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An extended trace identity and applications

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Abstract

For the loop algebras in the form of non-square matrices, their commuting operations can be used to set up linear isospectral problems. In order to look for the Hamiltonian structures of the corresponding integrable evolution hierarchies of equations, an extended trace identity is obtained by means of commutators, which undoes the constraint on the known trace identity proposed by Tu [Guizhang Tu. The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J Math Phys 1989;30(2):330–8], and has an obvious simplicity comparing with the quadratic-form identity given by Guo and Zhang [Fukui Guo, Yufeng Zhang. The quadratic-form identity for constructing the Hamiltonian structure of integrable systems. J Phys A 2005;38:8537–48] with the aspect of applications.

1. Introduction

In the section, we briefly recall the applications of the loop algebra $\widetilde{A}_1$. The necessity of improving the trace identity is proposed. Researching for integrable systems has been an important aspect in soliton theory [1]. Tu Guizhang [2] constructed the 2×2 loop algebra $\widetilde{A}_1$:

$$\widetilde{A}_1 = \{A = (a_{ij}(\lambda))_{2\times 2}, \quad a_{ij}(\lambda) = a_{ij}\lambda^m, \quad m = 0, \pm 1, \pm 2, \ldots, \mathrm{tr}(A) = 0\}$$

from which the isospectral Lax pairs could be established to further generate the zero curvature equation. It followed that the hierarchies of soliton equations were obtained, such as the AKNS hierarchy, the TA hierarchy, the BPT hierarchy and so on. Then their Hamiltonian structures were worked out by employing the trace identity [2,3]. In Refs. [4–6], a type of loop algebra $\widetilde{G}_0$ once was proposed which was used to get the expanding integrable systems, i.e., integrable couplings, of the AKNS hierarchy, the KN hierarchy, etc. However, their Hamiltonian structures could not be obtained by using the trace identity [2], it could not be reached that whether they were Liouville integrable or not. In another hand, a simple method for obtaining multi-component Lax integrable systems was presented with the help of the loop algebra $\widetilde{G}_1$ in the vector form in Ref. [5]. Since $\widetilde{G}_1$ is not presented in the square-matrix form, their Hamiltonian structures could not be obtained by use of the trace identity as well.

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In this paper, Section 2 presents two forms of the loop algebras and the trace identity as well as the quadratic-form identity. In Section 3, a kind of loop algebra $\widetilde{G}$ presented by the column vector form and a commutator are constructed, respectively. In Section 4, an extended trace identity under the framework of the loop algebra $\widetilde{G}$ is constructed. As its application, the Hamiltonian structure of an integrable system is exhibited.

2. Two types of loop algebras

We present two forms of the loop algebras. One is the square-matrix form $\widetilde{A}_{N-1}$, another is the s-dimensional column vector form $\widetilde{G}$. First, we introduce the loop algebra $\widetilde{A}_{N-1}$ and the resulting Lax pairs. $\widetilde{A}_{N-1}$ refers to

$$\widetilde{A}_{N-1} = \{A = (a_{ij}(\lambda))_{N\times N}, \quad a_{ij}\lambda^m, \quad m = 0, \pm 1, \pm 2, \ldots, \mathrm{tr}(A) = 0\}, \tag{1}$$

where the commuting operation is defined as $[A, B] = AB - BA, \quad A, B \in \widetilde{A}_{N-1}$. The linear isospectral problem related to $\widetilde{A}_{N-1}$ is given by

$$\begin{cases} \psi_x = U\psi, \quad U = e_0 + \sum\limits_{i=1}^{p} u_i e_i, \quad \lambda_t = 0, \quad \psi = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}, \\ \psi_t = V\psi, \quad \{e_0, \ldots, e_p\} \subset \widetilde{A}_{N-1}, \quad u = (u_1, \ldots, u_p)^{\mathrm{T}} \end{cases} \tag{2}$$

whose compatibility condition is the zero curvature equation

$$U_t - V_x + [U, V] = 0. \tag{3}$$

The corresponding stationary zero curvature equation is that

$$V_x = [U, V]. \tag{4}$$

If the rank numbers of λ and $u_i (1 \leqslant i \leqslant p)$ can be defined, denoted by $\mathrm{rank}(\lambda)$, rank (u_i), respectively, such that $\mathrm{rank} e_0 = \mathrm{rank} u_i e_i = \alpha, 1 \leqslant i \leqslant p$, then U is called the same-rank, noted by

$$\mathrm{rank}(U) = \mathrm{rank}(\partial) = \alpha, \tag{5}$$

where $\partial = \frac{\partial}{\partial x}$. Take a solution of Eq. (4) to be $V = \sum_{m\geqslant 0} V_m \lambda^{-m}$, if $\mathrm{rank}(V_m)$ can be defined such that $\mathrm{rank}(V_m\lambda^{-m}) = \eta = \mathrm{const.}, m \geqslant 0$, then we call V the same-rank, denoted by $\mathrm{rank}(V) = \eta$. Let two same-rank solutions V_1 and V_2 of Eq. (4) be linear dependent, i.e.

