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# An efficient $p$-version multigrid solver for fast hierarchical vector finite element analysis

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ABSTRACT

In this paper, a kind of $p$-type multigrid (MG) method is applied to solve the large sparse linear systems arising from the application of hierarchical tangential vector finite element method (TVFEM) for the analysis of electromagnetic devices. Several waveguide problems are analyzed and our numerical results show that the $p$-type MG method can greatly save iterations and CPU time when compared with the preconditioned conjugate gradient iterative methods.



## 1. Introduction

The finite element methods have been applied to the analysis of problems in electromagnetics for more than 30 years. It can combine geometrical adaptability and material generality for modeling arbitrary geometry and materials of any composition. As a result, a large number of research papers can be found in literature [1–3]. In recent years, the tangential vector finite element (TVFE) method [1] has gained more and more importance in the analysis of electromagnetic problems. This is due to the TVFE method (TVFEM) has many advantages compared with the traditional node-based finite elements. With the TVFEM, enforce continuity of vector components across element interfaces is automatically satisfied, Dirichlet boundary conditions can be easily enforced and spurious modes can be eliminated. Because of these special features, vector elements seem to be ideally suited for the analysis of vector partial differential equations (PDEs) in electromagnetics.

The most commonly used TVFEM is the $H_0$(curl) TVFEM [4], which provides a constant value along each edge. As a result, it is often called the edge-based TVFEM. For electric-large and complex problems, the application of $H_0$(curl) TVFEM will lead to a large sparse linear system that is difficult to solve with iterative methods. The slow convergence of iterative solvers is mainly caused from the ill-conditioning of the stiff matrix due to the over-sampling of the low frequency physical modes [5]. The ill-conditioning of the stiff matrix makes any conventional iterative method unpredictable. Taking the most powerful incomplete Cholesky preconditioned conjugate gradient method (ICCG) as an example, breakdown may occur during the incomplete IC factorization process due to the ill-conditioning [6]. Thus, more efficient iterative solvers should be developed in the analysis of microwave problems with the TVFEM.

In recent years, the multigrid-type (MG) methods have gained much attention in solving large sparse linear systems resulting from the discretization of PDEs [7]. The basic idea of this kind of methods is: slow convergence of iterative solvers is mainly caused by the low frequency component of the solution. If the problem was discretized by different levels of meshes with different densities, the low frequency error can be eliminated through correcting the solution on the much coarser grids. Originally, the MG algorithms were primarily used to deal with scalar, second-order elliptic PDEs on structured grids. With its development by many researchers, MG methods have grown to solve more and more problems. Now they have been regarded as one of the most promising iterative methods for solving systems of linear equations arising from the discretization of PDEs by either the finite difference or the $h$-version finite element method. Recently, some researchers developed the edge element-based geometric MG methods and have proven to be very efficient for solving systems of equations resulting from the finite element analysis of electromagnetic fields [8–10]. The MG methods have shown $O(n)$ complexity, and the convergence rate is mesh-independent for many problems [7]. For comparison, the widely spread ICCG method requires $O(n^2)$ operations for three-dimensional (3D) problems [11]. These properties make the MG method outperformed any other iterative methods in the 3D electromagnetic simulations.

Instead of using different levels of meshes, the MG method can be implemented through using different order elements with the

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help of hierarchical basis [12,13]. The linear system resulted from the application of hierarchical order finite element method has the hierarchical property and can be solved with the MG method by correcting the solution on different order spaces. To distinguish these two methods, the former is usually called the $h$-version MG method, while the latter is called the $p$-version MG method. In the $p$-version MG, the high order spaces correspond to fine grids which are as in the $h$-version MG method. And the low order spaces correspond to coarse grids. This means we can use less grid points in high order method. Specifically, the $p$-version MG method can avoid the complicated and time-consuming mesh refinement process of the $h$-version method, which always generates a very large linear system. In addition, the solution of the prolongation and projection operators in the $h$-version MG method is very sophisticated and may need much CPU time and storage costs for many problems. Contrary, the $p$-version MG method usually has very simple prolongation and projection operators and is very easy for application compared with the $h$-version MG method.

The $p$-version MG method is not only useful for high frequency electromagnetic fields, but also efficient for low frequency eddy current problems and complicated static fields as well. Otherwise, it has been mentioned in [21,22] that MG method always results in a computationally expensive algorithm in high dimensional problem. In fact, the sparse matrix obtained by hierarchical basis has large condition number. It makes the iterative solver more difficult. But the efficient algorithm could be got according to the trait of the high order basis. It needs much more research on this aspect. Fortunately, some investigators have proved that the $p$-version method can get much faster convergence rates than the $h$-version method [14]. In [15], a kind of $p$-version multigrid method is suggested for the hierarchical finite element method, which was called the multi-$p$ in the paper. In this paper, we extend this method to solve the 3D time-harmonic electric-based vector Helmholtz equations in electromagnetics.

