent for the Sobolev trace embedding, see [8,16,18]. Problem (1.2) can be looked as a perturbation of (1.1).

We will use the Mountain Pass Theorem and Ekeland's variational principle to study the existence of multiple solutions for problem (1.2) under the appropriate assumptions on $h_1(x)$ and $h_2(x)$ which are different from that in [18]. It seems difficult to study the multiplicity of solutions for (1.2) by dint of Nehari manifold and fibering maps methods.

Since $\mathbb{R}^N_+$ is an unbounded domain, the loss of compactness of the Sobolev embedding $W^{1,p}(\mathbb{R}^N_+) \hookrightarrow L^q(\mathbb{R}^N_+)$ renders variational technique more delicate. To preserve this compactness in our problem, we need to impose some conditions on the weight functions $h_1(x)$ and $h_2(x)$.

Throughout this paper, we make the following assumptions.

(H_1) $V(x) \in L^\infty(\mathbb{R}^N_+)$ and $V(x) \geq v_0 > 0$ in $\mathbb{R}^N_+$; $h(x) \in L^{p'}(\mathbb{R}^N_+)$ with $p' = \frac{p}{p-1}$.

(H_2) Let $1 < q < p < r < p_*$ and $h_1(x) \in L^\infty(\partial\mathbb{R}^N_+) \cap L^{\alpha_1}(\partial\mathbb{R}^N_+)$, $h_2(x) \in L^\infty(\partial\mathbb{R}^N_+) \cap L^{\alpha_2}(\partial\mathbb{R}^N_+)$ with $\alpha_1 = \frac{p}{p-q}, \alpha_2 = \frac{p_*}{p_*-r}$.

(H_3) For $h \equiv 0$ in $\mathbb{R}^N_+$, we suppose that there exist non-empty domain $\Gamma_k = \{x \in \partial\mathbb{R}^N_+ | h_k(x) > 0\} \subset \partial\mathbb{R}^N_+$ with $\text{meas}(\Gamma_k) > 0, k = 1, 2$.

(H_4) For $h \not\equiv 0$ in $\mathbb{R}^N_+$, we suppose $\text{meas}(\Gamma_2) > 0$.

(H_5) For $h_2 \equiv 0$ in $\partial\mathbb{R}^N_+$, we suppose $\text{meas}(\Gamma_1) > 0$.

We note that the assumptions $(H_3) - (H_4)$ imply that the weight functions $h_1(x)$ and $h_2(x)$ are allowed to be sign-changing in $\partial\mathbb{R}^N_+$.

In recent years, the existence of solutions for the quasilinear elliptic equation with nonlinear boundary conditions on unbounded domain has received great attention, see [3,8,12,14-17] and the references therein. In particular, Pflüger [17] studied the elliptic boundary value problem

$$\begin{cases} -div(d(x)|\nabla u|^{p-2}\nabla u) = f(x, u), & x \in \Omega \\ d(x)|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + b(x)|u|^{p-2}u = g(x, u), & x \in \partial\Omega \end{cases} \tag{1.3}$$

where Ω is an unbounded domain in $\mathbb{R}^N$ with noncompact, smooth boundary $\partial\Omega$, and $0 < d_0 \leq d(x) \in L^\infty(\partial\Omega)$ and $b(x)$ is a positive and continuous function which satisfies

$$0 < c_1(1 + |x|)^{1-p} \leq b(x) \leq c_2(1 + |x|)^{1-p}, \quad x \in \partial\Omega. \tag{1.4}$$

Similar assumption on $b(x)$ can be also found in [3,7,14,16].

For problem (1.3), Pflüger used a weighted Sobolev space E. E is defined as the completion of $C_\delta^\infty(\Omega)$, which is the space of $C_0^\infty(\mathbb{R}^N)$ -functions restricted on Ω, and the norm is

$$\|u\|_E^p = \int_\Omega d(x)|\nabla u(x)|^p dx + \int_{\partial\Omega} (1+|x|^p)^{-1}|u(x)|^p d\sigma \tag{1.5}$$

Under this norm, the compact embeddings $E \hookrightarrow L^q(\Omega)$ and $E \hookrightarrow L^r(\partial\Omega)$ were established. By the assumption (1.4), one can obtain the equivalent norm for E

$$\|u\|_E^p = \int_\Omega d(x)|\nabla u|^p dx + \int_{\partial\Omega} b(x)|u|^p d\sigma \tag{1.6}$$

For the smooth exterior domain Ω in $\mathbb{R}^N$, which the boundary $\partial\Omega$ is compact, Filippucci et al.[12] considered the existence and nonexistence of solutions to the elliptic exterior problem

$$\begin{cases} -div(a(x)|\nabla u|^{p-2}\nabla u) + |u|^{q-2}u = \lambda h(x)|u|^{r-2}u, & x \in \Omega \\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + b(x)|u|^{p-2}u = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

where $a(x) \geq a_0 > 0, h(x) \geq 0$ in Ω and $b(x) > 0$ in $\partial\Omega$, and $h(x) \in L^\infty(\Omega) \cap L^{p_0}(\Omega)$.

By the variational method, they obtained the following main results.