$$V_1 = \gamma V_2, \quad \gamma = \mathrm{const.}, \tag{6}$$

then we have

Theorem 1 ([1–3]). *If the relations (5) and (6) all hold, then*

$$\frac{\delta}{\delta u_i}\langle V, U_\lambda\rangle = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^\gamma\left\langle V, \frac{\partial U}{\partial u_i}\right\rangle\right), \tag{7}$$

where V is a same-rank solution to Eq. (4), $\gamma = \mathrm{const.}, \quad \langle A, B\rangle = \mathrm{tr}(AB), \quad A, B \in \widetilde{A}_{N-1}$, *the identity (7) is called the trace identity, which has been proved in* Ref. [1].

Although we call the power series $V = \sum_{m\geqslant 0} V_m\lambda^{-m}$ the solution to Eq. (4), it is only required that the series makes the coefficients of the same-order powers of λ equal. Only part sum can be used in the integrable systems derived from Eq. (3) and some of the terms in V are required when using Eq. (7) to produce Hamiltonian structures. These facts are independent of the convergence or dispersion of V. That is why we do not discuss the convergence of V. The loop algebra $\widetilde{A}_1$ is often used for convenience, that is, take $\widetilde{A}_{N-1} = \widetilde{A}_1$ in Eqs. (2) and (7).

Let G be a s-dimensional Lie algebra with basis $e_1, e_2, \ldots, e_s$. Take $a = \sum_{k=1}^{s} a_k e_k, \; b = \sum_{k=1}^{s} b_k e_k \in G$. The commutator in G is given by $c = [a, b] = \sum_{k=1}^{s} c_k e_k$. The loop algebra $\widetilde{G}$ generated by G has the basis $e_k(m) = e_k\lambda^m, 1 \leqslant k \leqslant s, m \in \mathbf{Z}$(a integer set), the commuting operations read $[e_k(m), e_j(n)] = [e_k, e_j]\lambda^{m+n}$. The column vector form of $\widetilde{G}$ is given by

$$\widetilde{G} = \{a = (a_1, \ldots, a_s)^{\mathrm{T}}, \quad a_k = \sum_m a_{k,m}\lambda^m, \quad [a, b] = c = (c_1, \ldots, c_s)^{\mathrm{T}}\}. \tag{8}$$

The linear isospectral Lax pairs by using $\widetilde{G}$ can be taken as

$$\begin{cases} \psi_\partial = [U, \psi], \quad U, V, \psi \in \widetilde{G}, \\ \psi_t = [V, \psi], \quad \lambda_t = 0 \end{cases} \tag{9}$$

where $\partial = \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k}$, α_k are arbitrary constants, ψ_∂ denotes the derivative sum of ψ with aspect to x_k $(k = 1,2,\ldots,n)$. The compatibility condition of Eq. (9) is the zero curvature equation

$$U_t - V_\partial + [U, V] = 0 \tag{10}$$

whose resulting stationary auxiliary equation reads

$$V_\partial = [U, V]. \tag{11}$$

Take $U = U(\lambda, u) = U_0 + \sum_{i=1}^{p} u_i U_i, U_i \in \widetilde{G}, u = (u_1, \ldots, u_p)^{\mathrm{T}}$, assume $\operatorname{rank} U_0 = \operatorname{rank}(u_i U_i) = \alpha, 1 \leqslant i \leqslant p$, then U is called the same-rank, denoted by

$$\operatorname{rank}(U) = \operatorname{rank}(\partial) = \operatorname{rank}\left(\frac{\partial}{\partial x_k}\right) = \alpha, \quad 1 \leqslant k \leqslant n. \tag{12}$$

Let two same-rank solutions V_1 and V_2 satisfy the relation (6), then we see that