## 2. A $p$-version multigrid method

In analyzing the high frequency electromagnetic problems, the three-dimensional boundary value problems (BVP) in electromagnetics, described by the time-harmonic electric field-based Helmholtz equation and the corresponding boundary conditions, is often considered:

$$\begin{cases} \nabla \times (\mu_r^{-1} \nabla \times \vec{\mathbf{E}}(r)) - k_0^2 \varepsilon_r \vec{\mathbf{E}}(r) = 0 \\ \hat{n} \times \vec{\mathbf{E}}(r) = 0 \quad \text{on } \mathbf{\Gamma}_E \subset \partial\Omega \\ \hat{n} \times (\nabla \times \vec{\mathbf{E}}(r)) = 0 \quad \text{on } \mathbf{\Gamma}_H \subset \partial\Omega \end{cases} \tag{1}$$

where the notation $k_0$ stands for the wave number of electromagnetic wave, $\Omega$ and $\partial\Omega$ represent a bounded domain in $C^3$ and its boundary, $\hat{n}$ denotes the normal external to $\Omega$ along $\partial\Omega$ and $\mathbf{\Gamma}_E$ and $\mathbf{\Gamma}_H$ denotes the perfect electric conductor (PEC) and perfect magnetic conductor (PMC), respectively.

In this paper, we use the Galerkin method [1] to derive the finite element system. As tetrahedral elements have strong ability in modeling arbitrary structures, in our FEM analysis the tetrahedral elements are used for mesh discretization. Multiplying the Helmholtz equation in (1) by a weighting function $\vec{\mathbf{W}}$ and integrating over each finite element volume $\mathbf{\Omega_i}$, we can then obtain the weak form TVFEM formulation, which is of the following form:

$$\int_{\mathbf{\Omega_i}} (\nabla \times \vec{\mathbf{W}}_\mathbf{i}) \cdot \mu_r^{-1} (\nabla \times \vec{\mathbf{E}}_\mathbf{i})\,\mathrm{d}v - \int_{\mathbf{\Omega_i}} \vec{\mathbf{W}}_\mathbf{i} \cdot \varepsilon_r^{-1} \vec{\mathbf{E}}_\mathbf{i}\,\mathrm{d}v = 0$$

The above equation can be obtained for all the meshes in the volume of interest.

Setting the expansion functions equal to the weighting function $\vec{\mathbf{W}}$, the electric field $\vec{\mathbf{E}}_\mathbf{i}$ within the $i$th mesh can be expanded as

$$\vec{\mathbf{E}}_\mathbf{i} = \sum \mathbf{x_j} \vec{\mathbf{W}}_\mathbf{ij}$$

Here $\vec{\mathbf{W}}_\mathbf{ij}$ is the $j$th expansion function in the $i$th mesh. To implement the $p$-version MG method, the hierarchical high order basis functions are used. The hierarchical basis is constructed in such a way that low order FE spaces are included in high order FE spaces. As a result, the hierarchical FE basis has the natural wavelet property (it does not mean the basis is orthogonal). Most of the low frequency components of the solution are contained in the unknowns connected with the low order basis functions, while the high frequency components are contained in the unknowns connected with the high order basis functions. The $H_1$(curl) TVFE tetrahedral elements we use here are the hierarchical basis functions proposed by Webb [12], which provide linear tangential/quadratic normal variation along element edges and quadratic variation at element faces and inside the element, and are characterized by 20 linearly independent vector basis functions of the following form:

$$\begin{array}{ll} \mathbf{w}_{ij} = \zeta_i \nabla\zeta_j - \zeta_j \nabla\zeta_i & \text{(1 per edge)} \\ \mathbf{g}_{ij} = \zeta_i \nabla\zeta_j + \zeta_j \nabla\zeta_i & \text{(2 per edge)} \\ \mathbf{f}_{ijk} = \zeta_k(\zeta_i \nabla\zeta_j - \zeta_j \nabla\zeta_i) & \text{(3 per edge)} \end{array} \tag{2}$$

In the above formulation, $\mathbf{w}_{ij}$ is the basis functions that span the $H_0$(curl) finite element space. We can see that this kind of basis can be split into two groups—pure gradient basis functions and solenoid-like vector basis functions and are most appropriate for finite element implementations.

