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Asymptotic moment boundedness of the numerical solutions of stochastic differential equations

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ABSTRACT

Few papers look at the asymptotic boundedness of numerical solutions of stochastic differential equations (SDEs). One of the open questions is whether numerical approximations can reproduce the boundedness property of the underlying SDEs. In this paper, we give positive answer to this question. Firstly we discuss the asymptotic moment upper bound of the Itô type SDEs and show that the Euler–Maruyama (EM) method is capable to preserve the boundedness property for SDEs with the linear growth condition on both drift and diffusion coefficients. But under the weaker assumption, the one-sided Lipschitz, on the drift coefficient, the EM method fails to work. We then show that the backward EM method can work in this situation.

1. Introduction

Asymptotic properties of the solutions of SDEs have been widely studied in the past decades, particularly the stability theory has been attracting lots of attention (see for example, [1] and the references therein).

Due to the difficulty to find the explicit solutions to SDEs, different types of numerical methods have been introduced to approximate the underlying solutions (see, for example, [2–4]). Thus the study of the stability of the numerical methods has naturally boomed in recent years. We mention [5–7] here, as they are among those papers with original ideas. More recent works investigate the stability for different types of SDEs and different sorts of numerical methods, such as [8–16] and the references therein. We also mention some works on stochastic difference equations [17,18] as they are naturally related to discrete numerical solutions.

Another important asymptotic property of the SDE solutions, the asymptotic boundedness, has its own right. Unlike the stability property that requires the solutions be attracted by an equilibrium state, the boundedness property only requires the solutions stay within certain regime as time tends to infinity [19]. Works on the boundedness of the underlying SDE solutions can be found, such as [20,19,21,22] and their references therein. But there are few papers investigating the asymptotic boundedness of the numerical solutions.

The main purpose of this paper is to investigate the asymptotic moment boundedness of two classical numerical methods. We focus on two types of moment, small moment (i.e. pth moment with p much smaller than 1) and second moment. For the case of small moment, they do have some applications. For instance the stochastic permanence studied in the stochastic population model, see for example [23], in which the probability of the solution larger than some constant can be estimated by the small moment together with Markov's inequality. The case of second moment is widely studied for many different asymptotic properties. In this paper, we find that compared with the case of small moment stronger conditions are required in second moment but better results could be obtained (see Section 5 for details). In addition, thanks to Hölder's inequality the asymptotic pth moment boundedness for $1 < p < 2$ could be implied by the second moment boundedness.

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Our key aim in this paper is to answer the question: given that the solution of the underlying Itô type SDE is asymptotically bounded in moment, is there any numerical method that could preserve the boundedness property?

Due to the techniques used to deal with the small moment are much more complicated than those for the second moment, the majority of the paper is devoted to the case of small moment. This paper is constructed as follows. We briefly introduce the two classical numerical methods in Section 2. The main results of the small moment are developed in Sections 3 and 4. In each of these two sections we first present the results for the underlying true solution, the relative results for the numerical solution then follow. Section 3 is devoted to the asymptotic boundedness of the EM method under the linear growth condition, and Section 4 discusses the backward EM method applied to a set of SDEs on which the EM method fails to work. Section 5 discusses the results for the case of second moment. The last section summarizes the paper and discusses some possible further research.

2. Preliminary

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is increasing and right continuous, with $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets. Let $B(t) = (B_1(t), \ldots, B_m(t))$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. The inner product of x, y in $\mathbb{R}^n$ is denoted by $\langle x, y\rangle$. We denote $\max(a, b)$ and $\min(a, b)$ by $a \vee b$ and $a \wedge b$ respectively. In this paper, we consider the n-dimensional Itô SDE

$$dx(t) = f(x(t))dt + \sum_{i=1}^{m} g_i(x(t))dB_i(t), \quad t \geq 0, x(0) \in \mathbb{R}^n. \tag{2.1}$$

We assume that $f, g_1, \ldots, g_m\colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth enough for the SDE (2.1) to have a unique global solution on $[0, \infty)$ (see, for example, [1]).

