Soft MTL-algebras based on fuzzy sets

Jianming Zhan^{a,*}, Wiesław A. Dudek^{b}

^{a} Department of Mathematics, Hubei Institute for Nationalities,

Enshi, Hubei Province, 445000, P. R. China

^{b} Institute of Mathematics and Computer Science, Wrocław University of Technology,

Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

Abstract

In this paper, we deal with soft MTL-algebras based on fuzzy sets. By means of ∈-soft sets and q-soft sets, some characterizations of (Boolean, G- and MV-) filteristic soft MTL-algebras are investigated. Finally, we prove that a soft set is a Boolean filteristic soft MTL-algebra if and only if it is both a G-filteristic soft MTL-algebra and an MV-filteristic soft MTL-algebra.

1 Introduction

To solve complicated problems in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of

---

* Corresponding author. Tel/Fax: 0086-718-8437732.

E-mail addresses: zhanjianming@hotmail.com (J. Zhan), dudek@im.pwr.wroc.pl (W.A. Dudek).

these theories have their own difficulties which have been pointed out in [15]. Maji et al. [14] and Molodtsov [15] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [15] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, research on the soft set theory is progressing rapidly. Maji et al. [13] described the application of soft set theory to a decision making problem. They also studied several operations on the theory of soft sets. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [18, 19]. Jun [6] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras, and introduced the notions of soft BCK/BCI-algebras, and then investigated their basic properties [7]. Aktas et al. [1] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing some examples to clarify their differences.

The interest in foundation of Fuzzy Logic has been rapidly recently and several new algebras playing the role of the structures of truth values has been introduced. Hájek introduced the axiom system of basic logic (BL) for fuzzy propositional logic and defined the class of BL-algebras (see [5]). The logic MTL, Monoidal t-norm based logic was introduced by Esteva and Godo [3]. This logic is very interesting from many points of view. From the logic point of view, it can be regarded as a weak system of Fuzzy Logic. In connection with the logic MTL, Esteva and Godo [3] introduced a new algebra, called a MTL-algebra, and studied several basic properties. In the same times independently were introduced in [4] weak-BL algebras as commutative weak-pseudo-BL algebras. MTL-algebras and weak-BL algebras are the same algebras.

Based on the fuzzy set theory, Kim et al. in [9] studied the fuzzy structure of filters in MTL-algebras. As a continuation of the paper [9], Jun et al. [8] gave characterizations of fuzzy filters in MTL-algebras and investigated further properties of fuzzy filters in MTL-algebras. The other important results can be found in [17, 20].

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which was mentioned in [16], played a vital role to generate some different types of fuzzy subsets. It is worth pointing out that Bhakat and Das [2] initiated the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee\, q)$-fuzzy subgroup. In fact, the $(\in, \in \vee\, q)$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. With this objective in view, Ma et al. [10, 11, 12] discussed some kind of generalized fuzzy filters of MTL-algebras.

In this paper, we deal with soft MTL-algebras based on fuzzy sets. In Section 2, we recall some basic defnitions of MTL-algebras. In Section 3, we discuss the characterizations of filteristic soft MTL-algebras. In Section 4, we divide

into three parts. In Subsection 4.1, we investigate some characterizations of Boolean filteristic soft MTL-algebras. Some properties of MV- and G-filteristic soft MTL-algebras are invetigated in Subsection 4.2 and 4.3, respectively. Finally, we prove that a soft set is a Boolean filteristic soft MTL-algebra if nd only if it is both a G-filteristic soft MTL-algebra and an MV-filteristic soft MTL-algebra.

2   Preliminaries

By a *commutative, integral and bounded residuated lattice* we shall mean a lattice $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 0, 1)$ containing the least element 0 and the largest element $1 \neq 0$, and endowed with two binary operation $\odot$ (called product) and $\rightarrow$ (called residuum) such that
   (1) $\odot$ is associative, commutative and isotone,
   (2) $\forall x \in L$, $x \odot 1 = x$,
   (3)   the Galois correspondence holds, that is,
        $\forall x, y, z \in L$, $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z$.

In a commutative, integral and bounded residuated lattice, the following are true (see [17]):
   (1) $x \leq y \Leftrightarrow x \rightarrow y = 1$,
   (2) $0 \rightarrow x = 1$, $1 \rightarrow x = x$, $x \rightarrow (y \rightarrow x) = 1$,
   (3) $y \leq (y \rightarrow x) \rightarrow x$,
   (4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
   (5) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$, $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Based on the Hájek's results [5], Axioms of MTL and Formulas which are provable in MTL, Esteva and Godo [3] defined the algebras, so called MTL-algebras corresponding to the MTL-logic in the following way:

A *MTL-algebra* is a commutative, integral and bounded residuated lattice $L = (L, \leq, \wedge, \vee, \odot, \rightarrow, 1)$ satisfying the pre-linearity equation:

$$(x \rightarrow y) \vee (y \rightarrow x) = 1.$$

In a MTL-algebra, the following are true:

   (1) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$,
   (2) $x \odot y \leq x \wedge y$,
   (3) $x' = x'''$, $x \leq x''$, $x' \odot x = 0$,
   (4)   if $x \vee x' = 1$, then $x \wedge x' = 0$,
   where $x' = x \rightarrow 0$.

Throughout this paper, L is a MTL-algebra unless otherwise specified.
We cite below some notations, definitions and basic results which will be needed in the sequel.

A non-empty subset A of L is called a *filter* of L if it is closed undet the operation $\odot$ and for every $x \in A$, $x \leq y$ implies $y \in A$. It is easy to check that a non-empty subset A of L is a filter of L if and only if $1 \in A$ and for all $x \in A$ from $x \to y \in A$ it follows $y \in A$.

A filter A of L is called:
- a *Boolean filter* if $x \vee x' \in A$ for any $x \in L$,
- a *G-filter* if $x \odot x \to y \in A \Rightarrow x \to y \in A$ for any $x, y \in L$,
- an *MV-filter* if $x \to y \in A \Rightarrow ((y \to x) \to x) \to y \in A$ for any $x, y \in L$.

We now review some fuzzy logic concepts. A fuzzy set of L is a function $\mu : L \to [0, 1]$.

Now, we recall some the following concepts and results in [8, 9, 21].

Definition 2.1. A fuzzy set μ of L is called a *fuzzy filter* of L if
$(F1)$ $\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in L$,
$(F2)$ it is order-preserving, that is, $x \leq y \Rightarrow \mu(x) \leq \mu(y)$ for all $x, y \in L$.