Combining each individual sub-linear system together, we can obtain the resulting finite element system

$$[\mathbf{S_{EE}}] - \mathbf{k_0^2}[\mathbf{T_{EE}}] = \mathbf{0} \tag{3}$$

Here, $[\mathbf{S_{EE}}]$ and $[\mathbf{T_{EE}}]$ are defined as

$$[\mathbf{S_{EE}}]_\mathbf{ij} = \int_\mathbf{\Omega} (\nabla \times \vec{\mathbf{W}}_\mathbf{i}) \cdot \mu_r^{-1} (\nabla \times \vec{\mathbf{W}}_\mathbf{j})\,\mathrm{d}v$$

$$[\mathbf{T_{EE}}]_\mathbf{ij} = \int_\mathbf{\Omega} \vec{\mathbf{W}}_\mathbf{i} \cdot \varepsilon_r^{-1} \vec{\mathbf{W}}_\mathbf{j}\,\mathrm{d}v$$

After eliminating the boundary elements on the PECs, we finally obtain a large sparse algebraic linear system, which can be written in the following form:

$$\mathbf{A_p x_p} = \mathbf{b_p} \tag{4}$$

where $\mathbf{A_p} \in C^{n\times n}$, $\mathbf{b_p} \in C^n$, $\mathbf{p}$ is the order of the basis functions. To solve this linear system, the most powerful and promising method is the multigrid method. As we have known, the basic idea of the multigrid methods is to solve the problem on several level grids with different densities. The high frequency error components are damped on the fine grid after a few iterations, while the low frequency error components can be corrected on the coarser grid. In the process of MG method, for the projection of the unknown values from the high order space to low order space, the restriction operator is needed. On the contrary, to give an approximation to the high order space, a prolongation operator should be defined. The efficiency of the MG method strongly depends on the choice of the transfer operators. In the geometric MG method, the definition of the transfer operators is a difficult task. However, with the help of hierarchical basis, the transfer operators of the $p$-version MG method can be easily deduced and has very simple form. In order to explain the procedure that leads to the construction of these operators, here we consider a two

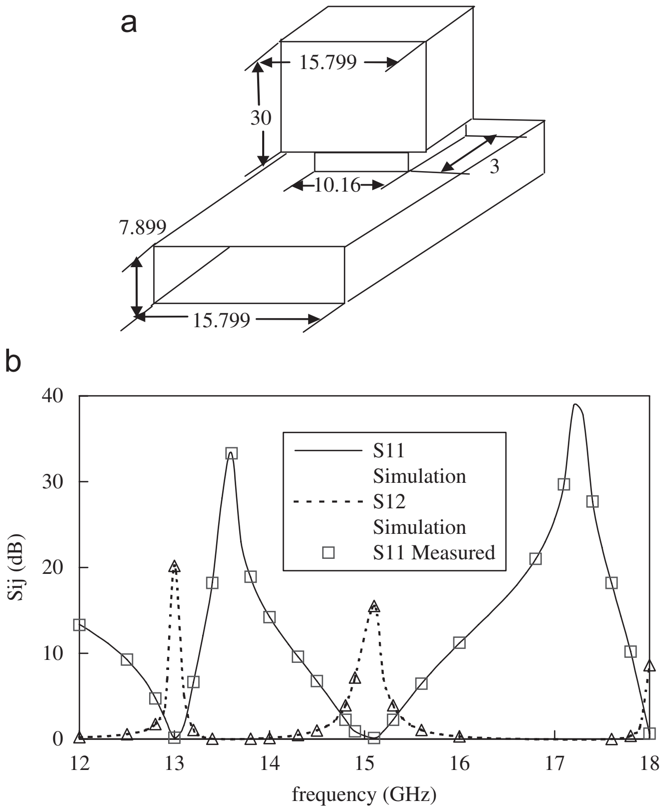


**Fig. 1.** (a) The geometry and dimensions of a short-circuited E-plane slot-coupled T-junction. (b) Magnitude of $S_{11}$, $S_{12}$ versus frequency for the short-circuited E-plane slot-coupled T-junction problem.

