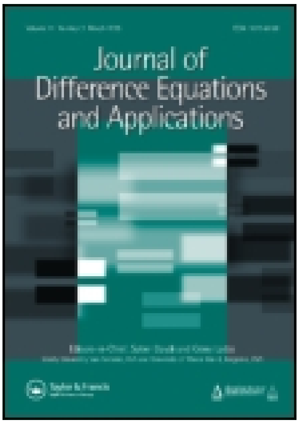

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A projected discrete Gronwall's inequality with sub-exponential growth

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A projected discrete Gronwall's inequality with sub-exponential growth

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Along with the increasing interest in (h, k)-dichotomy, more attentions are paid to sub-exponential growth in research of asymptotic behaviours. In this paper, we generalize a projected discrete Gronwall's inequality given in [J. Differ. Equ. Appl. 10 (2004), 661–689] to a general one, which may include both terms of sub-exponential growth inside the summation and non-monotonic terms outside the summation. We demonstrate our results with concrete non-monotonic functions and sub-exponential functions. We apply our results to estimating bounded solutions of a non-linear difference equation with an (h, k)-dichotomy.

Keywords: Gronwall's inequality; difference equation; sub-exponential growth; non-monotonicity; (h, k)-dichotomy

AMS (2000) Classification: 34A40; 34D09; 39A11

1. Introduction

Discrete inequalities of Gronwall type and related sum-difference inequalities play a fundamental role in the study of difference equations. Required in the discussion on existence, uniqueness, boundedness, stability, invariant manifolds and other dynamical behaviours of solutions for difference equations, in recent decades a great progress has been made in the theory of discrete inequalities (see e.g. [1–3,5–9,12,13] and references therein). A basic one of those known results is the discrete Gronwall's inequality

$$u(n) \leq p(n) + q(n)\sum_{k=k_0}^{n-1} f(k)u(k), \quad n \geq k_0. \tag{1.1}$$

As shown in Ref. ([1], Theorem 4.1.1, p. 182), the unknown function $u(n)$ in (1.1) is estimated by $u(n) \leq p(n) + q(n)\sum_{k=k_0}^{n-1} p(k)f(k)\prod_{\tau=k+1}^{n-1}(1 + q(\tau)f(\tau))$ for all $n \geq k_0$.

Recently, in order to investigate invariant manifolds for functional difference equations, Matsunaga and Murakami [5] discussed in the Appendix the discrete inequality

$$u(n) \leq eb^n + p\sum_{s=0}^{n-1} b^{n-s-1}u(s) + q\sum_{s=n}^{\infty} c^{-n+s+1}u(s), \quad n \in \mathbb{Z}_+, \tag{1.2}$$

where e, b, c, p and q are non-negative constants and $0 < b < 1$, $0 < c < 1$. Being a discrete analogue of the projected Gronwall's inequality, considered in Ref. ([4],

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Lemma 6.2, p. 110), inequality (1.2) can be regarded as a discrete projected Gronwall's inequality.

Along with the increasing interest in (h, k)-dichotomy [6,10,11], more attentions are paid to sub-exponential growth in research of asymptotic behaviours. On the basis of the projected Gronwall's inequality considered in Ref. ([4], Lemma 6.2, p. 110), efforts (see Refs [14–16]) have been made to extend the known results to sub-exponential growth, non-monotonicity and lower regularity of given functions. Obviously, such efforts to the discrete inequality (1.2) are also interesting to difference equations.

In this paper, we generalize (1.2), a discrete inequality including functions of exponential growth only, to the discrete inequality

$$u(n) \leq a(n) + \sum_{s=s_0}^{n-1} b(n,s)u(s) + \sum_{s=n}^{\infty} c(n,s)u(s), \quad n \geq s_0, \tag{1.3}$$

where s_0 is a fixed number in $\mathbb{Z}_+$, $a(n)$, $b(n, s)$ and $c(n, s)$ are non-negative functions, $a(n)$ may be non-monotonic and $b(n, s)$ and $c(n, s)$ may grow sub-exponentially. We first estimate u in (1.3) in Theorem 1 under some basic hypotheses, which are much weaker than the corresponding discrete analogies in Ref. [16] and relax $b(n, s)$ and $c(n, s)$ to be of general form in two variables. Since these basic hypotheses do not restrict $b(n, s)$ to the form of variable separation, our result Theorem 1 is proved in a different idea from Ref. [16]. If $b(n, s)$ is additionally bounded by a function in the form of variable separation, we can use the same idea as in Ref. [16] to give an estimate of u in Theorem 2, which is much easier to calculate than the estimate given in Theorem 1. We demonstrate our results with concrete non-monotonic functions and sub-exponential functions. Finally, we apply our results to a non-linear difference equation with an (h, k)-dichotomy.

Throughout this paper, we use the following notations $\mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$ and $\mathbb{R}_+ := \{n \in \mathbb{R} : n \geq 0\}$. For a function $F(k)$ defined on $\mathbb{Z}_+$, it is a convention [1] to set $\sum_{s=k_1}^{k_2} F(s) = 0$ and $\prod_{s=k_1}^{k_2} F(s) = 1$ if $k_1, k_2 \in \mathbb{Z}_+$ and $k_1 > k_2$, i.e. empty sum takes the value 0 and empty product takes the value 1. As usual, let Δ denotes the forward difference operator, i.e. $\Delta F(n) = F(n+1) - F(n)$.