(1). Let $p < r < q < p^*$. Then there exists $\lambda_0 > 0$ such that problem (1.7) has no nontrivial weak solution if $\lambda \leq \lambda_0$; and problem (1.7) has at least a nontrivial positive weak solution u if $\lambda \geq \lambda_0$;

(2). Let $p < q < r < p^*$. Then problem (1.7) has no nontrivial weak solution if $\lambda \leq 0$ and has at least a nontrivial weak solution if $\lambda > 0$.

When Ω is a bounded domain in $\mathbb{R}^N$, the function $b(x)$ in (1.7) is permitted to be zero and the many results of existence for (1.7) have been established, see [1,5-7,13,19] and the references therein.

Motivated by the results of the above works, we are interested in the existence of multiple solutions for (1.2). We will use the Mountain pass theorem and Ekeland's variational principle to prove the existence of multiple solutions for (1.2).

Our main results in this paper read as follows.

Theorem 1 Assume $(H_1) - (H_4)$ hold. Then there exists $\lambda_0, m_0 > 0$, such that the problem (1.2) admits at least two solutions in $W^{1,p}(\mathbb{R}_+^N)$ provided $\lambda \in (0, \lambda_0)$ and $\|h\|_{p'} \leq m_0 < c_1\lambda^{(p-1)/(r-q)}$, where the constant $c_1 > 0$ is independent of $\lambda > 0$.

Remark 1. If $h \equiv 0$ in $\mathbb{R}_+^N$, then we let $m_0 = 0$.

Theorem 2 Assume $(H_1) - (H_2)$ and (H_5) hold. Then, for any $\lambda > 0$ and $h \in L^{p'}(\mathbb{R}_+^N)$, the problem (1.2) admits at least one solution in $W^{1,p}(\mathbb{R}_+^N)$.

This paper is organized as follows. In Section 2, we set up the variational framework of the problem (1.2) and verify the conditions in the Mountain pass theorem. By the lemmas in Section 2, we give the proofs of our main results in Section 3.

2 Preliminaries

Let $E = W^{1,p}(\mathbb{R}_+^N)$ denote the usual Sobolev space. In this space, we introduce the norm

$$\|u\|_E = (\int_{\mathbb{R}_+^N} (|\nabla u|^p + V(x)|u|^p)dx)^{1/p}. \tag{2.1}$$

By the assumption (H_1), it is equivalent to the standard one. It is well known that the embedding $E \hookrightarrow L^q = L^q(\mathbb{R}^N_+)(p \le q \le p^* = \frac{pN}{N-p})$ is continuous and there is constant $S > 0$ such that

$$S\|u\|_q \le \|u\|_E, \quad \forall u \in E. \tag{2.2}$$

Here and in the sequel, we denote $\|u\|_q = (\int_{\mathbb{R}^N_+} |u|^q dx)^{1/q}$ for $q \ge 1$.

As in [15, 18], we set S_q as the best Sobolev trace constants for the embedding $E \hookrightarrow L^q(\partial\mathbb{R}^N_+)$ for $p \le q < p_* = \frac{p(N-1)}{N-p}$, where $E \hookrightarrow L^q(\partial\mathbb{R}^N_+)$ means by $E \subset L^q(\partial\mathbb{R}^N_+)$ with continuous injection and

$$S_q = \inf_{u \in E\setminus\{0\}} \frac{\int_{\mathbb{R}^N_+}(|\nabla u|^p + V(x)|u|^p)dx}{\left(\int_{\partial\mathbb{R}^N_+} |u|^q d\sigma\right)^{p/q}} = \inf_{u \in E\setminus\{0\}} \frac{\|u\|_E^p}{\|u\|^p_{L^q(\partial\mathbb{R}^N_+)}}. \tag{2.3}$$

Then, we have for $p \le q < p_*$,

$$\|u\|_{L^q(\partial\mathbb{R}^N_+)} \le S_q^{-1/p}\|u\|_E, \quad \forall u \in E. \tag{2.4}$$

Lemma 1 (Boundary Trace embedding Theorem [9,10]) Let Ω be a domain in $\mathbb{R}^N$ satisfying the uniform C^1-regularity condition. Suppose that there exists a simple $(1,p)$-extension operator T for Ω, and $1 < p < N$ and $p \le q < p_* = \frac{p(N-1)}{N-p}$. Then we have $E \hookrightarrow L^q(\partial\Omega)$. If $p = N$, then the embedding still holds for $p \le q < \infty$.

Definition 1. we say that $u \in E$ is a solution to problem (1.2) if for any $\varphi \in E$, there holds

$$\int_{\mathbb{R}^N_+}(|\nabla u|^{p-2}\nabla u\nabla\varphi + V(x)|u|^{p-2}u\varphi)dx = \int_{\mathbb{R}^N_+} h\varphi dx + \int_{\partial\mathbb{R}^N_+}(\lambda h_1|u|^{q-2}u + h_2|u|^{r-2}u)\varphi d\sigma \tag{2.5}$$

Let $J(u) : E \to \mathbb{R}$ be the energy functional of problem (1.2), defined by

$$J(u) = \frac{1}{p}\|u\|_E^p - \int_{\mathbb{R}^N_+} h(x)udx - \frac{\lambda}{q}\int_{\partial\mathbb{R}^N_+} h_1(x)|u|^q d\sigma - \frac{1}{r}\int_{\partial\mathbb{R}^N_+} h_2(x)|u|^r d\sigma \tag{2.6}$$

Then, we see that the functional $J(u) \in C^1(E,\mathbb{R})$ under the assumptions $(H_1) - (H_5)$ and for any $\varphi \in E$,

$$\begin{aligned}\langle J'(u), \varphi\rangle = &\int_{\mathbb{R}^N_+}(|\nabla u|^{p-2}\nabla u\nabla\varphi + V(x)|u|^{p-2}u\varphi)dx - \int_{\mathbb{R}^N_+} h\varphi dx \\ &- \int_{\partial\mathbb{R}^N_+}(\lambda h_1|u|^{r-2}u + h_2|u|^{q-2}u)\varphi d\sigma\end{aligned} \tag{2.7}$$

We will make use of the Mountain pass theorem in [2].