Theorem 2.2. *A fuzzy set μ of L is a fuzzy filter of L if and only if*
$(F3)$ $\mu(1) \geq \mu(x)$,
$(F4)$ $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$
is satisfied for all $x, y \in L$.

Definition 2.3. A fuzzy filter μ of L is called a *fuzzy Boolean filter* of L if $\mu(x \vee x') = \mu(1)$ holds for all $x \in L$.

Theorem 2.4. *Let μ be a fuzzy filter of L, then the following are equivalent:*
(i) μ *is Boolean,*
(ii) $\mu(x \to z) \geq \min\{\mu(x \to (z' \to y)), \mu(y \to z)\}$,
(iii) $\mu(x) \geq \mu((x \to y) \to x)$.

Definition 2.5. A fuzzy filter μ of L is called
- a *fuzzy MV-filter* if $\mu(x \to y) \geq \mu(((y \to x) \to x) \to y)$,
- a *fuzzy G-filter* if $\mu(x \odot x \to y) \geq \mu(x \to y)$
for all $x, y \in L$.

3 Filteristic soft MTL-algebras

Molodtsov [15] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A \subset E$.

A pair (F, A) is called a *soft set* over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (F, A).

Definition 3.1. Let (F, A) be a soft set over L. Then (F, A) is called a *filteristic soft MTL-algebra over L* if $F(x)$ is a filter of L for all $x \in A$, for our convenience, the empty set $\emptyset$ is regarded as a filter of L.

Example 3.2. Let $L = [0, 1]$ and define a product $\odot$ and a residuum $\rightarrow$ on L as follows:

$$x \odot y = \begin{cases} x \wedge y & \text{if } x + y > 0.5, \\ 0 & \text{otherwise,} \end{cases} \qquad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{1 - x, y\} & \text{otherwise,} \end{cases}$$

for all $x, y \in L$. Then L is an MTL-algebra.

Let (F, A) be a soft set over L, where $A = (0, 1]$ and $F : A \rightarrow \mathcal{P}(L)$ be a set-valued function defined by

$$F(x) = \begin{cases} L & \text{if } 0 < x \leq 0.5, \\ \{1\} & \text{if } 0.5 < x \leq 0.6, \\ \emptyset & \text{if } 0.8 < x \leq 1. \end{cases}$$

Thus, $F(x)$ is a filter of L for all $x \in A$, and so (F, A) is a filteristic soft MTL-algebra over L.

For a fuzzy set μ in any MTL-algebra L and $A \subseteq [0, 1]$ we can consider two set-valued functions

$$F : A \rightarrow \mathcal{P}(L), \quad t \mapsto \{x \in L \mid x_t \in \mu\}$$

and

$$F_q : A \rightarrow \mathcal{P}(L), \quad t \mapsto \{x \in L \mid x_t \, \mathrm{q} \, \mu\}.$$

Then (F, A) and (F_q, A) are called an $\in$-*soft set* and q-*soft set* over L, respectively.

Theorem 3.3. *Let μ be a fuzzy set of L and let (F, A) be an $\in$-soft set over L with $A = (0, 1]$. Then (F, A) is a filteristic soft MTL-algebra over L if and only if μ is a fuzzy filter of L.*

Proof. Let μ be a fuzzy filter of L and $t \in A$. If $x \in F(t)$, then $x_t \in \mu$, and so $1_t \in \mu$, i.e., $1 \in F(t)$. Let $x, y \in L$ be such that $x, x \rightarrow y \in F(t)$. Then $x_t \in \mu$ and $(x \rightarrow y)_t \in \mu$, and so $y_{\min\{t,t\}} = y_t \in \mu$. Hence $y \in F(t)$. This proves that (F, A) is a filteristic soft MTL-algebra over L.

Conversely, assume that (F, A) is a filteristic soft MTL-algebra over L. If there exists $a \in L$ such that $\mu(1) < \mu(a)$, then we can choose $t \in A$ such that $\mu(1) < t \leq \mu(a)$. Thus, $1_t \overline{\in} \mu$, i.e., $1 \overline{\in} F(t)$. This is a contradiction. Hence, $\mu(1) \geq \mu(x)$, for all $x \in L$. If there exist $a, b \in L$ such that $\mu(b) < s \leq \min\{\mu(a \rightarrow b), \mu(a)\}$. Then $(a \rightarrow b)_s \in \mu$ and $a_s \in \mu$, but $b_s \overline{\in} \mu$, that is, $a \rightarrow b \in F(s)$ and $a \in F(s)$, but $b \overline{\in} F(s)$, contradiction, and so, $\mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\}$, for all $x, y \in L$. Therefore, μ is a fuzzy filter of L. $\square$

Theorem 3.4. *Let μ be a fuzzy set of L and (F_q, A) a q-soft set over L with $A = (0, 1]$. Then the following are equivalent:*

    (i) μ is a fuzzy filter of L,

    (ii) $\forall t \in A$ each non-empty $F_q(t)$ is a filter of L.

Proof. Let μ be a fuzzy filter of L and let $F_q(t) \neq \emptyset$ for any $t \in A$. If $1 \overline{\in} F_q(t)$, then $1_t \overline{q} \mu$, and so $\mu(1) + t < 1$. Then $\mu(x) + t \leq \mu(1) + t < 1$ for all $x \in L$, and so $F_q(t) = \emptyset$, contradiction. Hence $1 \in F_q(t)$.

Let $x, y \in L$ be such that $x \to y \in F_q(t)$ and $x \in F_q(t)$. Then $(x \to y)_t q\mu$ and $x_t q\mu$, or equivalently, $\mu(x \to y) + t > 1$ and $\mu(x) + t > 1$. Thus,

$$\mu(y) + t \geq \min\{\mu(x \to y), \mu(x)\} + t = \min\{\mu(x \to y) + t, \mu(x) + t\} > 1,$$

and so $y_t q\mu$, i.e., $y \in F_q(t)$. Hence $F_q(t)$ is a filter of L.

Conversely, assume that the condition (ii) holds. If $\mu(1) < \mu(a)$ for some $a \in L$, then $\mu(1) + t \leq 1 < \mu(a) + t$ for some $t \in A$. Thus, $a_t q\mu$, and so $F_q(t) \neq \emptyset$. Hence $1 \in F_q(t)$, and so $1_t q\mu$, i.e., $\mu(1) + t > 1$, contradiction. Hence $\mu(1) \geq \mu(x)$ for all $x \in L$.