Theorem 2 [7] the quadratic-form identity. *Let* Eq. (12) *hold. Two same-rank solutions of* Eq. (11) *possess* Eq. (6). *Set* $[a,b]^{\mathrm{T}} = a^{\mathrm{T}} R(b), \quad a, b \in \widetilde{G}$, *the constant matrix* $F = (f_{ij})_{s\times s}$ *meets*

$$F = F^{\mathrm{T}}, R(b)F = -(R(b)F)^{\mathrm{T}}. \tag{13}$$

Define a quadratic functional as follows:

$$\{a, b\} = a^{\mathrm{T}} F b, \quad \forall a, b \in \widetilde{G}, \tag{14}$$

then the following identity holds

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \quad i = 1, \ldots, p \tag{15}$$

where γ *is a constant to be determined*, V *is a same-rank solution of* Eq. (11). Eq. (15) *is called the quadratic-form identity. The theorem has been proved in* Ref. [7]. Eqs. (9) and (15) *eliminate the constraint on* Eqs. (2) and (7). *When* $\widetilde{G} = \widetilde{A}_1$, Eq. (15) *is completely consistent to* Eq. (7).

3. Commutator

In this section, a new commutator is defined such that V_s (below) becomes a Lie algebra. Let $\widetilde{G}$ be the loop algebra (8),

$$[a, b] = Q(a)b, \quad \forall a, b \in \widetilde{G}, \tag{16}$$

since $[a, b]$ is known, $Q(a)$ is a determined $s \times s$ matrix. In terms of the bi-linearity of $[a, b]$, Q is a linear operator in a. From $[a, b] = -[b, a]$, we see that

$$Q(a)b = -Q(b)a, \quad a, b \in \widetilde{G}. \tag{17}$$

It is easy to find that Q has the relation with $R(a)$ presented in Theorem 2

$$Q(a) = -R^{\mathrm{T}}(a), \quad a \in \widetilde{G}. \tag{18}$$

Definition. Let V_s be a s-dimensional linear space, M_s be a matrix set. Q is an operator from V_s to M_s, and $Q(a) \in M_s$, $a \in V_s$. If Q is linear and meets

$$Q(Q(a)b) = [Q(a), Q(b)] = Q(a)Q(b) - Q(b)Q(a), \quad \forall a, b \in V_s, \tag{19}$$

then Q is called the commutator in V_s. All the Q's in V_s are regarded as a set, denoted by $K(V_s, M_s)$.

Theorem 3. *Let* V_s *be a s-dimensional linear space,* $a,b \in V_s$, *then* V_s *is a Lie algebra with the commuting operation if and only if there exists* $Q \in K(V_s, M_s)$ *to satisfy*

$$[a, b] = Q(a)b. \tag{20}$$

Proof. Let V_s be a Lie algebra together with the commutative operation $[a, b] = Q(a)b$, then Q is a linear operator from V_s to M_s and satisfies Eq. (17). Consider the Jacobi identity

$$[a,[b,c]]+[b,[c,a][c,[a,b]]=Q(a)Q(b)c-Q(b)Q(a)c-Q([a,b])c=0,\quad a,b,c\in V_s.$$

Since c is arbitrary, then $Q([a,b]) = Q(a)Q(b) - Q(b)Q(a) = [Q(a),Q(b)]$, $a,b \in V_s$. In terms of $Q([a,b]) = Q(Q(a)b)$, (4) indeed holds, $Q \in K(V_s, M_s)$.

On the contrary, let $Q \in K(V_s, M_s)$, $\forall a,b \in V_s$. Regard $[a, b] = Q(a)b$ as a commutator in V_s. Due to c being linear, $[a, b]$ is bilinear. From (19), $Q(Q(a)b) = -Q(Q(b)a)$, thus, $Q(a)b = -Q(b)a$ due to Q being a linear operator, which implies that $[a, b]$ is anti-symmetric. Again by using (19), $[a, b] = Q(a)b$ satisfies the Jacobi identity, i.e., the commutative operation $[a, b] = Q(a)b$ makes V_s become a Lie algebra.

The proof is completed. □

Corollary. *Let*

$$Q(a)=\begin{pmatrix}\alpha_1a_2+\alpha_2a_3 & -\alpha_1a_1+\alpha_3a_3 & -\alpha_2a_1-\alpha_3a_2\\ \beta_1a_2+\beta_2a_3 & -\beta_1a_1+\beta_3a_3 & -\beta_2a_1-\beta_3a_2\\ \gamma_1a_2+\gamma_2a_3 & -\gamma_1a_1+\gamma_3a_3 & -\gamma_2a_1-\gamma_3a_2\end{pmatrix},\tag{21}$$

where $a = (a_1, a_2, it\ a_3)^{\mathrm{T}}$, $\alpha_i,\beta_i,\gamma_i$ *are all constants, then* $Q \in K(V_s,M_s)$ *if and only if*

$$\begin{cases}\alpha_3\beta_1-\alpha_1\beta_3=\alpha_2\gamma_3-\alpha_3\gamma_2,\\ \alpha_1\beta_2-\alpha_2\beta_1=\beta_2\gamma_3-\beta_3\gamma_2,\\ \alpha_1\gamma_2-\alpha_2\gamma_1=\beta_3\gamma_1-\beta_1\gamma_3.\end{cases}\tag{22}$$