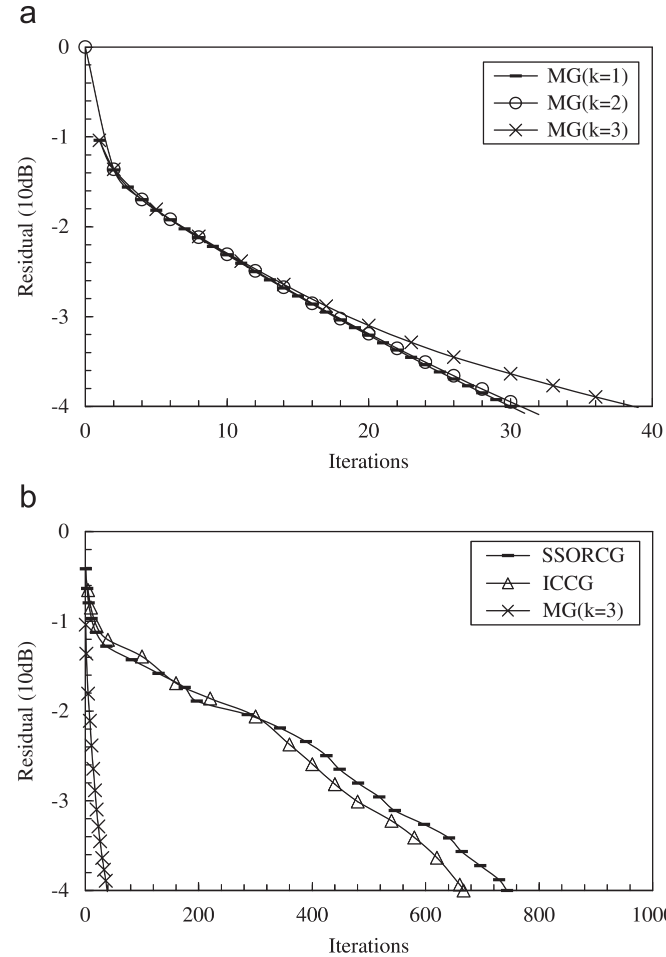


**Fig. 2.** (a) Convergence curves of the *p*-version MG method for the short-circuited E-plane slot-coupled T-junction when the operating frequency is 12 GHz. (b) Comparison of different iterative solvers for the short-circuited E-plane slot-coupled T-junction problem when the operating frequency is 12 GHz.

level MG method. The $H_0$(curl) and $H_1$(curl) TVFEM equations are given by

$$\mathbf{A_0 x_0} = \mathbf{b_0}$$

$$\mathbf{A_1 x_1} = \mathbf{b_1} \tag{5}$$

Here we regard each finite-dimensional vector space as a level, the $H_0$(curl) equations in (5) can be seen as an approximation of TVFE space on the coarser grid. While the $H_1$(curl) equations can be seen as an approximation on the finer grid. The linear system of the $H_1$(curl) TVFEM can be written into block form as

$$\begin{bmatrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{x_{(0)}} \\ \mathbf{x_{(1)}} \end{bmatrix} = \begin{bmatrix} \mathbf{b_{(0)}} \\ \mathbf{b_{(1)}} \end{bmatrix} \tag{6}$$

From the construction process of TVFEM, we can see that $\mathbf{A}_{11}$ is totally identical with the $H_0$(curl) TVFE stiff matrix $\mathbf{A}_0$. Denoting $\mathbf{I_0}$ an $n_0 \times n_0$ identity matrix, where $n_0$ is the order of $\mathbf{A}_0$, we have

$$\mathbf{A_0} = [\mathbf{I_0}, \mathbf{0}]\mathbf{A_1}[\mathbf{I_0}, \mathbf{0}]^{\mathbf{T}}$$

Then the restriction operator $\mathbf{I_0^1}$ and the projection operator $\mathbf{I}_1^0$ can be deduced:

$$\mathbf{I_0^1} = [\mathbf{I_0}, \mathbf{0}]^{\mathbf{T}}, \quad \mathbf{I_1^0} = [\mathbf{I_0}, \mathbf{0}] \tag{7}$$

The algorithm of the two-level V-cycle *p*-version multigrid method can be described as follows:

1. Solve the equation $\mathbf{A_0 x_0} = \mathbf{b_0}$ to find an initial solution $[\mathbf{x_{(0)}}, \mathbf{0}]$.
2. Perform $k$ iterations of the linear system $\mathbf{A}_1\mathbf{x}_1 = \mathbf{b}_1$ using Gauss–Seidel smoother, and compute the residual vector $\mathbf{r_1} = \mathbf{b_1} - \mathbf{A_1 x}$.
3. Compute the residual error $\varepsilon = \|\mathbf{r_1}\|/\|\mathbf{b_1}\|$, if the residual error is within the tolerance, stop; else continue.
4. Solve equation $\mathbf{A_0}\varepsilon_\mathbf{0} = \mathbf{I_1^0 r_1}$ on the coarser grid.
5. Correct solution vector on the fine level $\mathbf{x_1^{(k)}} = \mathbf{x_1^{(k)}} + \mathbf{I_0^1}\varepsilon_\mathbf{0}$ and go back to step 2 with the initial vector $\mathbf{x_1^{(k)}}$.

As the linear systems on the low order space are of very small size compared with that on the high order space, the cost of solving them is comparatively much smaller. In this paper, we use the multifrontal solver [16] to solve the linear system in step (4) of the above algorithm.