If there exist $a, b \in L$ such that $\mu(b) < \min\{\mu(a \to b), \mu(a)\}$. Then

$$\mu(b) + s \leq 1 < \min\{\mu(a \to b), \mu(a)\} + s$$

for some $s \in A$. Hence $(a \to b)_s q\mu$ and $a_s q\mu$, i.e., $a \to b \in F_q(s)$ and $a \in F_q(s)$. Since $F_q(s)$ is a filter of L, we have $b \in F_q(s)$, and so $b_s q\mu$, that is, $\mu(b) + s > 1$, contradiction. Hence $\mu(y) \geq \min\{\mu(x \to y), \mu(x)\}$, for all $x, y \in L$. Therefore μ is a fuzzy filter of L. $\square$

Definition 3.5. A fuzzy set μ of L is an $(\in, \in \vee\, q)$-*fuzzy filter* of L if for all $x, y \in L$ it satisfies:

    $(F5)$ $\mu(1) \geq \min\{\mu(x), 0.5\}$,

    $(F6)$ $\mu(y) \geq \min\{\mu(x \to y), \mu(x), 0.5\}$.

Theorem 3.6. *Let μ be a fuzzy set of L and (F, A) be an $\in$-soft set over L with $A = (0, 0.5]$. Then the following are equivalent:*

    (i) μ is an $(\in, \in \vee\, q)$-fuzzy filter of L,

    (ii) (F, A) is a filteristic soft MTL-algebra over L.

Proof. Let μ be an $(\in, \in \vee\, q)$-fuzzy filter of L. For any $t \in A$, we have $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in F(t)$ by Definition 3.5. Hence $\mu(1) \geq \min\{\mu(x), 0.5\} \geq \min\{t, 0.5\} = t$, which implies, $1_t \in \mu$, and so $1 \in F(t)$. If $x \to y \in F(t)$ and $x \in F(t)$, then $(x \to y)_t \in \mu$ and $x_t \in \mu$, that is, $\mu(x \to y) \geq t$ and $\mu(x) \geq t$. Now, by $(F6)$, we have

$$\mu(y) \geq \min\{\mu(x \to y), \mu(x), 0.5\} \geq \min\{t, 0.5\} = t,$$

which implies, $y_t \in \mu$, and so $y \in F(t)$. Thus, (F, A) is a filteristic soft MTL-algebra over L.

Now assume that the condition (ii) holds. If there exists $a \in L$ such that $\mu(1) < \min\{\mu(a), 0.5\}$, then $\mu(1) < t \leq \min\{\mu(a), 0.5\}$ for some $t \in A$. It

6

follows that $1_t\overline{\in}\mu$, i.e., $1\overline{\in}F(t)$, contradiction. Hence $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$. If there exist $a, b \in L$ such that $\mu(b) < \min\{\mu(a \to b), \mu(a), 0.5\}$, then taking $t = \frac{1}{2}(\mu(b) + \min\{\mu(a \to b), \mu(a), 0.5\})$, we have $t \in A$ and

$$\mu(b) < t < \min\{\mu(a \to b), \mu(a), 0.5\},$$

which implies, $a \to b \in F(t), a \in F(t)$, but $b\overline{\in}F(t)$, contradiction. It follows from Definition 3.5 that μ is an $(\in, \in \vee q)$-fuzzy filter of L. $\square$

Definition 3.7. [10] A fuzzy set μ of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy filter* of L if and only if for $x, y \in L$ it satisfies:

$(F7)$ $\max\{\mu(1), 0.5\} \geq \mu(x)$,
$(F8)$ $\max\{\mu(y), 0.5\} \geq \min\{\mu(x \to y), \mu(x)\}$.

Theorem 3.8. *Let μ be a fuzzy set of L and (F, A) be an $\in$-soft set over L with $A = (0.5, 1]$. Then the following are equivalent:*

(i) μ *is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L,*
(ii) (F, A) *is a filteristic soft MTL-algebra over L.*

Proof. Let μ be an $(\overline{\in}, \overline{\in}\vee\overline{q})$-fuzzy filter of L. For any $t \in A$, by Definition 3.7, we have $\mu(x) \leq \max\{\mu(1), 0.5\}$ for all $x \in F(t)$. Thus, $t \leq \mu(x) \leq \max\{\mu(1), 0.5\} = \mu(1)$, which implies $1_t \in \mu$, i.e., $1 \in F(t)$.

Let $x, y \in L$ be such that $x \to y \in F(t)$ and $x \in F(t)$, then $(x \to y)_t \in \mu$ and $x_t \in \mu$, i.e., $\mu(x \to y) \geq t$ and $\mu(x) \geq t$. It follows from Definition 3.7 that

$$t \leq \min\{\mu(x \to y), \mu(x)\} \leq \max\{\mu(y), 0.5\} = \mu(y),$$

which implies, $y_t \in \mu$, i.e., $y \in F(t)$. Hence $F(t)$ is a filter of L for all $t \in A$, and so (F, A) is a filteristic soft MTL-algebra over L.

Now, assume that (F, A) is a filteristic soft MTL-algebra over L. If there exists $a \in L$ such that $\mu(a) \geq \max\{\mu(1), 0.5\}$, then $\mu(a) \geq t > \max\{\mu(1), 0.5\}$ for some $t \in A$, and so $\mu(1) < t$. Thus, $1\overline{\in}F(a)$. Contradiction. Hence $\mu(x) \leq \max\{\mu(1), 0.5\}$ for all $x \in L$. If there exist $a, b \in L$ such that $\min\{\mu(a \to b), \mu(a)\} \geq t > \max\{\mu(b), 0.5\}$ for some $t \in A$, then $(a \to b)_t$ and a_t are in μ. But $b_t\overline{\in}\mu$, therefore $a \to b, a \in F(t)$. This is a contradiction since $b\overline{\in}F(t)$. It follows from Definition 3.7 that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L. $\square$

Next, we give the following two important results by q-soft sets.

Theorem 3.9. *Let μ be a fuzzy set of L and (F_q, A) be a q-soft set over L with $A = (0, 0.5]$. Then (F_q, A) is a filteristic soft MTL-algebra over L if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L.*

Proof. Let (F_q, A) be a filteristic soft MTL-algebra over L, then $F_q(t)$ is a filter of L for all $t \in A$. If $\max\{\mu(1), 0.5\} < \mu(a)$ for some $a \in L$, then $\max\{\mu(1), 0.5\} + t \leq 1 < \mu(a) + t$ for some $t \in A$. Thus, $1_t\overline{q}\mu$, which is impossible. Hence $\max\{\mu(1), 0.5\} \geq \mu(x)$ for all $x \in L$.