Lemma 2.(Mountain Pass Theorem) Let E be a real Banach space. Suppose $J \in C^1(E,\mathbb{R})$ satisfies (PS) condition with $J(0) = 0$. In addition,

(A_1) there are $\rho, \alpha > 0$ such that $J(u) \ge \alpha$ when $\|u\|_E = \rho$,

(A_2) there is $e \in E, \|e\|_E > \rho$ such that $J(e) < 0$.

Define

$$\Gamma = \{\gamma \in C^1([0,1], E)|\gamma(0) = 0, \gamma(1) = e\}. \tag{2.8}$$

Then

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} J(\gamma(t)) \ge \alpha \tag{2.9}$$

is a critical value of $J(u)$.
Lemma 3. Let $1 < q < p < r < p_* = \frac{p(N-1)}{N-p}$. Assume $(H_1) - (H_4)$. Then there exist $\lambda_0, m_0 > 0$ such that $J(u)$ satisfies $(A_1) - (A_2)$ in Lemma 2 provided $\lambda \in (0, \lambda_0)$ and $\|h\|_{p'} \le m_0 < c_1\lambda^{(p-1)/(r-q)}$, , where the constant $c_1 > 0$ is independent of λ.
Proof. Without loss of generality, we assume $h \not\equiv 0$. It follows from Hölder inequality and (2.4) that

$$\int_{\partial\mathbb{R}^N_+} |h_1||u|^q dx \le \|h_1\|_{L^{\alpha_1}(\partial\mathbb{R}^N_+)} \|u\|^q_{L^p(\partial\mathbb{R}^N_+)} \le S_p^{-q/p} \|h_1\|_{L^{\alpha_1}(\partial\mathbb{R}^N_+)} \|u\|^q_E \tag{2.10}$$

with $\alpha_1 = \frac{p}{p-q}$. Similarly, we have

$$\int_{\partial\mathbb{R}^N_+} |h_2||u|^r dx \le \|h_2\|_{L^\infty(\partial\mathbb{R}^N_+)} \|u\|^r_{L^r(\partial\mathbb{R}^N_+)} \le S_r^{-r/p} \|h_2\|_{L^\infty(\partial\mathbb{R}^N_+)} \|u\|^r_E \tag{2.11}$$

Moreover, it follows from Young's inequality with $\epsilon > 0$ and (2.2) that

$$\int_{\mathbb{R}^N_+} |h||u| dx \le \|h\|_{p'} \|u\|_p \le S^{-1/p} \|h\|_{p'} \|u\|_E \le \epsilon \|u\|^p_E + C_\epsilon \|h\|^{p'}_{p'} \tag{2.12}$$

Thus,

$$\begin{aligned} J(u) &\ge \frac{1}{p}\|u\|^p_E - \lambda\beta_1\|u\|^q_E - \beta_2\|u\|^r_E - \epsilon\|u\|^p_E - C_\epsilon\|h\|^{p'}_{p'} \\ &\ge \frac{1}{2p}\|u\|^p_E - \lambda\beta_1\|u\|^q_E - \beta_2\|u\|^r_E - C_\epsilon\|h\|^{p'}_{p'} \end{aligned} \tag{2.13}$$

with $0 < \epsilon \le 1/2p$ and $\beta_1 = S_p^{-q/p}\|h_1\|_{L^{\alpha_1}(\partial\mathbb{R}^N_+)}$, $\beta_2 = S_r^{-r/p}\|h_2\|_{L^\infty(\partial\mathbb{R}^N_+)}$. We now denote

$$g(z) = \lambda\beta_1 z^{q-p} + \beta_2 z^{r-p}, \quad z > 0. \tag{2.14}$$

To verify (A_1) in Lemma 2, it suffices to show that $g(z_1) < 1/2p$ for some $z_1 = \|u\|_E > 0$. Note that $g(z) \to +\infty$ whenever $z \to 0^+$ or $z \to +\infty$. Then $g(z)$ has a minimum at $z_1 > 0$. In order to find z_1, we have

$$g'(z) = \lambda\beta_1(q-p)z^{q-p-1} + \beta_2(r-p)z^{r-p-1},$$

so that

$$g'(z_1) = 0, \text{ and } z_1 = \left(\frac{\lambda\beta_1(p-q)}{\beta_2(r-p)}\right)^{1/(r-q)} \equiv \lambda^{1/(r-q)}\beta_0^{1/(r-q)} > 0.$$

where β_0 is independent of λ. Moreover, $g(z_1) < 1/2p$ implies that

$$\psi(\lambda) \equiv g(z_1) = \beta_1(r-q)(r-p)^{-1}\lambda^{(r-p)/(r-q)}\beta_0^{(q-p)/(r-q)} < 1/2p. \tag{2.15}$$