The convergence of this algorithm is very good when it is used to solve positive definite linear system. But the discretization of Helmholtz equation with vector finite element is not positive definite linear system. So the above algorithm cannot ensure the stability of the solving. This mainly because of the coarse grids rectification by using of multigrid method could introduce the new residual error component of high frequency. On the other hand, the coarse grids rectification would not dispel the residual error component of low frequency very well if the grids are too coarse. So it has been indicated by many research that the algorithm cannot converge well when the scale of the grid is too large. But if the scale of the grid is fine enough this algorithm has excellent equality.

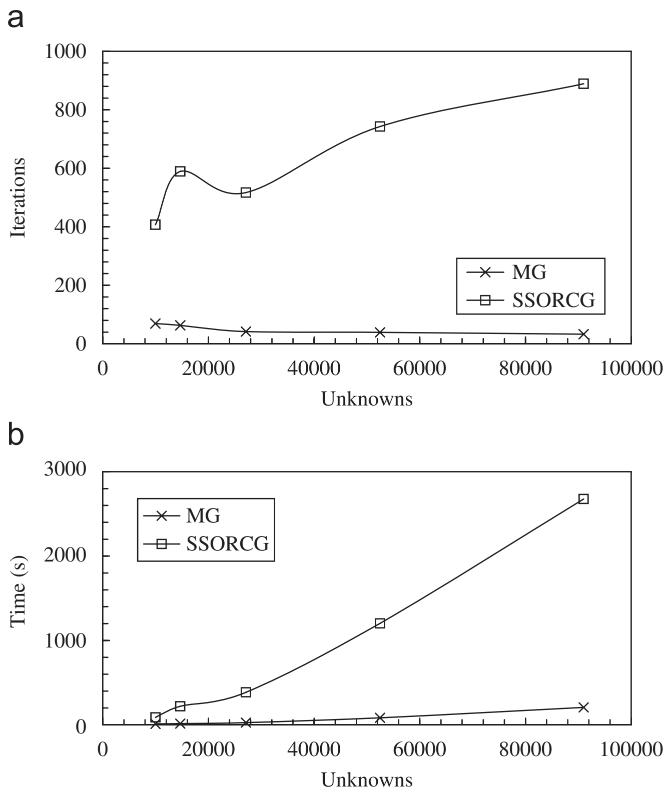


**Fig. 3.** (a) Iterations of the SSORCG and $p$-version MG method versus unknowns for the short-circuited E-plane slot-coupled T-junction problem when the operating frequency is 12 GHz. (b) CPU time used by the SSORCG and $p$-version MG method versus unknowns for the short-circuited E-plane slot-coupled T-junction problem when the operating frequency is 12 GHz (the lowest time of MG is 28 s).

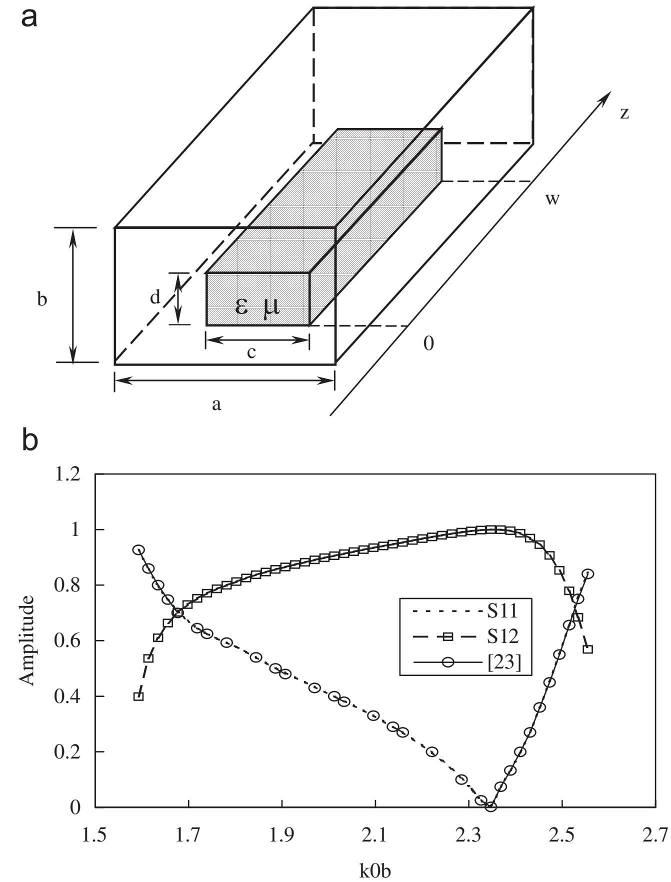


**Fig. 4.** (a) Configuration of the rectangular waveguide filled with partial-height dielectric. (b) Reflection and transmission characteristics of the dielectric-loaded waveguide problem.

In the next section, the $p$-version MG method is applied to solve the hierarchical TVFEM equations in electromagnetics. Two waveguide discontinuity problems are investigated.