If there exist $a, b \in L$ such that $\max\{\mu(b), 0.5\} < \min\{\mu(a \to b), \mu(a)\}$. Then $\max\{\mu(b), 0.5\} + s \leq 1 < \min\{\mu(a \to b), \mu(a)\} + s$ for some $s \in A$. Hence

7

$(a \rightarrow b)_s q \mu$ and $a_s q \mu$, i.e., $a \rightarrow b \in F_q(s)$ and $a \in F_q(s)$. Since $F_q(s)$ is a filter of L, we have $b \in F_q(s)$, and so $b_s q \mu$, that is, $\mu(b) + s > 1$, contradiction. Hence $\max\{\mu(y), 0.5\} \geq \min\{\mu(x \rightarrow y), \mu(x)\}$, for all $x, y \in L$. Therefore μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L.

Conversely, let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L. For any $t \in A$. By Definition 3.7, we have $\mu(x) \leq \max\{\mu(1), 0.5\}$ for all $x \in F_q(t)$, and so $\max\{\mu(1), 0.5\} + t \geq \mu(x) + t > 1$. Hence $\mu(1) + t > 1$, that is, $1 \in F_q(t)$. Let $x, y \in L$ be such that $x \rightarrow y \in F_q(t)$ and $x \in F_q(t)$. Then $(x \rightarrow y)_t q \mu$ and $x_t q \mu$, or equivalently, $\mu(x \rightarrow y) + t > 1$ and $\mu(x) + t > 1$. By Definition 3.7, we have

$$\max\{\mu(y), 0.5\} + t \geq \min\{\mu(x \rightarrow y), \mu(x)\} + t = \min\{\mu(x \rightarrow y) + t, \mu(x) + t\} > 1,$$

and so $y_t q \mu$, i.e., $y \in F_q(t)$. Hence $F_q(t)$ is a filter of L, and so (F_q, A) is a filteristic soft MTL-algebra over L. $\qquad \square$

Theorem 3.10. *Let μ be a fuzzy set of L and (F_q, A) be a q-soft set over L with $A = (0.5, 1]$. Then (F_q, A) is a filteristic soft MTL-algebra over L if and only if μ is an $(\in, \in \vee \, q)$-fuzzy filter of L.*

Proof. Let (F_q, A) be a filteristic soft MTL-algebra over L. Then $F_q(t)$ is a filter of L for all $t \in A$. If $\mu(1) < \min\{\mu(a), 0.5\}$ for some $a \in L$, then $\mu(1) + t \leq 1 < \min\{\mu(a), 0.5\} + t$ for some $t \in A$. Thus, $1_t \overline{q} \mu$, contradiction. Hence $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in L$.

If there exist $a, b \in L$ such that $\mu(b) < \min\{\mu(a \rightarrow b), \mu(a), 0.5\}$. Then $\mu(b) + s \leq 1 < \min\{\mu(a \rightarrow b), \mu(a), 0.5\} + s$ for some $s \in A$. Hence $(a \rightarrow b)_s q \mu$ and $a_s q \mu$, i.e., $a \rightarrow b \in F_q(s)$ and $a \in F_q(s)$. Since $F_q(s)$ is a filter of L, we have $b \in F_q(s)$, and so $b_s q \mu$, that is, $\mu(b) + s > 1$, contradiction. Hence $\mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x), 0.5\}$, for all $x, y \in L$. Therefore μ is an $(\in, \in \vee \, q)$-fuzzy filter of L.

Conversely, let μ be an $(\in, \in \vee \, q)$-fuzzy filter of L. By Definition 3.5, we have $\mu(1) \geq \min\{\mu(x), 0.5\}$ for all $x \in F_q(t)$, and so $\mu(1) + t \geq \min\{\mu(x), 0.5\} + t = \min\{\mu(x) + t, 0.5 + t\} > 1$. Hence $\mu(1) + t > 1$, that is, $1 \in F_q(t)$.

Now, let $x, y \in L$ be such that $x \rightarrow y \in F_q(t)$ and $x \in F_q(t)$. Then $(x \rightarrow y)_t q \mu$ and $x_t q \mu$, or equivalently, $\mu(x \rightarrow y) + t > 1$ and $\mu(x) + t > 1$. Thus

$$\begin{aligned} \mu(y) + t & \geq \min\{\mu(x \rightarrow y), \mu(x), 0.5\} + t \\ & = \min\{\mu(x \rightarrow y) + t, \mu(x) + t, 0.5 + t\} > 1, \end{aligned}$$

and so $y_t q \mu$, i.e., $y \in F_q(t)$. Hence $F_q(t)$ is a filter of L, and consequently, (F_q, A) is a filteristic soft MTL-algebra over L. $\qquad \square$

Definition 3.11. For $0 < \alpha < \beta \leq 1$ a fuzzy set μ of L is called a *fuzzy filter with thresholds* $(\alpha, \beta]$ if for all $x, y \in L$:

(F9) $\max\{\mu(1), \alpha\} \geq \min\{\mu(x), \beta\}$,

(F10) $\max\{\mu(y), \alpha\} \geq \min\{\mu(x \rightarrow y), \mu(x), \beta\}$.

Theorem 3.12. *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (\alpha, \beta] \subset (0, 1]$ is a filteristic soft MTL-algebra over L if and only if μ is a fuzzy filter with thresholds $(\alpha, \beta]$.*

Proof. Let (F, A) be a filteristic soft MTL-algebra as in Theorem. If there exists $a \in L$ such that $\max\{\mu(1), \alpha\} < \min\{\mu(a), \beta\}$, then $\max\{\mu(1), \alpha\} < t \leq \min\{\mu(a), \beta\}$ for some $t \in (\alpha, \beta]$. Thus $1\overline{\in}F(t)$ which is a contradiction. If there exist $a, b \in L$ such that $\max\{\mu(b), \alpha\} < t \leq \min\{\mu(a \rightarrow b), \mu(a), \beta\}$. Hence $(a \rightarrow b)_t \in \mu, a_t \in \mu$. But $b_t\overline{\in}\mu$, therefore $a \rightarrow b \in F(t), a \in F(t)$. This also is a contradiction since $b\overline{\in}F(t)$. Consequently, μ is a fuzzy filter with thresholds $(\alpha, \beta]$ of L.