2. Main result

Consider inequality (1.3) and suppose that

(H_1): $a : \mathbb{Z}_+ \to \mathbb{R}_+$ is bounded and $a_* := \inf_{n \geq s_0} a(n)$,

(H_2): the functions $b(n, s)$ and $c(n, s)$ are defined for all integers $0 \leq s \leq n < \infty$ and for all integers $0 \leq n \leq s < \infty$, respectively, and both are non-negative and

(H_3): $\eta(n) := \sum_{s=s_0}^{n-1} b(n,s) + \sum_{s=n}^{\infty} c(n,s)$ is well defined for all $n \geq s_0$ and $\eta := sup_{n \geq s_0} \eta(n) < 1$.

Our main result is the following:

THEOREM 1. *Suppose that* (H_1–H_3) *hold. Then any non-negative bounded function* u *satisfying* (1.3) *is estimated by*

$$u(n) \leq \frac{\tilde{a}(n)}{1-\eta} + \frac{1}{(1-\eta)^2} \sum_{s=s_0}^{n-1} \left(\prod_{\tau=s+1}^{n-1} \xi(\tau) \right) (\tilde{a}(s) - a_*) \tilde{b}(s+1, s), \quad \forall n \geq s_0,$$

where $\tilde{a}(n) := \sup_{\tau \ge n} a(\tau)$, $\tilde{b}(n,s) := \sup_{\tau \ge n} b(\tau, s)$, $\xi(n) := r(n) + \tilde{b}(n+1,n)/(1-\eta)$, $r(n) := \max_{s_0 \le s \le n} B(n,s)$, *and*

$$B(n,s) := \begin{cases} \tilde{b}(n+1,s)/\tilde{b}(n,s), & \tilde{b}(n,s) \neq 0, \\ 0, & \tilde{b}(n,s) = 0. \end{cases}$$

Remark 1. When $a(n) = e\rho_1^n$, $b(n,s) = \alpha\rho_1^{n-s-1}$ and $c(n,s) = \beta\rho_2^{-n+s+1}$, inequality (1.3) is just what Matsunaga and Murakami considered in Ref. [5]. In this case, it is obvious that $r(n) = \rho_1$ for all $n \ge s_0$ when $\alpha \neq 0$ and $r(n) = 0$ for all $n \ge s_0$ when $\alpha = 0$. Hence their result on inequality (1.2), where $a(n)$, $b(n,s)$ and $c(n,s)$ all grow exponentially, is a special case in our Theorem 1.

Before proving the theorem, we need the following lemma.

Lemma 1. *Suppose that* $(H_1 – H_3)$ *hold. Then for an arbitrary bounded solution* u *of inequality* (1.3) *there exists a non-negative bounded solution* v *of the inequality*

$$v(n) \le a(n) - a_* + \sum_{s=s_0}^{n-1} b(n,s)v(s) + \sum_{s=n}^{\infty} c(n,s)v(s), \quad \forall n \ge s_0, \tag{2.4}$$

such that $u(n) - v(n) \le a_*/(1-\eta)$ *for all* $n \ge s_0$.

Proof. Let u be an arbitrary bounded solution of (1.3) on $\mathbb{Z}_+(s_0) := \{n \in \mathbb{Z}_+ : n \ge s_0\}$. Let $S := \{n \in \mathbb{Z}_+(s_0) : \ u(n) \ge a_*/(1-\eta)\}$, which is allowed to be empty in some cases. Define

$$v(n) := \begin{cases} u(n) - a_*/(1-\eta), & n \in S, \\ 0, & n \in \mathbb{Z}_+(s_0) \backslash S, \end{cases} \tag{2.5}$$

which is bounded and non-negative. It follows that

$$u(n) - v(n) \le \frac{a_*}{1-\eta}, \quad \forall n \ge s_0, \tag{2.6}$$

the same inequality as in the result of our lemma. In the sequel, we only need to prove that v satisfies inequality (2.4). From (H_3), we see that the infinite sums $\sum_{s=s_0}^{n-1} b(n,s)v(s)$ and $\sum_{s=n}^{\infty} c(n,s)v(s)$ in (2.4) are both well defined for all $n \ge s_0$. Inequality (2.4) holds naturally for $n \in \mathbb{Z}_+(s_0)\backslash S$ because $v(n) \equiv 0$ by (2.5). For $n \in S$, substituting (2.6) into (1.3), we obtain

$$\begin{aligned} v(n) + \frac{a_*}{1-\eta} &\le a(n) + \sum_{s=s_0}^{n-1} b(n,s)\left(v(s) + \frac{a_*}{1-\eta}\right) + \sum_{s=n}^{\infty} c(n,s)\left(v(s) + \frac{a_*}{1-\eta}\right) \\ &\le a(n) + \sum_{s=s_0}^{n-1} b(n,s)v(s) + \sum_{s=n}^{\infty} c(n,s)v(s) + \frac{a_*\eta}{1-\eta}, \quad \forall n \ge s_0, \end{aligned}$$