Let us recall the two numerical methods we will use below. The reader is referred to [3,4] for more details on the numerical methods. The Euler–Maruyama (EM) method applied to (2.1) is defined by

$$Y_{k+1} = Y_k + f(Y_k)\Delta t + \sum_{i=1}^{m} g_i(Y_k)\Delta B_{i,k}, \qquad Y_0 = x(0), \tag{2.2}$$

for $k = 0, 1, \ldots,$ where Δt is the timestep and $\Delta B_{i,k} = B_i((k + 1)\Delta t) - B_i(k\Delta t)$ is the Brownian increment.

The backward EM method (or the drift implicit EM method) is defined by

$$Y_{k+1} = Y_k + f(Y_{k+1})\Delta t + \sum_{i=1}^{m} g_i(Y_k)\Delta B_{i,k}, \qquad Y_0 = x(0), \tag{2.3}$$

for $k = 0, 1, \ldots.$

3. Euler–Maruyama in small moment

We begin by imposing the linear growth condition on both drift and diffusion coefficients of the SDE (2.1):

$$|f(x)|^2 \vee |g_i(x)|^2 \leq K|x|^2 + \alpha \quad \forall x \in \mathbb{R}^n \text{ and } 1 \leq i \leq m, \tag{3.1}$$

where K and α are positive constants. In this section, we will be concerned with the asymptotic boundedness in small moment of the solution $x(t)$ of (2.1) and the preservation of this property using the EM method.

3.1. Asymptotic boundedness

We first give a sufficient condition for the asymptotic small moment boundedness of the SDE solution. It should be emphasized that a more general sufficient condition exists (see, for example, Theorem 5.2, p. 157 in [19]). The condition we employ in Theorem 3.1 is in line with the one for the boundedness of the numerical solution in Theorem 3.2, and it is still an open question that whether there exists a numerical method that could recover the asymptotic boundedness of the underlying SDE solution under the more general condition (for example, the condition given in Theorem 5.2 of [19]).

Theorem 3.1. *Let (3.1) hold. If there exists a positive constant D such that for any $x \in \mathbb{R}^n$*

$$\frac{\langle x, f(x)\rangle + \frac{1}{2}\sum_{i=1}^{m}|g_i(x)|^2}{D + |x|^2} - \frac{\sum_{i=1}^{m}\langle x, g_i(x)\rangle^2}{(D + |x|^2)^2} \leq -\lambda + \frac{P_1(|x|)}{D + |x|^2} + \frac{P_3(|x|)}{(D + |x|^2)^2}, \tag{3.2}$$

where λ is a positive constant and $P_i(|x|)$ is a polynomial of $|x|$ with degree i, then there exists a $p^* \in (0, 1)$ such that for all $0 < p < p^*$ the solution of (2.1) obeys

$$\limsup_{t\to\infty} \mathbb{E}(|x(t)|^p) \le C, \quad \forall x(0) \in \mathbb{R}^n, \tag{3.3}$$

where C is a positive constant dependent on K, α, p, D, but independent of $x(0)$.

Following the same technique used in Theorem 5.2 in [19], by choosing the Lyapunov function $V = (D + |x(t)|^2)^{p/2}$, it is straightforward to prove this theorem. So we omit it here. Now we give the result for the EM solution.

Theorem 3.2. *Let (3.1) and (3.2) hold. Then for any $\varepsilon \in (0, \lambda)$, there exists a pair of constants $p^* \in (0, 1)$ and $\Delta t^* \in (0, 1)$ such that for $\forall p \in (0, p^*)$ and $\forall \Delta t \in (0, \Delta t^*)$, the EM solution (2.2) satisfies*

$$\limsup_{k\to\infty} \mathbb{E}|Y_k|^p \le \frac{C_2'}{p(\lambda - \varepsilon)}, \quad \forall Y_0 \in \mathbb{R}^n, \tag{3.4}$$

where C_2' is a constant dependent on K, α, p and D, but independent of Y_0 and Δt.