Then, we take λ_0 such that $\psi(\lambda_0) < 1/2p$ and $\psi(\lambda) < \psi(\lambda_0) < 1/2p$ for $\lambda \in (0, \lambda_0)$. Thus, it follows from (2.13) and (2.15) that there exist $m_0, \alpha > 0$ such that $J(u) \geq \alpha$ with $\lambda \in (0, \lambda_0), \|u\|_E = z_1 = \rho$ and $\|h\|_{p'} \leq m_0 < c_1\lambda^{(p-1)/(r-q)}$ for each $h \in L^{p'}(\mathbb{R}^N_+)$, where the constant $c_1 > 0$ is independent of λ. Thus (A_1) in Lemma 2 is true.

We now verify (A_2) in Lemma 2. Let $\Gamma_2 = \{x \in \partial\mathbb{R}^N_+ | h_2(x) > 0\}$. By the assumptions $(H_3) - (H_4)$, Γ_2 is a non-empty domain. Take a bounded surface $\Gamma_2^0 \subset \Gamma_2 \subset \partial\mathbb{R}^N_+$ and a ball domain B_a in $\mathbb{R}^N$ with the center in $\partial\mathbb{R}^N_+$ and the radius $a > 0$ such that $\Gamma_2^0 \subset \partial(B_a \cap \mathbb{R}^N_+)$. Choose $\varphi_1 \in C_0^2(B_a), \varphi_1 \geq 0$ and $\varphi_1 > 0$ in $\Gamma_2^0 \subset \partial(\mathrm{supp}\varphi_1 \cap \mathbb{R}^N_+) \subset \Gamma_2$. Let $\varphi_1(x) = 0$, $x \in B_a^c \equiv \overline{\mathbb{R}^N_+} \setminus (\overline{B}_a \cap \overline{\mathbb{R}^N_+})$. Then, $\int_{\partial\mathbb{R}^N_+} h_2|\varphi_1|^r d\sigma = \int_{\Gamma_2} h_2|\varphi_1|^r d\sigma \geq \int_{\Gamma_2^0} h_2|\varphi_1|^r d\sigma > 0$ and

$$J(t\varphi_1) = \frac{t^p}{p}\|\varphi_1\|_E^p - \frac{\lambda t^q}{q}\int_{\partial\mathbb{R}^N_+} h_1|\varphi_1|^q d\sigma - \frac{t^r}{r}\int_{\partial\mathbb{R}^N_+} h_2|\varphi_1|^r d\sigma - t\int_{\mathbb{R}^N_+} h\varphi_1 dx \tag{2.16}$$

and $J(t\varphi_1) \to -\infty$ as $t \to +\infty$ since $q < p < r$. Therefore, there exists t_1 large enough, such that $J(t_1\varphi_1) < 0$. Then, we take $e = t_1\varphi_1 \in E$ and $J(e) < 0$ and (A_2) in Lemma 2 is true. This completes the proof of Lemma 3. □

Lemma 4. Let $(H_1) - (H_2)$ hold. If $\{u_n\}$ is a bounded sequence in E. Then there exists a subsequence (still denoted by $\{u_n\}$) and $v \in W^{1,p}(\mathbb{R}^N_+)$ such that as $n \to \infty$,

$$\int_{\partial\mathbb{R}^N_+} h_1|u_n|^q d\sigma \to \int_{\partial\mathbb{R}^N_+} h_1|v|^q d\sigma, \quad \int_{\partial\mathbb{R}^N_+} h_2|u_n|^r d\sigma \to \int_{\partial\mathbb{R}^N_+} h_2|v|^r d\sigma. \tag{2.17}$$

Proof. Let $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$ and

$$\begin{aligned} &\Omega_k = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}^+ | \ |x| < k\}, \\ &\partial\Omega_k = B_k = \{x = (x', 0), \ x' \in \mathbb{R}^{N-1} | \ |x'| < k\}, \qquad B_k^c = \mathbb{R}^{N-1} \backslash \overline{B}_k \end{aligned} \tag{2.18}$$

with $k = 1, 2, \cdots$.

Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N_+)$, then $\{u_n\}$ is bounded in $W^{1,p}(\Omega_k)$ for $\forall k \geq 1$. By the Sobolev compact embedding theorem in the bounded domain Ω_1, $\{u_n\}$ has a subsequence $\{u_n^1\}$ which converges v_1 in $L^\alpha(\Omega_1) \cap L^r(\partial\Omega_1)$ with $1 < \alpha < p^* = \frac{pN}{N-p}$ and $1 < r < p_* = \frac{p(N-1)}{N-p}$. Let

$$u_n^1 \to v_1 \quad \text{strongly in } L^\alpha(\Omega_1) \text{ and } L^r(\partial\Omega_1).$$

Likewise, the subsequence $\{u_n^1\}$ is bounded in $W^{1,p}(\Omega_2)$ so that it has a subsequence $\{u_n^2\}$ which converges v_2 in $L^\alpha(\Omega_2) \cap L^r(\partial\Omega_2)$. Let

$$u_n^2 \to v_2 \quad \text{strongly in } L^\alpha(\Omega_2) \cap L^r(\partial\Omega_2).$$