Example. $V_s = R_3 = \{a = a_1i + a_2j + a_3k = (a_1, a_2, a_3)^{\mathrm{T}}\}$, the vector product of a and b in R_3 presents

$$a\times b=\det\begin{pmatrix}i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}=(a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1)^T=\begin{pmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}=Q(a)b.\tag{23}$$

Since Q meets Eq. (22), R_3 is a Lie algebra along with the commutative operation $[a, b] = a \times b$.

4. Extended trace identity

In the section, the major result, i.e. the extended trace identity is presented. Then an application example is given. Take the linear isospectral problem to be (9), which can be written as by means of Q in $\widetilde{G}$

$$\begin{cases}\psi_\partial=Q(U)\psi,\\ \psi_t=Q(V)\psi,\end{cases}\tag{24}$$

whose corresponding zero curvature equation is

$$Q(U_t)-Q(V_\partial)+[Q(U),Q(V)]=0.\tag{25}$$

The stationary zero curvature equation reads

$$Q(V_\partial)=[Q(U),Q(V)].\tag{26}$$

According to the linearity of Q and Eq. (19), Eq. (25) is equivalent to Eq. (10), and Eq. (26) equivalent to Eq. (11). In terms of the linearity of Q, we have $\mathrm{rank}(Q(U)) = \mathrm{rank}(U)$, $\mathrm{rank}(Q(V)) = \mathrm{rank}(V)$. $V_1 = \gamma V_2$ is equivalent to

$$Q(V_1)=\gamma Q(V_2).\tag{27}$$

Since Eqs. (24) and (2) have the common structure, according to the Theorem 1, we conclude that

Theorem 4. *Let the relations (5) and (6) hold, then the identity holds as follows*:

$$\frac{\delta}{\delta u_i}\langle Q(V), Q(U_\lambda)\rangle = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^\gamma\left\langle Q(V), Q\left(\frac{\partial U}{\partial u_i}\right)\right\rangle\right), \quad i = 1,\dots,p \tag{28}$$

where γ is a constant to be determined, V is a same-rank solution to Eq. (11), *Q is a commutator in $\widetilde{G}$. We call* Eq. (28) *the extended trace identity. The proof of the theorem is similar to* Theorem 1. *Here we omit it.* Theorem 1 *only suits for the loop algebra $\widetilde{A}_{N-1}$ in the square-matrix forms, while* Theorem 4 *eliminates the constraint.*

From the above discussions, the steps to generate the Hamiltonian integrable hierarchies of equations are as follows:

(1) Solving Eq. (11), the part sum of V, i.e. V can be expressed as the form $V = \sum_{m=0}^{n}(a_{m1}, a_{m2},\dots,a_{ms})^{\mathrm{T}}\lambda^{n-m}$, consists of the polynomial $V^{(n)}$ in λ. The zero curvature equation

$$U_t - V^{(n)}_{\partial} + [U, V^{(n)}] = 0$$

determines the integrable hierarchy

$$u_{t_n} = K_n(u).$$

(2) By making use of Theorem 4 to obtain J and the Hamiltonian functions H_n, the Hamiltonian structure of the obtained integrable hierarchy is given by

$$u_{t_n} = K_n(u) = J\frac{\delta H_n}{\delta u}.$$

Comparing Theorem 4 and Theorem 2, we find that Theorem 4 promotes computational simplicity. But there are some examples to imply that Theorem 4 cannot be applied instead of Theorem 2. That is to say, the two theorems play their own roles in generating Hamiltonian structures. Applying Theorem 4 or Theorem 2 can give rise to the Hamiltonian structures of the integrable hierarchies presented in [4–7]:

$$\text{For } \widetilde{A}_1 = \left\{A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1\end{pmatrix} = a_1h + a_2e + a_3f = (a_1, a_2, a_3)^{\mathrm{T}}\right\},$$

$$Q(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ -2a_2 & 2a_1 & 0 \\ 2a_3 & 0 & -2a_1\end{pmatrix}. \tag{29}$$

If let

$$\begin{cases} A = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1\end{pmatrix} \leftrightarrow a = (a_1, a_2, a_3)^{\mathrm{T}}, \\ B = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1\end{pmatrix} \leftrightarrow b = (b_1, b_2, b_3)^{\mathrm{T}}, \\ \langle Q(a), Q(b)\rangle = 4\langle A, B\rangle, \end{cases} \tag{30}$$

then Theorem 4 and Theorem 2 play the same roles when using $\widetilde{A}_1$.