On the other hand, if μ is a fuzzy filter with thresholds $(\alpha, \beta]$, then, by $(F9)$, we have $\max\{\mu(1), \alpha\} \geq \min\{\mu(x), \beta\}$ for all $x \in F(t)$. Thus, $\max\{\mu(1), \alpha\} \geq \min\{\mu(x), \beta\} \geq \min\{t, \beta\} = t > \alpha$, which implies, $\mu(1) \geq t$, i.e., $1_t \in \mu$. Hence $1 \in F(t)$. Let $x, y \in L$ be such that $x \rightarrow y \in F(t)$ and $x \in F(t)$. Thus, $(x \rightarrow y)_t \in \mu$ and $x_t \in \mu$, i.e., $\mu(x \rightarrow y) \geq t$ and $\mu(x) \geq t$. By $(F10)$, we have $\max\{\mu(y), \alpha\} \geq \min\{\mu(x \rightarrow y), \mu(x), \beta\} \geq \min\{t, \beta\} = t > \alpha$, and so $\mu(y) \geq t$, i.e., $y_t \in \mu$, and so $y \in F(t)$. Therefore, (F, A) is a filteristic soft MTL-algebra over L. $\square$

4   Boolean (MV-, G-) filteristic soft MTL-algebras

In this section divided in three parts we describe some types of generalized fuzzy filters of MTL-algebras introduced in [11]. In the first part we characterize Boolean filteristic soft MTL-algebras; in the second – G-filtersistic soft MTL-algebras; in third – MV-filtersistic soft MTL-algebras which are natural generalizations of Boolean filters, G-filters and MV-filters, respectively. Finally, we describe relationship between these soft MTL-algebras.

4.1   Boolean filteristic soft MTL-algebras

We start with the following definition.

Definition 4.1.1.   A soft set (F, A) over L is called a *Boolean filteristic soft MTL-algebra over L* if $F(x)$ is a Boolean filter of L for all $x \in A$. The empty set is treatment as a Boolean filter.

Example 4.1.2.   Consider the set $L = \{0, a, b, 1\}$ with two operations defined by the following tables:

$\odot$	0	a	b	1		$\rightarrow$	0	a	b	1
0	0	0	0	0		0	1	1	1	1
a	0	a	$\overline{a}$	a		a	0	1	1	1
b	0	a	a	b		b	0	b	1	1
1	0	a	b	1		1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$, where $\wedge$ and $\vee$ are min and max operations, respectively, is an MTL-algebra.

Let (F, A) be a soft set over L, where $A = (0, 1]$ and $F : A \to \mathcal{P}(L)$ be a set-valued function defined by

$$F(x) = \begin{cases} \{0, a, b, 1\} & \text{if} \quad 0 < x \le 0.4, \\ \{1, a, b\} & \text{if} \quad 0.4 < t \le 0.8, \\ \emptyset & \text{if} \quad 0.8 < t \le 1. \end{cases}$$

Thus, $F(x)$ is a Boolean filter of L for all $x \in A$, and so (F, A) is a Boolean filteristic soft MTL-algebra over L.

The following proposition is obvious.

Proposition 4.1.3. *A Boolean filteristic MTL-algebra is a filteristic MTL-algebra.*

Theorem 4.1.4. *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (0, 1]$ is a Boolean filteristic soft MTL-algebra over L if and only if μ is a fuzzy Boolean filter of L.*

Proof. Let (F, A) an $\in$-soft set (F, A) over L with $A = (0, 1]$. If it is a Boolean filteristic soft MTL-algebra over L, then, by Proposition 4.1.3, it is a filteristic soft MTL-algebra over L, and so μ is a fuzzy filter of L (Theorem 3.3). If there exist $a, b, c \in L$ such that $\mu(a \to c) < s \le \min\{\mu(a \to (c' \to b)), \mu(b \to c)\}$ for some $s \in A$. Then $(a \to b)_s \mu$ and $a_s \in \mu$, but $b_s \overline{\in} \mu$, that is, $a \to (c' \to b) \in F(s)$ and $b \to c \in F(s)$. Thus $a \to c \overline{\in} F(s)$, which is a contradiction. Therefore, μ is a fuzzy Boolean filter of L.

Conversely, if μ is a fuzzy Boolean filter of L, then it is also a fuzzy filter of L and, by Theorem 3.3, (F, A) is a filteristic soft MTL-algebra over L. Let $x, y, z \in L$ be such that $x \to (z' \to y), y \to z \in F(t)$. Then $(x \to (z' \to y))_t \in \mu$ and $(y \to z)_t \in \mu$. Hence, by Theorem 2.4, we obtain $\mu(x \to z) \ge \min\{\mu(x \to (z' \to y)), \mu(y \to z)\} \ge t$, and so $x \to z \in F(t)$. This proves (Theorem 2.4) that (F, A) is a Boolean filteristic soft MTL-algebra over L. $\square$

Theorem 4.1.5. *Let μ be a fuzzy set of L. If (F_q, A), where $A = (0, 1]$, is a q-soft set over L, then μ is a fuzzy Boolean filter if and only if each non-empty $F_q(t)$ is a Boolean filter.*

Proof. Let μ be a fuzzy Boolean filter of L. Then, by Theorem 3.4, $F_q(t)$ is a filter of L. Let $x, y, z \in L$ be such that $x \to (z' \to y) \in F_q(t)$ and $y \to z \in F_q(t)$. Then $(x \to (z' \to y))_t q \mu$ and $(y \to z)_t q \mu$, or equivalently, $\mu(x \to (z' \to y)) + t > 1$ and $\mu(y \to z) + t > 1$. Since μ is a fuzzy Boolean of L, we have

$$\mu(x \to z) + t \ge \min\{\mu(x \to (z' \to y)), \mu(y \to z)\} + t$$
$$= \min\{\mu(x \to (z' \to z)) + t, \mu(y \to z) + t\} > 1,$$

and so $(x \to z)_t q$, i.e., $x \to z \in F_q(t)$. This proves (Theorem 2.4) that $F_q(t)$ is a Boolean filter of L.