Since $\{u_n^2\}$ is a subsequence of $\{u_n^1\}$. thus, $v_2 = v_1$ in $\overline{\Omega}_1$. Continuing this line of reasoning, we obtain a sequence v_k with the following properties:

$$\begin{aligned} &v_k \in L^\alpha(\Omega_k) \cap L^r(\partial\Omega_k), \quad k = 1, 2, \cdots \\ &v_k(x) = v_1(x), \ \text{ a.e. in } \overline{\Omega}_1 \\ &v_k(x) = v_2(x), \ \text{ a.e. in } \overline{\Omega}_2 \\ &\cdots\cdots\cdots\cdots \\ &v_k(x) = v_{k-1}(x), \ \text{ a.e. in } \overline{\Omega}_{k-1} \end{aligned}$$

It is clear that $v_k \to v$ a.e. in $\mathbb{R}^N_+$, where

$$v(x) = v_k(x), \quad x \in \overline{\Omega}_k, \quad \text{for } k = 1, 2, \cdots .$$

By a diagonal process, we take $\{u^m_m\}$ which is a subsequence of $\{u_n\}$. Thus, we have

$$u^m_m \to v \quad \text{strongly in } L^\alpha(\Omega_k) \cap L^r(\partial\Omega_k), \qquad k = 1, 2, \cdots .$$

Without loss of generality, we assume that the subsequence $\{u^m_m\}$ is $\{u_n\}$ itself. So,

$$u_n \to v, \quad \text{strongly in } L^\alpha(\Omega_k) \cap L^r(\partial\Omega_k), \;\; k = 1, 2, \cdots .$$

This implies that for $n \to \infty$,

$$u_n \to v, \quad \text{a.e.} \quad \text{in } \partial\Omega_k, \quad k = 1, 2, \cdots . \tag{2.19}$$

Now, we claim

$$\lim_{k\to\infty} \sup_{u \in E\setminus\{0\}} \frac{\|u\|_{L^r(B^c_k, |h_2|)}}{\|u\|_E} = 0. \tag{2.20}$$

Indeed, it follows Hölder inequality that

$$\|u\|^r_{L^r(B^c_k,|h_2|)} = \int_{B^c_k} |h_2||u|^r d\sigma \le (\int_{B^c_k} |h_2|^{\lambda_1} d\sigma)^{1/\lambda_1} (\int_{B^c_k} |u|^{r\lambda_2} d\sigma)^{1/\lambda_2}$$

with $\lambda_1 = \alpha_2 + \frac{r}{2\tau}, \lambda_2 = 1 + \frac{\tau}{r} < \frac{p_*}{r}$, $2\tau = p_* - r > 0$ and $\alpha_2 = \frac{p_*}{p_* - r}$. By the assumption (H_2) and (2.4), we have

$$\|u\|^r_{L^r(B^c_k,|h_2|)} \le \|h_2\|^{r/2\tau\lambda_1}_{L^\infty(\partial\mathbb{R}^N_+)} (\int_{B^c_k} |h_2|^{\alpha_2} d\sigma)^{1/\lambda_1} S^{-r/p}_{r+\tau} \|u\|^r_E. \tag{2.21}$$

The fact $h_2(x) \in L^{\alpha_2}(\partial\mathbb{R}^N_+)$ gives that

$$\lim_{k\to\infty} \|h_2\|_{L^{\alpha_2}(B^c_k)} = 0.$$

Then, (2.21) implies that

$$\frac{\|u\|_{L^r(B^c_k,|h_2|)}}{\|u\|_E} \le S^{-1/p}_{r+\tau} \|h_2\|^{1/2\tau\lambda_1}_{L^\infty(\partial\mathbb{R}^N_+)} (\int_{B^c_k} |h_2|^{\alpha_2} d\sigma)^{1/r\lambda_1} \to 0, \quad \text{as } k \to \infty. \tag{2.22}$$

This gives (2.20). Similarly, if $1 < q < p$, we have

$$\|u\|^q_{L^q(B^c_k,|h_1|)} = \int_{B^c_k} |h_1||u|^q d\sigma \le S^{-q/p}_p \|u\|^q_E \|h_1\|_{L^{\alpha_1}(B^c_k)} \tag{2.23}$$

with $\alpha_1 = \frac{p}{p-q}$. The assumption (H_2) implies

$$\lim_{k\to\infty} \|h_1\|_{L^{\alpha_1}(B^c_k)} = 0$$

and then

$$\lim_{k\to\infty}\sup_{u\in E\setminus\{0\}}\frac{\|u\|_{L^q(B_k^c,|h_1|)}}{\|u\|_E}=0. \tag{2.24}$$

In the following, we show that

$$\int_{\partial\mathbb{R}_+^N} h_2|u_n|^r d\sigma \to \int_{\partial\mathbb{R}_+^N} h_2|v|^r d\sigma \quad \text{as } n\to\infty. \tag{2.25}$$

Since the sequence $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}_+^N)$, we can assume(up to a subsequence) that $u_n \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}_+^N)$ and $\|v\|_E, \|u_n\|_E \le C_0$ for some constant $C_0>0$ and all $n\ge 1$..