from which we obtain the same as in (2.4). The proof is completed. □

Proof of Theorem 1. Since $(H_1 – H_3)$ hold, by Lemma 1 we see that there is a non-negative bounded solution v of inequality (2.4) such that

$$u(n) \leq \frac{a_*}{1-\eta} + v(n), \quad \forall n \geq s_0. \tag{2.7}$$

Thus the estimation of u is reduced to the estimation of v. Let

$$w(n) := \sup_{s \geq n} v(s). \tag{2.8}$$

Then w is non-increasing and $w(n) \geq v(n)$. On the other hand, let $\varepsilon > 0$ be given. Then for every $n \geq s_0$ there exists an integer $n_\varepsilon \geq n$ such that $w(n) - \varepsilon < v(n_\varepsilon)$. Thus, from (2.4) we get

$$\begin{aligned} w(n) - \varepsilon < v(n_\varepsilon) &\leq a(n_\varepsilon) - a_* + \sum_{s=s_0}^{n_\varepsilon - 1} b(n_\varepsilon, s) w(s) + \sum_{s=n_\varepsilon}^{\infty} c(n_\varepsilon, s) w(s) \\ &\leq a(n_\varepsilon) - a_* + \sum_{s=s_0}^{n_\varepsilon - 1} b(n_\varepsilon, s) w(s) + w(n) \left(\sum_{s=n}^{n_\varepsilon - 1} b(n_\varepsilon, s) + \sum_{s=n_\varepsilon}^{\infty} c(n_\varepsilon, s) \right) \\ &\leq a(n_\varepsilon) - a_* + \sum_{s=s_0}^{n_\varepsilon - 1} b(n_\varepsilon, s) w(s) + \eta(n_\varepsilon) w(n) \\ &\leq \tilde{a}(n) - a_* + \sum_{s=s_0}^{n_\varepsilon - 1} \tilde{b}(n, s) w(s) + \eta w(n), \end{aligned} \tag{2.9}$$

where we note that $\tilde{b}(n,s)$ is well defined for all integers $s_0 \leq s \leq n < \infty$ because the assumption (H_3) implies that $\sup_{n \geq s} b(n,s) < \infty$ for $s \in \mathbb{Z}_+$. Clearly, it follows from (2.9) that

$$w(n) \leq \frac{\tilde{a}(n) - a_*}{1-\eta} + \frac{1}{1-\eta} \sum_{s=s_0}^{n-1} \tilde{b}(n,s) w(s), \tag{2.10}$$

since $\varepsilon > 0$ is arbitrary and $0 \leq \eta < 1$.

In order to estimate $w(n)$ from (2.10), we cannot use the idea as for (2.16) and (2.17) in Ref. [16] because $\tilde{b}(n,s)$ may not be separable. Let

$$z(n) = \frac{1}{1-\eta} \sum_{s=s_0}^{n-1} \tilde{b}(n,s) w(s). \tag{2.11}$$

Inequality (2.10) can be rewritten as

$$w(n) \leq \frac{\tilde{a}(n) - a_*}{1-\eta} + z(n). \tag{2.12}$$

From (2.11) we can calculate

$$\begin{aligned}\Delta z(n) = z(n+1) - z(n) &= \frac{1}{1-\eta}\sum_{s=s_0}^{n-1}(\tilde{b}(n+1,s) - \tilde{b}(n,s))w(s) + \frac{\tilde{b}(n+1,n)}{1-\eta}w(n)\\ &\le \frac{r(n)-1}{1-\eta}\sum_{s=s_0}^{n-1}\tilde{b}(n,s)w(s) + \frac{\tilde{b}(n+1,n)}{1-\eta}w(n)\\ &\le (r(n)-1)z(n) + \frac{\tilde{b}(n+1,n)}{1-\eta}\left(\frac{\tilde{a}(n)-a_*}{1-\eta} + z(n)\right), \end{aligned} \tag{2.13}$$

where we note the definition of $r(n)$ given in the theorem and we apply inequality (2.12). Re-arranging terms in (2.13), we get

$$z(n+1) - \xi(n)z(n) \le \frac{(\tilde{a}(n) - a_*)\tilde{b}(n+1,n)}{(1-\eta)^2}. \tag{2.14}$$

where $\xi(n) := r(n) + \tilde{b}(n+1,n)/(1-\eta)$. Define

$$\tilde{\xi}_{\varepsilon_1}(n) = \xi(n) + \varepsilon_1,$$

where $\varepsilon_1 > 0$ is an arbitrary constant. Since $z(n)$ is non-negative, from inequality (2.14) we get

$$z(n+1) - \tilde{\xi}_{\varepsilon_1}(n)z(n) \le \frac{(\tilde{a}(n) - a_*)\tilde{b}(n+1,n)}{(1-\eta)^2}. \tag{2.15}$$

Note that $\tilde{\xi}_{\varepsilon_1}(n) > 0$ for all $n \ge s_0$ because $\xi(n)$ is non-negative and ε_1 is positive. It is reasonable to multiply (2.15) by $\prod_{s=s_0}^{n}\tilde{\xi}_{\varepsilon_1}(s)^{-1}$ to get

$$\Delta\left(\prod_{s=s_0}^{n-1}\tilde{\xi}_{\varepsilon_1}(s)^{-1}z(n)\right) \le \left(\prod_{s=s_0}^{n}\tilde{\xi}_{\varepsilon_1}(s)^{-1}\right)\frac{(\tilde{a}(n) - a_*)\tilde{b}(n+1,n)}{(1-\eta)^2}.$$