Conversely, assume that each non-empty $F_q(t)$ is a Boolean filter of L. Then μ is a fuzzy filter of L by Theorem 3.4. If there exist $a, b, c \in L$ such that $\mu(a \to c) < \min\{\mu(a \to (c' \to b)), \mu(b \to c)\}$. Then $\mu(a \to c) + s \leq 1 < \min\{\mu(a \to (c' \to b)), \mu(b \to c)\} + s$ for some $s \in A$. Hence $(a \to (c' \to b))_s q\mu$ and $(b \to c)_s q\mu$, but $(a \to c)_t \overline{q}\mu$, i.e., $a \to (c' \to b) \in F_q(s)$ and $b \to c \in F_q(s)$, but $a \to c \overline{\in} F_q(t)$, contradiction. Therefore μ is a fuzzy Boolean filter of L. $\square$

Definition 4.1.6. An $(\in, \in \vee q)$-fuzzy filter μ of L is called an $(\in, \in \vee q)$-*fuzzy Boolean filter* of L if

$$\mu(x \to z) \geq \min\{\mu(x \to (z' \to y)), \mu(y \to z), 0.5\}$$

holds for all $x \in L$.

Theorem 4.1.7. *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (0, 0.5]$ is a Boolean filteristic soft MTL-algebra if and only if μ is an $(\in, \in \vee \mathrm{q})$-fuzzy Boolean filter of L.*

Proof. Let an $\in$-soft set (F, A), where $A = (0, 0.5]$, be a Boolean filteristic soft MTL-algebra. If there exist $a, b, c \in L$ such that

$$\mu(a \to c) < \min\{\mu(a \to (c' \to b)), \mu(b \to c), 0.5\},$$

then for

$$t = \frac{1}{2}(\mu(a \to c) + \min\{\mu(a \to (c' \to b), \mu(b \to c), 0.5\})$$

we have $t \in A$ and

$$\mu(a \to c) < t < \min\{\mu(a \to (c' \to b)), \mu(b \to c), 0.5\},$$

which implies $a \to (c' \to b) \in F(t)$ and $b \to c \in F(t)$. This is a contradiction since $a \to c \overline{\in} F(t)$. So, μ is an $(\in, \in \vee \mathrm{q})$-fuzzy Boolean filter of L.

Conversely, if μ is an $(\in, \in \vee \mathrm{q})$-fuzzy Boolean filter of L, then, by Theorem 3.6, (F, A) is a filteristic soft MTL-algebra. Moreover, if $x, y, z \in L$ be such that $x \to (z' \to y) \in F(t)$ and $y \to z \in F(t)$ for some $t \in A$, then $\mu(x \to (z' \to y)) \geq t$ and $\mu(y \to z) \geq t$. Thus,

$$\mu(x \to z) \geq \min\{\mu(x \to (z' \to y)), \mu(y \to z), 0.5\} \geq \min\{t, 0.5\} = t,$$

which implies $(x \to z)_t \in \mu$, and so $x \to z \in F(t)$. Hence, (F, A) is a Boolean filteristic soft MTL-algebra over L. $\square$

Definition 4.1.8. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy Boolean filter* of L if it satisfies:

$$\max\{\mu(x \to z), 0.5\} \geq \min\{\mu(x \to (z' \to y)), \mu(y \to z)\}$$

for all $x \in L$.

Theorem 4.1.9. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (0.5, 1]$ the following conditions are equivalent:*

    (i) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy Boolean filter of L,*

    (ii) *(F, A) is a Boolean filteristic soft MTL-algebra of L.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy Boolean filter of L, then μ is also an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L and, by Theorem 3.8, (F, A) is a filteristic soft MTL-algebra. Let $x, y, z \in L$ be such that $x \rightarrow (z' \rightarrow y) \in F(t)$ and $y \rightarrow z \in F(t)$ for some $t \in A$, then $(x \rightarrow (z' \rightarrow y))_t \in \mu$ and $(y \rightarrow z)_t \in \mu$, i.e., $\mu(x \rightarrow (z' \rightarrow y)) \geq t$ and $\mu(y \rightarrow z) \geq t$. Thus,

$$t \leq \min\{\mu(x \rightarrow (z' \rightarrow y)), \mu(y \rightarrow z)\} \leq \max\{\mu(x \rightarrow z), 0.5\} = \mu(x \rightarrow z),$$

which implies $(x \rightarrow z)_t \in \mu$, i.e., $x \rightarrow z \in F(t)$. Hence $F(t)$ is a Boolean filter of L, and so (F, A) is a Boolean filteristic soft MTL-algebra over L.

    Conversely, assume that (F, A) is a Boolean filteristic soft MTL-algebra over L. If for some $t \in A$ there exist $a, b, c \in L$ such that

$$\min\{\mu(a \rightarrow (c' \rightarrow b)), \mu(b \rightarrow c)\} \geq t > \max\{\mu(a \rightarrow c), 0.5\},$$

then $(a \rightarrow (c' \rightarrow c))_t \in \mu$ and $(b \rightarrow c)_t \in \mu$ but $(a \rightarrow c)_t \overline{\in} \mu$. This means that $a \rightarrow (c' \rightarrow b) \in F(t)$, $b \rightarrow c \in F(t)$, but $a \rightarrow c \overline{\in} F(t)$, which is a contradiction. Therefore, μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy Boolean filter of L. $\qquad\square$

    Now, we give the following two important characterizations of Boolean filterstic q-soft sets.

Theorem 4.1.10. *Let μ be a fuzzy set of L. Then a q-soft set (F_q, A) over L with $A = (0, 0.5]$ is a Boolean filteristic soft MTL-algebra over L if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy Boolean filter of L.*

    The proof is similar to the proof of Theorem 3.9.

Theorem 4.1.11. *Let μ be a fuzzy set of L. Then a q-soft set (F_q, A) of L with $A = (0.5, 1]$ is a Boolean filteristic soft MTL-algebra over L if and only if μ is an $(\in, \in \vee q)$-fuzzy Boolean filter of L.*

    The proof is similar to the proof of Theorem 3.10.

    As a consequence of Theorems 3.6, 4.1.5 and 4.1.7 we obtain

Theorem 4.1.12. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (\alpha, \beta]$, where $0 < \alpha < \beta \leq 1$, the following conditions are equivalent:*

    (i) *μ is a fuzzy Boolean filter with thresholds $(\alpha, \beta]$ of L,*

    (ii) *(F, A) is a Boolean filteristic soft MTL-algebra over L.*

4.2   MV-filteristic soft MTL-algebras

In this subsection, we characterize MV-filteristic soft MTL-algebras by fuzzy MV-filters.

Definition 4.2.1.   A soft set (F, A) over L is called an *MV-filteristic soft MTL-algebra over L* if $F(x)$ is an MV-filter of L for all $x \in A$. The empty set is treatment as an MV-filter of L.