## 3. Numerical results

To verify the efficiency of the $p$-version MG method, we applied the proposed $p$-version MG method, the SSOR preconditioned conjugate gradient iterative solver (SSORCG) [17] and the ICCG [6] solver to the following test examples: (A) an E-plane slot-coupled T-junction and (B) a rectangular waveguide filled with a partial-height dielectric. The norm of the relative residual in the PCG termination criterion was set to −40 dB.

In our 3D TVFE simulation, the $TE_{10}$ mode is imposed at the waveguide input port. The scatter parameters are extracted with an efficient method proposed in [18]. The first example we analyzed here with the $H_1$(curl) TVFEM is a E-plane slot-coupled T-junction. The configuration and dimensions are drawn in Fig. 1(a). At first, we discretize the domain into 9234 tetrahedrons. As a result, a large, sparse complex linear system with a total of 52 472 unknowns can be obtained, with about 38 elements per row on average. The size of the local matrix corresponding to the $H_0$(curl) TVFE space is 8985, and the nonzero elements per row is about 14. We can see that the $H_0$(curl) TVFE system is very small compared with the $H_1$(curl) TVFE system. Our calculations of the amplitude of scatter parameters with the $H_1$(curl) hierarchical TVFEM is shown in Fig. 1(b). Compared with the measured results in Ref. [19], we can find that our results meet well with the measured data. The efficiency of the $p$-version MG method at $f = 12$ GHz is tested atfirst. In step 2 of the above $p$-version algorithm, we take $k = 1, 2, 3$ for the Gauss–Seidel smoother. In the MG method, as the operation is complex, so only the number of Gauss–Seidel iteration is counted for simplicity. The convergence curves are drawn in Fig. 2(a). We can see from the figure that with the increase of $k$, the iteration number increases gradually. However, as the correction on the low order space take much time, the CPU time used for $k = 1$, 2, 3 is, respectively, 157, 91, 80 s. The comparison of the $p$-version MG method with the SSORCG, ICCG solver is drawn in Fig. 2(b), from which we can draw the conclusion that the $p$-version multigrid method can reach convergence much faster than the preconditioned conjugate gradient method. The CPU time used by these three methods is respectively: 1066 s by SSORCG, 1720 s by ILU0CG and 91 s by the $p$-version multigrid method. Compared with the ICCG and SSORCG iteration, about 11.7 and 18.9 times of CPU time can be saved by the $p$-version multigrid method.

Furthermore, to investigate the convergence property of the $p$-version MG method for different sizes of linear systems, we discretize the problem with different meshes, and obtained a sequence of linear systems with unknowns varying from nearly 10 000 to nearly 100 000. The number of iterations and CPU time are shown in Fig. 3(a) and (b). We can see that the number of iterations of the $p$-version MG method decrease slowly along with the increased number of unknowns. This is because on the finer meshes the low order TVFE space correction may provide a more exact approximation to the low frequency eigenvalues. Contrarily, the number of iterations of the SSORCG method increased with the increasing of unknowns. The CPU time needed for both methods escalated with the increasing of the size of the linear system. However, the CPU time needed by the $p$-version method escalated much more slowly than SSORCG.