Summing up the above inequality from $n = s_0$ to $n = N-1$ and noting that $z(s_0) = 0$, which is observed from (2.11) and the conventions shown in the end of the Introduction, we get

$$z(N) \le \frac{1}{(1-\eta)^2}\sum_{n=s_0}^{N-1}\left(\prod_{\tau=n+1}^{N-1}\tilde{\xi}_{\varepsilon_1}(\tau)\right)(\tilde{a}(n) - a_*)\tilde{b}(n+1,n). \tag{2.16}$$

Passing to the limit as $\varepsilon_1 \to +0$ in (2.16), by the definition of $\tilde{\xi}_{\varepsilon_1}(n)$, we get

$$z(N) \le \frac{1}{(1-\eta)^2}\sum_{n=s_0}^{N-1}\left(\prod_{\tau=n+1}^{N-1}\xi(\tau)\right)(\tilde{a}(n) - a_*)\tilde{b}(n+1,n). \tag{2.17}$$

Thus, we obtain an estimate of $w(n)$ from (2.12) and (2.17) directly. Since $v(n) \le w(n)$, we finally obtain the result of the theorem from (2.7). This completes the proof. □

If $b(n,s)$ satisfies an additional condition

(A): $b(n,s) \le p_1(n)p_2(s)$ for all integers $0 \le s \le n < \infty$, where $p_1(n)$ and $p_2(n)$ are both non-negative functions defined on $\mathbb{Z}_+$ and $p_1(n)$ is non-increasing,

the estimation will be much easier.

Theorem 2. *Suppose that* $(H_1–H_3)$ *hold and* $b(n, s)$ *satisfies* (A). *Then any non-negative bounded function u satisfying* (1.3) *is estimated by*

$$u(n) \leq \frac{\tilde{a}(n)}{1-\eta} + \frac{p_1(n)}{(1-\eta)^2}\left[\sum_{s=s_0}^{n-1}(\tilde{a}(s)-a_*)p_2(s)\prod_{\tau=s+1}^{n-1}\left(1+\frac{p_1(\tau)p_2(\tau)}{1-\eta}\right)\right]$$

for all $n \geq s_0$, *where* $\tilde{a}(n) := \sup_{s\geq n} a(s)$.

Proof. Since $(H_1–H_3)$ hold, by Lemma 1, there is a non-negative bounded solution v of inequality (2.4) such that

$$u(n) \leq \frac{a_*}{1-\eta} + v(n), \quad \forall n \geq s_0. \tag{2.18}$$

In order to estimate v in (2.18), let $w(n) := \sup_{s\geq n} v(s)$ and $\tilde{b}(n, s) := \sup_{\tau\geq n} b(\tau, s)$. Similarly to the proof of Theorem 1, we can deduce that $w(n)$ satisfies inequality (2.10). By the assumption (A),

$$\tilde{b}(n,s) \leq \left\{\sup_{\tau\geq n} p_1(\tau)\right\} p_2(s) = p_1(n)p_2(s)$$

since p_1 is non-increasing. It follows from (2.10) that

$$w(n) \leq \frac{\tilde{a}(n)-a_*}{1-\eta} + \frac{p_1(n)}{1-\eta}\sum_{s=s_0}^{n-1} p_2(s)w(s). \tag{2.19}$$

Thus, we can apply the discrete version of the well-known Gronwall's inequality, shown in Ref. ([1], Theorem 4.1.1, p. 182) and usually called the discrete Gronwall's inequality, to inequality (2.19) and obtain

$$w(n) \leq \frac{\tilde{a}(n)-a_*}{1-\eta} + \frac{p_1(n)}{(1-\eta)^2}\left[\sum_{s=s_0}^{n-1}(\tilde{a}(s)-a_*)p_2(s)\prod_{\tau=s+1}^{n-1}\left(1+\frac{p_1(\tau)p_2(\tau)}{1-\eta}\right)\right].$$

This gives the result by the definition of $w(n)$ and the relation (2.18) between $u(n)$ and $v(n)$. This completes the proof. □

Remark 2. When functions $b(n, s)$ and $c(n, s)$ in (1.3) is of the special forms $b(n-s-1)$ and $c(-n+s+1)$, respectively and

$$b(n) \leq b(0)\rho^n, \quad \forall n \geq 0,$$

with $\rho \in (0, 1)$, the conditions $(H_1–H_3)$ and (A) are satisfied, where $p_1(n) = b(0)p^{\,n}$ and $p_2(n) = \rho^{-n-1}$. By our Theorem 2, we obtain a discrete result in a similar form to the result in Ref. [16] for integral inequalities. Actually, using the same idea, one can generalize [16] to a case with real functions of two variables inside the integrals, where $b(t-s)$ is replaced with $b(t, s)$ in variable separation but of sub-exponential growth and $c(s-t)$ is replaced with the general $c(t, s)$ of two variables.

3. Sub-exponential examples

In this section, we demonstrate our theorems with non-monotonic functions and sub-exponential functions.