Example 4.2.2.   Let $L = \{0, a, b, 1\}$ be a chain with operations defined by the following two tables:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$, where $\wedge$ and $\vee$ are min and max operations, respectively, is an MTL-algebra.

Let (F, A) be a soft set over L, where $A = (0, 1]$ and $F : A \to \mathcal{P}(L)$ be a set-valued function defined by

$$
F(x) = \begin{cases} \{0, a, b, 1\} & \text{if} \quad 0 < x \leq 0.4, \\ \{1\} & \text{if} \quad 0.4 < t \leq 0.8, \\ \emptyset & \text{if} \quad 0.8 < t \leq 1. \end{cases}
$$

Thus, $F(x)$ is an MV-filter of L for all $x \in A$, and so (F, A) is an MV-filteristic soft MTL-algebra over L.

From the above definitions, we can get the following:

Proposition 4.2.3.   *Every MV-filteristic MTL-algebra is a filteristic MTL-algebra, but the converse may not be true.*

Theorem 4.2.4.   *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (0, 1]$ is an MV-filteristic soft MTL-algebra over L if and only if μ is a fuzzy MV-filter of L.*

The proof is similar to the proof of Theorem 4.1.4.

Theorem 4.2.5.   *For a fuzzy set μ of L and a q-soft set (F_q, A) over L with $A = (0, 1]$ the following conditions are equivalent:*
(i) μ is a fuzzy MV-filter of L,
(ii) each non-empty $F_q(t)$ is an MV-filter of L.

The proof is similar to the proof of Theorem 4.1.5.

Definition 4.2.6. An $(\in, \in \vee q)$-fuzzy filter μ of L is called an $(\in, \in \vee q)$-*fuzzy MV-filter* of L if

$$\mu(((y \to x) \to x) \to y) \geq \min\{\mu(x \to y), 0.5\}$$

is satisfied for all $x, y \in L$.

Theorem 4.2.7. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (0, 0.5]$ the following conditions are equivalent:*
  (i) *μ is an $(\in, \in \vee$ q$)$-fuzzy MV-filter of L,*
  (ii) *(F, A) is an MV-filteristic soft MTL-algebra over L.*

The proof is similar to the proof of Theorem 4.1.7.

Definition 4.2.8. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy MV-filter* of L if

$$\max\{\mu(((y \to x) \to x) \to y), 0.5\} \geq \min\{\mu(x \to y)\}$$

is satisfied for all $x, y \in L$.

Theorem 4.2.9. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (0.5, 1]$ the following conditions are equivalent:*
  (i) *μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy MV-filter of L,*
  (ii) *(F, A) is an MV-filteristic soft MTL-algebra over L.*

The proof is similar to the proof of Theorem 4.1.9.

Theorem 4.2.10. *Let μ be a fuzzy set of L. Then a q-soft set (F_q, A) over L with $A = (0, 0.5]$ is an MV-filteristic soft soft MTL-algebra if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy MV-filter of L.*

The proof is similar to the proof of Theorem 3.9.

Theorem 4.2.11. *Let μ be a fuzzy set of L. Then a q-soft set (F_q, A) over L with $A = (0.5, 1]$ is a filteristic soft MV-algebra over L if and only if μ is an $(\in, \in \vee$ q$)$-fuzzy MV-filter of L.*

The proof is similar to the proof of Theorem 3.10.

As a consequence of Theorems 3.7, 4.2.5 and 4.2.7 we obtain

Theorem 4.2.12. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (\alpha, \beta]$, where $0 < \alpha < \beta \leq 1$, the following conditions are equivalent:*
  (i) *μ is a fuzzy MV-filter with thresholds $(\alpha, \beta]$ of L,*
  (ii) *(F, A) is an MV-filteristic soft MTL-algebra over L.*

From Theorems 4.1.6, 4.1.7, 4.1.8, 4.2.6, 4.2.7, 4.2.9 and Theorem 3.20 in [8] it follows

14

Theorem 4.2.13. *Let μ be a fuzzy set of L. If an $\in$-soft set (F, A) over L with $A = (\alpha, \beta] \subset (0, 1]$ is a Boolean filteristic soft MTL-algebra, then it also is an MV-filteristic soft MTL-algebra, but the converse may not be true.*

4.3   G-filteristic soft MTL-algebras

Now, we describe filteristic soft MTL-algebras connected with G-filters.

Definition 4.3.1.   A soft set (F, A) over L is called a *G-filteristic soft MTL-algebra over* L if $F(x)$ is a G-filter of L for all $x \in A$. The empty set is regarded as a G-filter of L.

Since G-filter is a filter every G-filteristic MTL-algebra is a filteristic MTL-algebra, but the converse is not be true in general.

Example 4.3.2.   Consider on $L = [0, 1]$ two operations $\odot$ and $\rightarrow$ defined by the following tables:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	0	a
b	0	c	b	c	d	b
c	0	c	c	c	0	c
d	0	0	d	0	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	b	b	d	1
b	0	a	1	a	d	1
c	d	1	1	1	d	1
d	a	1	1	1	1	1
1	0	a	b	c	d	1

Then $L(\wedge, \vee, \odot, \rightarrow, 0, 1)$ is a MTL-algebra.

Let (F, A) be a soft set over L, where $A = (0, 1]$ and $F : A \rightarrow \mathcal{P}(L)$ be a set-valued function defined by

$$F(x) = \begin{cases} \{0, a, b, c, d, 1\} & \text{if} \quad 0 < x \leq 0.4, \\ \{1, a\} & \text{if} \quad 0.4 < t \leq 0.8, \\ \emptyset & \text{if} \quad 0.8 < t \leq 1. \end{cases}$$

Thus, $F(x)$ is a G-filter of L for all $x \in A$, and so (F, A) is a G-filteristic soft MTL-algebra over L.

In a similar way as Theorem 4.1.4 we can prove

Theorem 4.3.3.   *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (0, 1]$ is a G-filteristic soft MTL-algebra over L if and only if μ is a fuzzy G-filter of L.*

Theorem 4.3.4.   *Let μ be a fuzzy set of L. If (F_q, A), where $A = (0, 1]$, is a q-soft set over L, then μ is a fuzzy G-filter if and only if each non-empty $F_q(t)$ is a G-filter.*

Proof. The proof is similar to the proof of Theorem 4.1.5. $\square$

Definition 4.3.5. An $(\in, \in \vee q)$-fuzzy filter μ of L is called an $(\in, \in \vee q)$-*fuzzy G-filter* if

$$\mu(x \to y) \geq \min\{\mu(x \odot x \to y), 0.5\}$$

holds for all $x, y \in L$.