In the following, we take an example to illustrate the applications of the extended trace identity.

Example. Take $V_3 = R_3$, U and V in Eq. (9) are presented:

$$U = (\lambda, q, r)^{\mathrm{T}}, \quad V = \sum_{m\geqslant 0} V_m\lambda^{-m} = \sum_{m\geqslant 0}(V_{1m}, \quad V_{2m}, V_{3m})^{\mathrm{T}}\lambda^{-m}.$$

Solving Eq. (11) yields

$$\begin{cases} V_{1m\partial} = qV_{3m} - rV_{2m}, \\ V_{2,m+1} = qV_{1m} + V_{3m\partial}, \\ V_{3,m+1} = rV_{1m} - V_{2m\partial}, \\ V_{2,0} = V_{3,0} = 0, V_{1,0} = \beta = \text{const.}, \\ \text{rank}(\lambda) = \text{rank}(q) = \text{rank}(r) = \text{rank}(U) = \text{rank}(\partial) = 1, \\ \text{rank}(\mathrm{V_m}) = \mathrm{m}, \quad \text{rank}(\mathrm{V}) = 0. \end{cases} \tag{31}$$

Denote $V^{(n)} = \sum_{m=0}^{n} V_m\lambda^{n-m}$, $V^{(n)}_{-} = \lambda^n V - V^{(n)}$, then

$$U_t - V_{\partial}^{(n)} + [U, V^{(n)}] = 0$$

determines the integrable hierarchy

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -V_{3,n+1} \\ V_{2,n+1} \end{pmatrix}. \tag{32}$$

Comparing the coefficients of λ^{-n-1} in Eq. (28) gives

$$\frac{\delta}{\delta u_i}\langle Q(V_{n+1}), Q(U_\lambda)\rangle = (-n+\gamma)\left\langle Q(V_n), Q\left(\frac{\partial U}{\partial u_i}\right)\right\rangle,$$

where $u_1 = q$, $u_2 = r$. Thus,

$$\frac{\delta}{\delta u}(-2V_{1,n+1}) = (-n+\gamma)\begin{pmatrix} -2V_{2n} \\ 2V_{3n} \end{pmatrix}.$$

Taking $n = 1$ gives $\gamma = 0$. Thus, the system (32) can be written as:

$$\begin{cases} u_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} V_{2,n+1} \\ V_{3,n+1} \end{pmatrix} = J\begin{pmatrix} V_{2,n+1} \\ V_{3,n+1} \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \\ H_{n+1} = -\frac{V_{1,n+2}}{n+1}, n \geqslant 0. \end{cases} \tag{33}$$

From Eq. (31), the recurrence operator L satisfies that

$$\begin{pmatrix} V_{2,n+1} \\ V_{3,n+1} \end{pmatrix} = L\begin{pmatrix} V_{2n} \\ V_{3n} \end{pmatrix}, \quad L = \begin{pmatrix} -q\partial^{-1}r & \partial + q\partial^{-1}q \\ -\partial - r\partial^{-1}r & r\partial^{-1}q \end{pmatrix}.$$

Therefore, the hierarchy (33) can be written as:

$$u_t = JL^n\begin{pmatrix} \beta q \\ \beta r \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \quad n \geqslant 0. \tag{34}$$

It is easy to verify that $JL = L^*J$. Hence, the Hamiltonian functions of the hierarchy (34) are involutive to each other, and each Hamiltonian function is its common conserved density. Thus, the hierarchy (34) is a $(1+n)$-dimensional Liouville integrable hierarchy of equations. Because the AKNS hierarchy was derived from the loop algebra $\widetilde{A}_1$, while the hierarchy (34) is obtained by using the loop algebra $\widetilde{G}$ presented in the paper, they have various commuting operations. Hence, they have various recurrence operators. In terms of Ref. [2], the recurrence operator of the AKNS hierarchy is presented as $L = \begin{pmatrix} \frac{\partial}{2} - r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & -\frac{\partial}{2} + q\partial^{-1}r \end{pmatrix}$, which is different from that in the hierarchy (34).

In this paper, we defined a commutator and constructed the extended trace identity, which is a powerful tool for generating Hamiltonian structures of the soliton equations. As its application, we obtained the Hamiltonian structure of the system (34) by using the Lie algebra V_3. We can construct various higher-dimensional Lie algebras G to generate more interesting integrable hierarchies and the resulting Hamiltonian structures by using the extended trace identity. Therefore, the method proposed in this paper has extensive applications.

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