Example 1. Consider inequality (1.3) with functions

$$a(n) = \frac{1+\cos n\pi}{n+1} + \lambda, \quad b(n,s) = \frac{\mu(4s+1+\cos(n+1)\pi)}{4n^2},$$

and $c(n,s) = c^{-n+s+1}(1-\cos 2(-n+s+1)\alpha)$, where λ, μ, α and c are positive constants such that $0 < c < 1$ and

$$\frac{\mu}{2} + \frac{c}{1-c} - \frac{c\cos 2\alpha - c^2}{1+c^2-2c\cos 2\alpha} < 1. \tag{3.20}$$

Obviously, $a(n)$ and $b(n,s)$ are both sub-exponential, $b(n, s)$ is not of variable separation, and none of $a(n)$, $b(n, s)$ and $c(n,s)$ is monotone. Note that conditions (H_1) and (H_2) are satisfied because $a(n) \le \lambda + 2/3$ for $n \ge 1$, $a_* := \inf_{n\ge1} a(n) = \lambda$ and the non-negative functions $b(n,s)$ and $c(n,s)$ are well defined for all integers $1 \le s \le n < \infty$ and for all integers $1 \le n \le s < \infty$, respectively.

In order to verify (H_3), we first note that

$$\sup_{n\ge0} \sum_{s=1}^{n-1} b(n,s) = \frac{\mu}{2}. \tag{3.21}$$

In fact, by the assumption of $b(n, s)$,

$$\sum_{s=1}^{n-1} b(n,s) = \begin{cases} \mu(n-1)/(2n), & \text{as } n = 2k \text{ and } k \in \mathbb{Z}_+, \\ \mu(n^2-1)/(2n^2), & \text{as } n = 2k+1 \text{ and } k \in \mathbb{Z}_+. \end{cases} \tag{3.22}$$

On the other hand,

$$\sum_{s=n}^{\infty} c(n,s) = \sum_{s=n}^{\infty} c^{-n+s+1} - \lim_{m\to\infty} S(n,m), \tag{3.23}$$

where $S(n,m) := \sum_{s=n}^{m} c^{-n+s+1}\cos 2(-n+s+1)\alpha$. Since

$$\begin{aligned} 2c\cos(2\alpha)S(n,m) &= \sum_{s=n}^{m} 2c^{-n+s+2}\cos 2\alpha \cos 2(-n+s+1)\alpha \\ &= \sum_{s=n}^{m} c^{-n+s+2}[\cos 2(-n+s+2)\alpha + \cos 2(-n+s)\alpha] \\ &= [c^{-n+m+2}\cos 2(-n+m+2)\alpha + S(n,m) - c\cos 2\alpha] + [c^2 + c^2S(n,m) \\ &\quad - c^{-n+m+3}\cos 2(-n+m+1)\alpha], \end{aligned}$$

we have

$$S(n,m) = \frac{c^{-n+m+2}[c\cos 2(-n+m+1)\alpha - \cos 2(-n+m+2)\alpha] + c\cos 2\alpha - c^2}{1+c^2-2c\cos 2\alpha}.$$

It follows from (3.23) that

$$\sum_{s=n}^{\infty} c(n,s) = \frac{c}{1-c} - \frac{c\cos 2\alpha - c^2}{1+c^2-2c\cos 2\alpha}. \tag{3.24}$$

Thus, by (3.22), (3.24) and (3.20) we get

$$\eta := \sup_{n\geq 0}\left\{\sum_{s=1}^{n-1} b(n,s) + \sum_{s=n}^{\infty} c(n,s)\right\} = \frac{\mu}{2} + \frac{c}{1-c} - \frac{c\cos 2\alpha - c^2}{1+c^2-2c\cos 2\alpha} < 1,$$

which verifies (H_3). Furthermore, $b(n,s) \leq p_1(n)p_2(s)$, where $p_1(n) = 1/n^2, p_2(s) = \mu(s+1/2)$ i.e., condition (A) is satisfied. Thus we can apply Theorem 2 to obtain

$$\begin{aligned} u(n) \leq{}& \frac{(n+1)\lambda+2}{(n+1)(1-\eta)} + \frac{\mu}{n^2(1-\eta)^2}\left\{\sum_{s=1}^{(n-2)/2}\left(\frac{4s+1}{2s+1}\right)\prod_{\tau=2s+1}^{n-1}\left(1+\frac{(2\tau+1)\mu}{2\tau^2(1-\eta)}\right)\right. \\ &\left. + \sum_{s=0}^{(n-2)/2}\left(\frac{4s+3}{2s+3}\right)\prod_{\tau=2s+2}^{n-1}\left(1+\frac{(2\tau+1)\mu}{2\tau^2(1-\eta)}\right)\right\}, \quad \text{for even}\, n, \end{aligned} \tag{3.25}$$

$$\begin{aligned} u(n) \leq{}& \frac{(n+2)\lambda+2}{(n+2)(1-\eta)} + \frac{\mu}{n^2(1-\eta)^2}\left\{\sum_{s=1}^{(n-1)/2}\left(\frac{4s+1}{2s+1}\right)\prod_{\tau=2s+1}^{n-1}\left(1+\frac{(2\tau+1)\mu}{2\tau^2(1-\eta)}\right)\right. \\ &\left. + \sum_{s=0}^{(n-3)/2}\left(\frac{4s+3}{2s+3}\right)\prod_{\tau=2s+2}^{n-1}\left(1+\frac{(2\tau+1)\mu}{2\tau^2(1-\eta)}\right)\right\}, \quad \text{for odd}\, n, \end{aligned} \tag{3.26}$$

since

$$\tilde{a}(n) := \sup_{\tau\geq n} a(\tau) = \begin{cases} \lambda + 2/(n+1), & n \text{ is even}, \\ \lambda + 2/(n+2), & n \text{ is odd}. \end{cases}$$