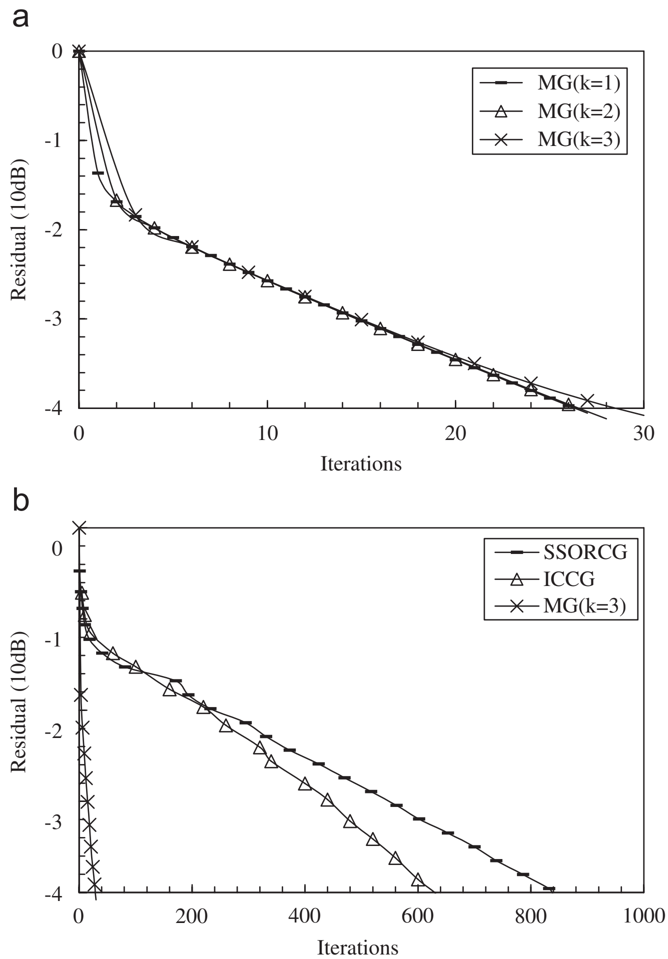


**Fig. 5.** (a) Convergence history of the *p*-version MG method for the rectangular waveguide filled with partial-height dielectric when the operating frequency is 90 MHz. (b) Comparison of different iterative solvers for the rectangular waveguide filled with partial-height dielectric when the operating frequency is 90 MHz.

This shows the great advantage of the *p*-version MG method when compared with the preconditioned conjugate gradient methods.

Another example is a rectangular waveguide filled with a partial-height dielectric [20], as is shown in Fig. 4(a). Some parameters of the waveguide are $a=2$, $b=1$, $c=0.888$, $d=0.399$, $w=0.8$ (mm), $\varepsilon_r=6.0$. With $H_1$(curl) hierarchical TVFEM, a total of 20 894 unknowns can be obtained. Our calculations of the amplitude of $|S_{11}|$ and $|S_{12}|$ parameters by the $H_1$(curl) hierarchical TVFEM are showed by the dashed curve line in Fig. 4(b). The comparison with results from Ref. [20] is made and excellent agreement is found.

Fig. 5(a) shows the iterative curves for different *k* of the *p*-version iterative method for the rectangular waveguide with operating frequency $f = 90$ MHz. In Fig. 5(b), the convergence characteristics of residual error versus iterations are given for the various iterative methods. In this example, the *p*-version MG method again showed great superiority over the PCG method. The CPU time used is: 435 s by SSORCG, 609 s by ILU0CG and 21 s by the *p*-version multigrid method. Compared with the ICCG and SSORCG iteration, about 29.0 and 20.7 times of CPU time can be saved by the *p*-version multigrid method.

The comparison of iteration numbers and CPU time of the *p*-version MG method and the SSORCG method for different meshes are shown in Fig. 6(a) and (b). In this example, a quite similar result can be obtained with the E-plane slot-coupled T-junction problem, which again shows the great superiority of the *p*-version MG method.

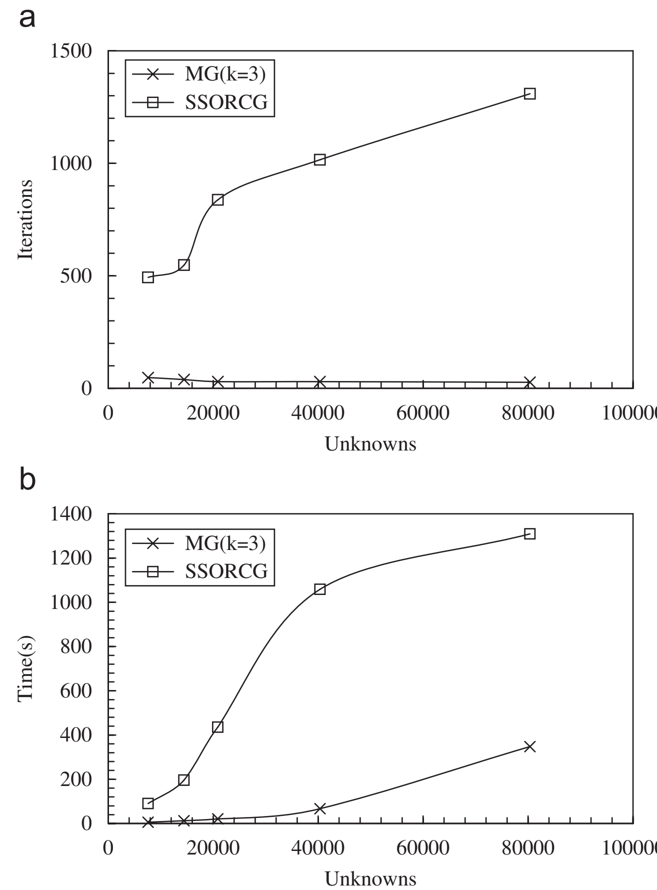


**Fig. 6.** (a) Iterations of the SSORCG and *p*-version MG method versus unknowns for the rectangular waveguide filled with partial-height dielectric when the operating frequency is 90 MHz. (b) CPU time used by the SSORCG and *p*-version MG method versus unknowns for the rectangular waveguide filled with partial-height dielectric when the operating frequency is 90 MHz (the lowest time of MG is 22 s).