By (2.20), we know that for any $\varepsilon>0$, there exists $k_\varepsilon>0$ so large that

$$\|u_n\|_{L^r(B_{k_\varepsilon}^c,|h_2|)} \le C_0^{-1}\varepsilon\|u_n\|_E \le \varepsilon, \quad \text{for } n=1,\ 2,\cdots$$

and

$$\|v\|_{L^r(B_{k_\varepsilon}^c,|h_2|)} \le \varepsilon.$$

Since the embedding $W^{1,p}(\Omega_{k_\varepsilon}) \hookrightarrow L^r(B_{k_\varepsilon})$ is compact (see [6,7]) and $h_2\in L^\infty(\partial\mathbb{R}_+^N)$, we have

$$\lim_{n\to\infty}\|u_n-v\|_{L^r(B_{k_\varepsilon})}=\lim_{n\to\infty}\|u_n-v\|_{L^r(B_{k_\varepsilon},|h_2|)}=0.$$

Thus, there exists $N_1>0$, when $n>N_1$,

$$\|u_n-v\|_{L^r(B_{k_\varepsilon})}<\varepsilon.$$

So,

$$\begin{aligned}\|u_n-v\|_{L^r(\partial\mathbb{R}_+^N,|h_2|)} &\le \|h_2\|_{L^\infty(\partial\mathbb{R}_+^N)}\|u_n-v\|_{L^r(\partial\mathbb{R}_+^N)}\\ &\le \|h_2\|_{L^\infty(\partial\mathbb{R}_+^N)}(\|u_n\|_{L^r(B_{k_\varepsilon}^c)}+\|v\|_{L^r(B_{k_\varepsilon}^c)}+\|u_n-v\|_{L^r(B_{k_\varepsilon})}\le 3\varepsilon\|h_2\|_\infty.\end{aligned} \tag{2.26}$$

This shows that $u_n\to v$ in $L^r(\partial\mathbb{R}_+^N,|h_2|)$ as $n\to\infty$.

Similarly, we can prove

$$\int_{\partial\mathbb{R}_+^N} h_1|u_n|^q d\sigma \to \int_{\partial\mathbb{R}_+^N} h_1|v|^q d\sigma \quad \text{as } n\to\infty. \tag{2.27}$$

This completes the proof of Lemma 4. □

Remark 2. By Brezis-Lieb's Lemma in [4], it follows from (2.17) that

$$\int_{\partial\mathbb{R}_+^N}|h_1||u_n-v|^q d\sigma\to 0, \quad \int_{\partial\mathbb{R}_+^N}|h_2||u_n-v|^r d\sigma\to 0, \quad \text{as } n\to\infty. \tag{2.28}$$

Lemma 5. Assume $(H_1)-(H_2)$. Then $J(u)$ defined by (2.6) satisfies (PS) condition on E.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of $J(u)$ in E, that is ,

$$J(u_n)\to c, \quad J'(u_n)\to 0 \quad \text{in } E^* \text{ as } n\to\infty. \tag{2.29}$$

We first claim that $\{u_n\}$ is bounded in E. Using (2.10), it follows that for n large enough

$$\begin{aligned} c+1+\|u_n\|_E &\geq J(u_n)-r^{-1}\langle J'(u_n),u_n\rangle \\ &= (\frac{1}{p}-\frac{1}{r})\|u_n\|_E^p+(\frac{1}{r}-\frac{1}{q})\int_{\partial\mathbb{R}_+^N}\lambda h_1|u_n|^q dx+(\frac{1}{r}-1)\int_{\mathbb{R}_+^N} hu_n dx \\ &\geq (\frac{1}{p}-\frac{1}{r})\|u_n\|_E^p+\lambda(\frac{1}{r}-\frac{1}{q})S_p^{-q/p}\|h_1\|_{L^{\alpha_1}(\partial\mathbb{R}_+^N)}\|u_n\|_E^q+(\frac{1}{r}-1)\|h\|_{p'}\|u_n\|_E. \end{aligned} \tag{2.30}$$

Since $1<q<p<r$, we conclude that $\{u_n\}$ is bounded in E. We now show that $\{u_n\}$ has a convergent subsequence in E. Denote

$$\begin{aligned} P_n=\langle J'(u_n),u_n-v\rangle &= \int_{\mathbb{R}_+^N}(|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-v)+V|u_n|^{p-2}u_n(u_n-v))dx \\ &\quad -\int_{\partial\mathbb{R}_+^N}(\lambda h_1|u_n|^{q-2}u_n+h_2|u_n|^{r-2}u_n)(u_n-v)dx-\int_{\mathbb{R}_+^N}h(u_n-v))dx. \end{aligned} \tag{2.31}$$

Then the fact $J'(u_n)\to 0$ in E^* implies that $P_n\to 0$ as $n\to\infty$. Moreover, The fact $u_n\rightharpoonup v$ in E implies that $Q_n\to 0$, where

$$Q_n=\int_{\mathbb{R}_+^N}(|\nabla v|^{p-2}\nabla v\nabla(u_n-v)+V|v|^{p-2}v(u_n-v))dx \tag{2.32}$$