Although Theorem 2 requires an additional condition (A), the resulted inequality is easier to be calculated and the proof is simpler than that of Theorem 1. For a precise estimate we usually enlarge $b(n,s)$ with $p_1(n), p_2(s)$ in (A) as exact as possible. In our Example 1, $\limsup_{n\to\infty} b(n,s)/p_1(n)p_2(s) = \limsup_{n\to\infty}(4s+1+\cos(n+1)\pi)/(4s+2) = 1$.

Example 2. Consider inequality (1.3) with functions

$$a(n) = \frac{1}{2^{2n}} + \lambda, \quad b(n,s) = \frac{\sigma_1 b^{2^n+2^{n-s}-2^s}}{1-b^{2^{n-s+1}}}, \quad c(n,s) = \frac{\sigma_2 c^{2^{s-n}}}{1-c^{2^{s-n+1}}},$$

where $0 < b < 1$, $0 < c < 1$ and λ, σ_1, σ_2 are positive constants such that

$$\max_{0\le n\le K} b(n,s) + \frac{\sigma_2 c}{1-c} < 1 \tag{3.27}$$

with $K := \max\{0, 1 + (\ln\ln 2 - \ln|\ln b|)/\ln 2\}$. Obviously, neither $b(n, s)$ nor $c(n, s)$ is of variable separation. Note that our this case includes stronger growth than exponential growth, because $\lim_{n\to\infty} b(n+1,s)/b(n,s) = 0$, i.e., the growth rate of $b(n, s)$ is stronger than exponential one. Thus those functions with stronger growth can be enlarged by functions of exponential growth and the known result in Ref. [5] can be applied to this example for an estimate of $u(n)$. However, if applying our Theorem 1, we can obtain a better estimate for $u(n)$ because our theorems are not restricted to the case of exponential growth and the stronger growth can be considered in the estimate.

In order to apply our Theorem 1, we note that $a(n) \le \lambda + 1/2$ for $n \ge 0$ and $a_* := \inf_{n\ge 0} a(n) = \lambda$, i.e. (H_1) holds. Moreover, $b(n, s)$ and $c(n,s)$ are both non-negative and are well defined for all integers $0 \le s \le n < \infty$ and for all integers $0 \le n \le s < \infty$, respectively, i.e. (H_2) holds. In order to verify (H_3), we first note that

$$\sum_{k=0}^{n} \frac{c^{2^k}}{1-c^{2^{k+1}}} = \sum_{k=0}^{n}\left(\frac{1}{1-c^{2^k}} - \frac{1}{1-c^{2^{k+1}}}\right) = \frac{1}{1-c} - \frac{1}{1-c^{2^{n+1}}},$$

implying that $\sum_{s=n}^{\infty} c(n,s) = \sigma_2 c/(1-c)$. On the other hand, although it is hard to calculate the sum $\varsigma(n) := \sum_{s=0}^{n-1} b(n,s)$ directly, we know

$$\begin{aligned}\varsigma(n+1) &= \sigma_1 \sum_{s=0}^{n-1} \frac{b^{2^{n+1}+2^{n+1-s}-2^s}}{1-b^{2^{n-s+2}}} + \sigma_1 \frac{b^{2^{n+1}+2-2^n}}{1-b^{2^2}} \\ &\le \left(\frac{b^{2^n+2}}{1+b^4} + b^{2^{n-1}}\right)\varsigma(n) \le 2b^{2^{n-1}}\varsigma(n),\end{aligned} \tag{3.28}$$

implying that $\varsigma(n+1) < \varsigma(n)$ for $n > K$, the number defined just after (3.27). Therefore, $\sup_{n\ge 0}\varsigma(n) = \max_{0\le n\le K}\varsigma(n)$ and, from the assumption (3.27),

$$\eta := \sup_{n\ge 0}\left\{\sum_{s=0}^{n-1} b(n,s) + \sum_{s=n}^{\infty} c(n,s)\right\} = \max_{0\le n\le K}\varsigma(n) + \frac{\sigma_2 c}{1-c} < 1,$$

i.e. (H_3) is also satisfied. By Theorem 1 we conclude that

$$\begin{aligned} u(n) \le{}& \frac{1+2^{2n}\lambda}{2^{2n}(1-\eta)} + \frac{\sigma_1 b^{2^n+2}}{(1-\eta)^2(1-b^4)}\left[\frac{b}{1+b^2} + \frac{\sigma_1 b^2}{(1-\eta)(1-b^4)}\right]^{n-1} \\ &\times \left\{\sum_{s=0}^{n-1} (2b)^{-2^s}\left[\frac{b}{1+b^2} + \frac{\sigma_1 b^2}{(1-\eta)(1-b^4)}\right]^{-s}\right\}, \quad \forall n \ge 0, \end{aligned} \tag{3.29}$$