## 4. Conclusions

In this paper, the *p*-version multigrid method is applied for the full-wave finite element analysis of microwave devices. For this suggested method, the linear system is derived from the application of the hierarchical TVFE method and the advantage of the hierarchical property of the stiff matrix is taken. The prolongation and projection operators are derived to be very simple. Our numerical results show that the proposed method is very efficient for the solution of hierarchical TVFEM for waveguide discontinuity problems.

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## References

[1] J.M. Jin, The Finite Element Method in Electromagnetics, Wiley, New York, 1993.
[2] J.S. Wang, R. Mittra, Finite element analysis of MMIC structures and electronic packages using absorbing boundary conditions, IEEE Trans. MTT 42 (3) (1994) 441–449.
[3] J.-F. Lee, R. Mittra, A note on the application of edge-element for modeling three-dimensional inhomogeneously-filled cavities, IEEE Trans. MTT 40 (9) (1992) 1767–1773.
[4] H. Whitney, Georнefric Integtnlioii Theory, Princeton University Press, Princeton, 1957.

[5] R. Dyczij-Edlinger, O. Biro, A joint vector and scalar potential formulation for driven high frequency problems using hybrid edge and nodal finite elements, IEEE Trans. MTT 44 (1) (1996) 15–23.
[6] R.S. Chen, X.W. Ping, E.K.N. Yung, C.H. Chan, et al., Application of diagonally perturbed incomplete factorization preconditioned conjugate gradient algorithms for edge finite element analysis of Helmholtz equations, IEEE Trans. Antennas Propag. 54 (5) (2006) 1604–1608.
[7] R. Hiptmair, Multigrid method for Maxwell's equations, SIAM J. Numer. Anal. 36 (1998) 204–225.
[8] V. Cingoski, R. Tokuda, S. Noguchi, H. Yamashita, Fast multigrid solution method for nested edge-based finite element meshes, IEEE Trans. Magn. 36 (2000) 1539–1542.
[9] B. Weiss, O. Biro, Edge element multigrid solution of nonlinear magnetostatic problems, Compel 20 (2) (2001) 357–365.
[10] M. Schinnerl, J. Shoberl, M. Kaltenbacher, Nested multigrid methods for the fast numerical computation of 3D magnetic fields, IEEE Trans. Magn. 36 (2000) 1557–1560.
[11] Y. Saad, Iterative Methods for Sparse Linear Systems, PWS Publishing Company, Boston, 1995.
[12] J.P. Webb, Hierarchical vector basis functions of arbitrary order for triangular and tetrahedral finite elements, IEEE Trans. Antennas Propag. 47 (8) (1999) 1244–1253.
[13] L.S. Andersen, J.L. Volakis, Hierarchical tangential vector finite elements for tetrahedral, IEEE MGWL 8 (3) (1998) 127–129.
[14] W. Daigang, J. Kesun, *P*-version adaptive computation of FEM, IEEE Trans. Magn. 30 (5) (1994) 3515–3518.
[15] N. Hu, I.N. Katz, Multi-*p* methods: iterative algorithms for the *p*-version of the finite element analysis, SIAM J. Sci. Comput. 16 (6) (1995) 1308–1329.
[16] R.S. Chen, E.K.N. Yung, J.M. Jin, Application of the multifrontal method to the vector FEM for analysis of microwave filters, Microwave Opt. Technol. Lett. 31 (6) (2001) 465–470.
[17] R.S. Chen, E.K.N. Yung, C.H. Chan, D.G. Fang, Application of SSOR preconditioned conjugate gradient algorithm to edge-FEM for 3-dimensional full wave electromagnetic boundary value problems, IEEE Trans. MTT 50 (4) (2002) 1165–1172.
[18] M.-J. Tsai, F. De Flaviis, et al., Modeling planar arbitrarily shaped microstrip elements in multilayered media, IEEE Trans. MTT 45 (3) (1997).
[19] T. Sieverding, F. Arndt, Field theoretic CAD of open or aperture matched T-junction coupled rectangular waveguide structures, IEEE Trans. MTT 40 (1992) 353–363.
[20] K. Ise, K. Inoue, M. Koshiba, Three-dimensional finite-element method with edge elements for electromagnetic waveguide discontinuities, IEEE Trans. MTT 39 (8) (1991) 1289–1295.
[21] R. Hiptmair, Multigrid method for H(div) in three dimensions, Electron. Trans. Num. Anal. 6 (1997) 133–152.
[22] J.J. Heys, T.A. Manteuffel, S.F. McCormick, L.N. Olson, Algebraic multigrid for higher-order finite elements, J. Comput. Phys. 204 (2005) 520–532.