Theorem 4.3.6. *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (0, 0.5]$ is a G-filteristic soft MTL-algebra if and only if μ is an $(\in, \in \vee q)$-fuzzy G-filter of L.*

Proof. The proof is similar to the proof of Theorem 4.1.7. $\qquad\qquad$ $\square$

Definition 4.3.7. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filter of L is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-*fuzzy G-filter* of L if

$$\max\{\mu(x \to y), 0.5\} \geq \min\{\mu(x \odot x \to y)\}$$

holds for all $x, y \in L$.

Theorem 4.3.8. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (0.5, 1]$ the following conditions are equivalent:*
  (i) μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy G-filter of L,
  (ii) (F, A) is a G-filteristic soft MTL-algebra over L.

Proof. The proof is analogous to the proof of Theorem 4.1.9. $\qquad$ $\square$

Also the proofs of the following two theorems are very similar to the proofs of Theorems 3.9 and 3.10, respectively.

Theorem 4.3.9. *Let μ be a fuzzy set of L. Then a q-soft set (F_q, A) over L with $A = (0, 0.5]$ is a G-filteristic soft MTL-algebra if and only if μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy G-filter.*

Theorem 4.3.10. *Let μ be a fuzzy set of L. Then a q-soft set (F_q, A) over L with $A = (0.5, 1]$ is a G-filteristic soft MTL-algebra if and only if μ is an $(\in, \in \vee q)$-fuzzy G-filter of L.*

As a consequence of Theorems 3.7, 4.3.4 and 4.3.6 we obtain

Theorem 4.3.11. *For a fuzzy set μ of L and an $\in$-soft set (F, A) over L with $A = (\alpha, \beta]$, where $0 < \alpha < \beta \leq 1$, the following conditions are equivalent:*
  (i) μ is a fuzzy G-filter with thresholds $(\alpha, \beta]$,
  (ii) (F, A) is a G-filteristic soft MTL-algebra over L.

Finally, we give the relationship between the filteristic soft MTL-algebras described above.

Theorem 4.3.12. *Let μ be a fuzzy set of L. If an $\in$-soft set (F, A) over L with $A = (\alpha, \beta] \subset (0, 1]$ is a Boolean filteristic soft MTL-algebra, then it also is a G-filteristic soft MTL-algebra, but the converse may not be true.*

Proof. It is a consequence of Theorems 4.1.6, 4.1.7, 4.1.8, 4.3.5, 4.3.6, 4.3.8 and Theorem 4.5 in [21]. □

Theorem 4.3.13. *Let μ be a fuzzy set of L. Then an $\in$-soft set (F, A) over L with $A = (\alpha, \beta] \subset (0, 1]$ is a Boolean filteristic soft MTL-algebra if and only if it is both an MV-filteristic soft MTL-algebra and a G-filteristic soft MTL-algebra.*

Proof. It is a consequence of Theorems 4.2.13, 4.3.12 and Theorem 4.5 in [21]. □

5. Conclusion

In this paper, we apply fuzzy and soft set theory to MTL-algebras. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of soft MTL-algebras and other algebraic structure.

In our future study of MTL-algebras, may be the following topics should be considered:

(1) To describe the soft MTL-algebras based on rough sets;

(2) To discuss the relations between soft MTL-algebras based on fuzzy sets and rough sets;

(3) To consider the soft implication-based fuzzy filters in MTL-algebras.

Acknowledgements

References

[1] H. Aktas, N. Cagman, Soft sets and soft groups, Inform. Sci., 177 (2007), 2726–2735.

[2] S. K. Bhakat, P. Das, $(\in, \in \vee \, \mathrm{q})$-fuzzy subgroup, Fuzzy Sets Syst., 80 (1996), 359–368.

[3] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous, t-norms, Fuzzy Sets Syst., 124 (2001), 271–288.

[4] P. Flondor, G. Georgescu, A. Iorgulescu, Pseudo-t-norms and pseudo-Bl algebras, Soft Computing, 5 (2001), 355–371.

[5] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Press, Dordrecht, 1998.

[6] Y.B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008), 1408–1413.

[7] Y.B. Jun, C.H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci., 178 (2008), 2466–2475.

[8] Y.B. Jun, Y. Xu, X.H. Zhang, Fuzzy filters of MTL-algebras, Inform. Sci., 175 (2005), 120–138.

[9] K.H. Kim, Q. Zhang, Y.B. Jun, On fuzzy filters of MTL-algebras, J. Fuzzy Math., 10 (2002), 981–989.

[10] X. Ma, J. Zhan, A note on "Generalized fuzzy filters of MTL-algebras", J. Multiple-Valued Logic and Soft Computing, 15 (2009), 1–4.

[11] X. Ma, J. Zhan, D. Xiang, Some types of generalized fuzzy filters of MTL-algebras, J. Intelligent and Fuzzy Systems, 20 (2009), in print.

[12] X. Ma, J. Zhan, Y. Xu, Generalized fuzzy filters of MTL-algebras, J. Multiple-Valued Logic and Soft Computing, 14 (2008), 119-128.

[13] P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, Comput. Math. Appl., 45 (2003), 555–562.

[14] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl., 44 (2002), 1077–1083.

[15] D. Molodtsov, Soft set theory - First results, Comput. Math. Appl., 37 (1999), 19–31.

[16] P. M. Pu, Y. M. Liu, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl., 76 (1980), 571–599.

[17] E. Turunen, Boolean deductive systems of MTL-algebras, Arch. Math. Logic, 40 (2001), 467–473.

[18] L.A. Zadeh, From circuit theory to system theory, Proc. Inst. Radio Eng., 50 (1962), 856–865.

[19] L.A. Zadeh, Toward a generalized theory of uncertainty (GTU)-an outine, Inform. Sci., 172 (2005), 1–40.

[20] X.H. Zhang, K.Y. Qin, W.A. Dudek, Ultra LI-ideals in lattice implication algebras and MTL-algebras, Czechoslovak Math. J., 57(132) (2007), 591–605.

[21] X.H. Zhang, Y.Q. Wang, Y.L. Liu, Fuzzy ultra filters and fuzzy G-filters of MTL-algebras, in: L.Wang and Y.Jin(Eds.): FSKD, LNAI 3613 (2005) 160–166.