where we notice that $r(n) = b^{2^n+1}/(1 + b^2)$ because $\tilde{b}(n,s) = b(n,s)$ and

$$b(n+1,s) = \sigma_1 \frac{b^{2^{n+1}+2^{n+1-s}-2^s}}{1-b^{2^{n+2-s}}} = \frac{b^{2^n} b^{2^{n-s}}}{1+b^{2^{n+1-s}}}\left(\sigma_1 \frac{b^{2^n+2^{n-s}-2^s}}{1-b^{2^{n+1-s}}}\right) = \frac{b^{2^n}}{1/b^{2^{n-s}}+b^{2^{n-s}}} b(n,s).$$

4. Application to difference equations

In this section, we apply our result to estimate solutions for the non-linear difference system

$$x(n+1) = A(n)x(n) + f(n, x(n)), \tag{4.30}$$

where $A(n)$ is a $d \times d$ matrix valued function on $\mathbb{Z}_+$ and $f(n,x)$ is an $\mathbb{R}^d$ valued function on $\mathbb{Z}_+ \times \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and a $d \times d$ matrix A denote by $|x|$ and $|A|$ its Euclidean norm and the corresponding norm, respectively. Suppose that

(S1): The linear system

$$x(n+1) = A(n)x(n) \tag{4.31}$$

admits an $(h,\, k)$-dichotomy for $n \geq 0$, i.e. as defined in Ref. [10], there exist a projection P and a positive constant c such that

$$\begin{cases} |U(n)PU^{-1}(m)| \leq ch(n)h(m)^{-1}, & n \geq m \geq 0, \\ |U(n)(I-P)U^{-1}(m)| \leq ck(n)^{-1}k(m), & m \geq n \geq 0, \end{cases} \tag{4.32}$$

where $U(n)$ is a fundamental matrix of Equation (4.31) and h, k are two positive non-increasing functions defined on $\mathbb{Z}_+$ such that $\lim_{n\to\infty} h(n) = 0$ and $\lim_{n\to\infty} k(n) = 0$.

(S2): $f : \mathbb{Z}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|f(n,x)| \leq \zeta(n)|x|$ for all $n \geq 0$ and $x \in \mathbb{R}^d$, where $\zeta(n)$ is a non-negative function.

(S3): The function $\eta(n) := c\{h(n)\sum_{s=0}^{n-1} h(s+1)^{-1}\zeta(s) + k(n)^{-1}\sum_{s=n}^{\infty} k(s+1)\zeta(s)\}$ is well defined for all $n \geq 0$ such that $\eta := sup_{n\geq 0}\eta(n) < 1$.

Corollary 1. *Suppose that* (*S*1–*S*3) *hold. Then every bounded solution* $x(n)$ *of system* (4.32) *satisfies*

$$|x(n)| \leq \frac{ch(n)|x(n_1)|}{(1-\eta)h(n_1)} + \frac{c^2 h(n)|x(n_1)|}{(1-\eta)^2 h(n_1)} \left\{ \sum_{s=n_1}^{n-1} \varpi(s) \prod_{\tau=s+1}^{n-1} \left(1 + \frac{c\varpi(\tau)}{1-\eta}\right) \right\} \tag{4.33}$$

for all $n \geq n_1$, *where* $n_1 \in \mathbb{Z}_+$ *is given and* $\varpi(n) := h(n)h(n+1)^{-1}\zeta(n)$. *Furthermore,* $|x(n)| \to 0$ *as* $n \to \infty$ *in the convergence rate of h if* $\sum_{n=0}^{\infty}\varpi(n) < \infty$.

Before proving this Corollary, we need the following lemma.

LEMMA 2. *Suppose that* ($S1$–$S3$) *hold. Then for every solution* $x(n)$ *of* (4.30) *which is bounded on* $\mathbb{Z}_+$ *there is an* $x_P \in P\mathbb{R}^d$ *such that*

$$x(n) = U(n)U^{-1}(0)x_P + \sum_{s=0}^{n-1} U(n)PU^{-1}(s+1)f(s, x(s))$$
$$- \sum_{s=n}^{\infty} U(n)(I-P)U^{-1}(s+1)f(s, x(s)).$$

The proof of this lemma can be referred to Ref. ([1], Theorems 5.6.8 and 5.8.6). Although the result given in Ref. [1] was obtained under the assumption of exponential dichotomies, there are no difference between exponential dichotomy cases and (h,k)-dichotomy cases because the equality in Lemma 2 is guaranteed by the summability of $\sum_{s=n}^{\infty} U(n)(I-P)U^{-1}(s+1)f(s, x(s))$, which follows from the assumption of Lemma 2 immediately.