Let $\|u_n\|_E\leq C_0$ for all $n\geq 1$. Then it follows from (2.4) that $\{\int_{\partial\mathbb{R}_+^N}|h_1||u_n|^q d\sigma\}$ is bounded. Moreover, we get from Hölder inequality and (2.28) that

$$\int_{\partial\mathbb{R}_+^N}|h_1||u_n|^{q-1}|u_n-v|dx\leq(\int_{\partial\mathbb{R}_+^N}|h_1||u_n-v|^q dx)^{1/q}(\int_{\partial\mathbb{R}_+^N}|h_1||u_n|^q d\sigma)^{(q-1)/q}\to 0 \tag{2.33}$$

as $n\to\infty$. Similarly, we have

$$\int_{\partial\mathbb{R}_+^N}|h_2||u_n|^{r-1}|u_n-v|dx\to 0,\quad \text{as } n\to\infty. \tag{2.34}$$

We now prove that

$$\int_{\mathbb{R}_+^N}h(x)(u_n-v)dx\to 0\quad\text{as } n\to\infty. \tag{2.35}$$

Since $h\in L^{p'}(\mathbb{R}_+^N)$, then for any $\varepsilon>0$, there exists $k_0\geq 1$ such that $k\geq k_0$

$$\int_{\Omega_k^c}|h(x)|^{p'}dx\leq\varepsilon \tag{2.36}$$

with $\Omega_k^c=\mathbb{R}_+^N\setminus\overline{\Omega}_k$ and Ω_k is given in (2.18). The compact embedding $W^{1,p}(\Omega_k)\hookrightarrow L^p(\Omega_k)$ implies that

$$\int_{\Omega_k}|u_n-v|^p dx\to 0\quad\text{as } n\to\infty. \tag{2.37}$$

Note that

$$\int_{\mathbb{R}_+^N}|h(u_n-v)|dx \leq (\int_{\Omega_k}|u_n-v|^p dx)^{\frac{1}{p}}(\int_{\Omega_k}|h|^{p'}dx)^{\frac{1}{p'}}+(\int_{\Omega_k^c}|u_n-v|^p dx)^{\frac{1}{p}}(\int_{\Omega_k^c}|h|^{p'}dx)^{\frac{1}{p'}} \quad (2.38)$$

Then (2.36)-(2.38) yield (2.35) as $n\to\infty$. Therefore, it follows from (2.31)-(2.35) that

$$T_n=\int_{\mathbb{R}_+^N}(|\nabla u_n|^{p-2}\nabla u_n-|\nabla v|^{p-2}\nabla v)\nabla(u_n-v)dx+\int_{\mathbb{R}_+^N}V(u_n-v)dx\to 0 \quad (2.39)$$

Using the standard inequality in $\mathbb{R}^N$ given by

$$\langle|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta\rangle\geq C_p|\xi-\eta|^p,\ \ p\geq 2 \quad (2.40)$$

and

$$\langle|\xi|^{p-2}\xi-|\eta|^{p-2}\eta,\xi-\eta\rangle\geq C_p|\xi-\eta|^2(|\xi|+|\eta|)^{p-2},\ \ 1<p<2, \quad (2.41)$$

we have from (2.39) that $\|u_n-v\|_E\to 0$ as $n\to\infty$. Thus $J(u)$ satisfies (PS) condition on E and we finish the proof of Lemma 5. □

3 Proofs of main results

Proof of Theorem 1. By Lemmas 3 and 5, $J(u)$ satisfies all assumptions in Lemma 2. Then there exists $u_1\in E$ such that u_1 is a solution of (1.1) by Lemma 2. Furthermore, $J(u_1)\geq\alpha>0$.

We now seek a second solution u_2. If $h\equiv 0$ in $\mathbb{R}_+^N$, we have from (H_3) that $\Gamma_1=\{x\in\partial\mathbb{R}_+^N|h_1(x)>0\}$ is a non-empty domain . Take a bounded domain $\Gamma_1^0\subset\Gamma_1$ and a ball domain B_b in $\mathbb{R}^N$ with then center in $\partial\mathbb{R}_+^N$ and the radius $b>0$ such that $\Gamma_1^0\subset\partial(B_b\cap\mathbb{R}_+^N)\subset\Gamma_1$ and meas$(B_b\cap\mathbb{R}_+^N)>0$. Choose $\varphi_2\in C_0^2(B_b),\varphi_2\geq 0$ and $\varphi_2>0$ in $\Gamma_1^0\subset\partial(\text{supp}\varphi_2\cap\mathbb{R}_+^N)\subset\Gamma_1$. Let $\varphi_2(x)=0$, $x\in B_b^c\equiv\overline{\mathbb{R}_+^N}\setminus(\overline{B}_b\cap\overline{\mathbb{R}_+^N})$. Then $\int_{\partial\mathbb{R}_+^N}h_1|\varphi_2|^q d\sigma=\int_{\Gamma_1}h_1|\varphi_2|^q d\sigma\geq\int_{\Gamma_1^0}h_1|\varphi_2|^q d\sigma>0$ and

$$\begin{aligned}J(t\varphi_2)&=\frac{t^p}{p}\|\varphi_2\|_E^p-\frac{\lambda t^q}{q}\int_{\partial\mathbb{R}_+^N}h_1|\varphi_2|^q dx-\frac{t^r}{r}\int_{\partial\mathbb{R}_+^N}h_2|\varphi_2|^r dx\\&\leq\frac{t^p}{p}\|\varphi_2\|_E^p-\frac{\lambda t^q}{q}\int_{\Gamma_1^0}h_1|\varphi_2|^q d\sigma-\frac{t^r}{r}\int_{\partial\mathbb{R}_+^N}h_2|\varphi_2|^r dx<0\end{aligned} \quad (3.1)$$