Proof of Corollary 1. By Lemma 2, every bounded solution $x(n)$ of system (4.32) satisfies

$$x(n) = U(n)PU^{-1}(n_1)x(n_1) + \sum_{s=n_1}^{n-1} U(n)PU^{-1}(s+1)f(s, x(s))$$
$$- \sum_{s=n}^{\infty} U(n)(I-P)U^{-1}(s+1)f(s, x(s)), \quad \forall n \geq n_1,$$

where $n_1 \in \mathbb{Z}_+$ is given arbitrarily. It follows that

$$|x(n)| \leq ch(n)h(n_1)^{-1}|x(n_1)| + c\sum_{s=n_1}^{n-1} h(n)h(s+1)^{-1}\zeta(s)|x(s)|$$
$$+ c\sum_{s=n}^{\infty} k(n)^{-1}k(s+1)\zeta(s)|x(s)|, \quad \forall n \geq n_1. \tag{4.34}$$

Then Theorem 2 is applicable to the inequality (4.34) and the estimate (4.33) can be obtained.

Furthermore, under the assumption $\sum_{n=0}^{\infty} \varpi(n) < \infty$, it is well-known that $\prod_{n=0}^{\infty}(1 + \theta\varpi(n)) < \infty$ for an arbitrarily given constant $\theta > 0$. It follows from (4.36) that $|x(n)| \leq Lh(n)$ for all $n \geq n_1$, where $L > 0$ is a constant. Hence $|x(n)| \to 0$ as $n \to \infty$ in the convergence rate of h. The proof is completed. □

By Corollary 1, a weaker condition for bounded solutions $x(n)$ of system (4.30) to approach 0 as $n \to \infty$ is that

$$\lim_{n\to\infty} h(n)\sum_{s=0}^{n-1} \varpi(s) \prod_{\tau=s+1}^{n-1} \left[1 + \frac{c\varpi(\tau)}{1-\eta}\right] = 0. \tag{4.35}$$

Furthermore, Corollary 1 covers the situation of exponential convergence rate, discussed in Ref. ([1], Theorem 5.8.6, pp. 273–274). In fact, consider the class

$$\mathcal{H} := \left\{\phi : \mathbb{Z}_+ \to \mathbb{R}_+ | \phi(n) > 0, \phi(n)\sum_{s=0}^{n-1} \phi(s+1)^{-1} \text{ is bounded}\right\}.$$

If $\zeta(n)$ is identical to a positive constant, then the assumption (S3) in our Corollary 1 implies that

$$g(n) := \sum_{s=0}^{n-1} h(s+1)^{-1} \leq Kh(n)^{-1}, \quad \forall n \in \mathbb{Z}_+, \tag{4.36}$$

where K is a positive constant and $K > 1$. This means that $h \in \mathcal{H}$. This further implies that $\Delta g(n) = h(n+1)^{-1} \geq (1/K)g(n+1)$, i.e. $(1 - 1/K)g(n+1) - g(n) \geq 0$. Hence,

$$\Delta\left[\left(1 - \frac{1}{K}\right)^n g(n)\right] \geq 0, \quad \forall n \in \mathbb{Z}_+. \tag{4.37}$$

Summing up (4.37) from $n_1 \geq 0$ to n, we get $g(n) \geq g(n_1)(1 - 1/K)^{n_1 - n}$. By (4.36) we obtain $Kh(n)^{-1} \geq g(n_1)(1 - 1/K)^{n_1 - n}$, i.e.

$$h(n) \leq \frac{K}{g(n_1)}\left(1 - \frac{1}{K}\right)^{n - n_1}.$$

It means that $h(n)$ tends to 0 as $n \to \infty$ in the exponential convergence rate. On the other hand, it is also easy to give an example of slower convergence rate. Consider system (4.30) with $A(n) := \operatorname{diag}\{\lambda_1(n), \ldots, \lambda_d(n)\}$, where

$$\lambda_j(n) \leq \frac{n^2}{(n+1)^2}, \quad \forall n \geq 0, \ j = 1, \ldots, \ell, \quad \lambda_j(n) \geq \frac{(n+2)^2(2n+1)}{(n+1)^2(2n+3)}, \\ \forall n \geq 0, \ j = \ell + 1, \ldots, d.$$

and suppose $|f(n,x)| \leq (1/(n+2)^2)|x|$. One can check that the linear system (4.31) admits an (h,k)-dichotomy, where $h(n) = 1/(n+1)^2$ and $k(n) = (2n+3)/(n+2)^2$. Obviously, h tends to 0 in a slower convergence rate than exponential one and $h \notin \mathcal{H}$. Moreover, (S2) and (S3) are satisfied with $\zeta(n) := 1/(n+2)^2$. Note the fact that $c = 1$, $\eta(n) = n/(n+1)^2 + 1/(2n+3)$ and hence $\eta = 9/20$. By Corollary 1,

$$|x(n)| \leq \frac{20|x(0)|}{11(1+n^2)} + \frac{400|x(0)|}{121(1+n^2)}\left\{\sum_{s=0}^{n-1} \frac{1}{(s+1)^2} \prod_{\tau=s+1}^{n-1}\left(1 + \frac{20}{11(\tau+1)^2}\right)\right\}.$$

As defined in Corollary 1, $\varpi(n) = 1/(n+1)^2$. It is clear that $\sum_{n=0}^{\infty} \varpi(n) < \infty$. Thus, Corollary 1 implies that $|x(n)|$ approaches 0 in the convergence rate of h.

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