for small $t>0$. If $h\neq 0$, we choose $\varphi_3\in E$ such that $\int_\Omega h\varphi_3 dx>0$ and then

$$J(t\varphi_3)=\frac{t^p}{p}\|\varphi_3\|_E^p-\frac{\lambda t^q}{q}\int_{\partial\mathbb{R}_+^N}h_1|\varphi_3|^q dx-\frac{t^r}{r}\int_{\partial\mathbb{R}_+^N}h_2|\varphi_3|^r dx-t\int_{\mathbb{R}_+^N}h\varphi_3 dx<0 \quad (3.2)$$

for small $t>0$. Then, for any open ball $B_\tau\subset E$, it follows from (3.1) and (3.2) that,

$$-\infty<c_\tau=\inf_{\overline{B}_\tau}J(u)<0. \quad (3.3)$$

where B_τ is a open ball in E centered at the origin with radius $\tau > 0$. Thus,

$$c_\rho = \inf_{u \in \overline{B}_\rho} J(u) < 0 \quad \text{and} \quad \inf_{u \in \partial B_\rho} J(u) > 0, \tag{3.4}$$

where $\rho > 0$ is given in Lemma 3. Letting $\varepsilon_n \downarrow 0$ such that

$$0 < \varepsilon_n < \inf_{u \in \partial B_\rho} J(u) - \inf_{u \in B_\rho} J(u). \tag{3.5}$$

Then, by Ekeland's variational principle in [11], there exists $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_\rho \leq J(u_n) < c_\rho + \varepsilon_n \tag{3.6}$$

and

$$J(u_n) < J(u) + \varepsilon_n \|u_n - u\|_E, \quad \forall u \in \overline{B}_\rho,\ u \neq u_n. \tag{3.7}$$

Then it follows from (3.4)-(3.6) that

$$J(u_n) < c_\rho + \varepsilon_n \leq \inf_{u \in B_\rho} J(u) + \varepsilon_n < \inf_{u \in \partial B_\rho} J(u), \tag{3.8}$$

so that $u_n \in B_\rho$. We now consider the functional $F : \overline{B}_\rho \to \mathbb{R}$ given by

$$F(u) = J(u) + \varepsilon_n \|u - u_n\|_E, \quad u \in \overline{B}_\rho. \tag{3.9}$$

Then (3.7) shows that $F(u_n) < F(u), u \in \overline{B}_\rho,\ u \neq u_n$ and thus u_n is a strict local minimum of $F(u)$. Moreover,

$$t^{-1}(F(u_n + tv) - F(u_n)) \geq 0, \quad \text{for small } t > 0 \text{ and } \forall v \in B_1. \tag{3.10}$$

Hence,

$$t^{-1}(J(u_n + tv) - J(u_n)) + \varepsilon_n \|v\|_E \geq 0. \tag{3.11}$$

Passing to the limit as $t \to 0^+$, it follows that

$$\langle J'(u_n), v \rangle + \varepsilon_n \|v\|_E \geq 0, \quad \forall v \in B_1. \tag{3.12}$$

Replacing v in (3.12) by $-v$, we get

$$-\langle J'(u_n), v \rangle + \varepsilon_n \|v\|_E \geq 0, \quad \forall v \in B_1. \tag{3.13}$$

So that $\|J'(u_n)\| \leq \varepsilon_n$. Therefore, there is a sequence $\{u_n\} \subset B_\rho$ such that $J(u_n) \to c_\rho < 0$, and $J'(u_n) \to 0$ in E^* as $n \to \infty$. By Lemma 5, $\{u_n\}$ has a convergent subsequence in E, still denoted by $\{u_n\}$, such that $u_n \to u_2$ in E. Thus u_2 is a solution of (1.1) with $J(u_2) < 0$. Then the proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. We use the Ekeland's variation principle to prove Theorem 2. When $h_2 = 0$, we have from (2.13) that

$$J(u) \geq \frac{1}{2p} \|u\|_E^p - \lambda S_p^{-q/p} \|h_1\|_{L^{\alpha_1}(\partial \mathbb{R}^N_+)} \|u\|_E^q - C_\epsilon \|h\|_{p'}^{p'}. \tag{3.14}$$

Since $1 < q < p$, then for any $\lambda > 0$ and $h \in L^{p'}(\mathbb{R}^N_+)$, it follows that there exist $\rho > 0, \alpha > 0$ such that $J(u) \geq \alpha$ with $\|u\|_E = \rho$. On the other hand, we get from (3.1)-(3.2) that $J(t\varphi_3) < 0$ for small $t > 0$. Then we have

$$c_\rho = \inf_{u \in \overline{B}_\rho} J(u) < 0 \quad \text{and} \quad \inf_{u \in \partial B_\rho} J(u) > 0. \tag{3.15}$$

Using the Ekeland's variational principle as in the proof of Theorem 1, we obtain that there admits a solution for problem (1.2). This completes the proof of Theorem 2. □

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