Proof. For the constant D in (3.2), we compute

$$D + |Y_{k+1}|^2 = D + |Y_k|^2 + 2\left\langle Y_k, f(Y_k)\Delta t + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right\rangle + \left|f(Y_k)\Delta t + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2.$$

Let

$$\xi_k = \frac{1}{D + |Y_k|^2}\left(2\left\langle Y_k, f(Y_k)\Delta t + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right\rangle + \left|f(Y_k)\Delta t + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2\right),$$

for any $p \in (0, 1)$ we have

$$|D + |Y_{k+1}|^2|^{p/2} = |D + |Y_k|^2|^{p/2}(1 + \xi_k)^{p/2}.$$

Clearly $\xi_k > -1$, recalling the fundamental inequality

$$(1 + u)^{p/2} \le 1 + \frac{p}{2}u + \frac{p(p-2)}{8}u^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!}u^3, \quad u > -1, \tag{3.5}$$

we have

$$|D + |Y_{k+1}|^2|^{p/2} \le |D + |Y_k|^2|^{p/2}\left(1 + \frac{p}{2}\xi_k + \frac{p(p-2)}{8}\xi_k^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!}\xi_k^3\right).$$

Hence the conditional expectation

$$\mathbb{E}(|D + |Y_{k+1}|^2|^{p/2}\big|\mathcal{F}_{k\Delta t}) \le |D + |Y_k|^2|^{p/2}\,\mathbb{E}\left(1 + \frac{p}{2}\xi_k + \frac{p(p-2)}{8}\xi_k^2 + \frac{p(p-2)(p-4)}{2^3 \times 3!}\xi_k^3|\mathcal{F}_{k\Delta t}\right). \tag{3.6}$$

Since $\Delta B_{i,k}, i = 1, \ldots, m$, is independent from each other and is independent of $\mathcal{F}_{k\Delta t}$, we have $\mathbb{E}(\Delta B_{i,k}|\mathcal{F}_{k\Delta t}) = \mathbb{E}(\Delta B_{i,k}) = 0$, $\mathbb{E}((\Delta B_{i,k})^2|\mathcal{F}_{k\Delta t}) = \mathbb{E}((\Delta B_{i,k})^2) = \Delta t$ and $\mathbb{E}(\Delta B_{i,k}\Delta B_{j,k}|\mathcal{F}_{k\Delta t}) = \mathbb{E}(\Delta B_{i,k}\Delta B_{j,k}) = \mathbb{E}(\Delta B_{i,k})\mathbb{E}(\Delta B_{j,k}) = 0$, for $i \ne j$. By (3.1) we can get

$$\begin{aligned}
\mathbb{E}(\xi_k|\mathcal{F}_{k\Delta t}) &= \mathbb{E}\left(\frac{1}{D + |Y_k|^2}\left(2\left\langle Y_k, f(Y_k)\Delta t + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right\rangle + \left|f(Y_k)\Delta t + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2\right)\Bigg|\mathcal{F}_{k\Delta t}\right)\\
&= \frac{1}{D + |Y_k|^2}\left(2\langle Y_k, f(Y_k)\rangle + \sum_{i=1}^m |g_i(Y_k)|^2\right)\Delta t + \frac{1}{D + |Y_k|^2}|f(Y_k)|^2\Delta t^2\\
&\le \frac{1}{D + |Y_k|^2}\left(2\langle Y_k, f(Y_k)\rangle + \sum_{i=1}^m |g_i(Y_k)|^2\right)\Delta t + K\Delta t^2 + \frac{C_2}{D + |Y_k|^2}\Delta t^2.
\end{aligned} \tag{3.7}$$

Similarly, we can show that

$$\mathbb{E}(\xi_k^2|\mathcal{F}_{k\Delta t}) \ge \frac{4}{(D + |Y_k|^2)^2}\sum_{i=1}^m \langle Y_k, g_i(Y_k)\rangle^2\Delta t - C_1\Delta t^2 - \frac{C_2}{(D + |Y_k|^2)^2}\Delta t^2, \tag{3.8}$$

and

$$\mathbb{E}(\xi_k^3|\mathcal{F}_{k\Delta t}) \le C_1\Delta t^2 + \frac{C_2}{(D+|Y_k|^2)^3}\Delta t^2, \tag{3.9}$$

where C_1 is a positive constant dependent on K, and C_2 is a positive constant dependent on α. C_1 and C_2 may change from line to line. Now consider the following two fractions,

$$\frac{(D+|Y_k|^2)^{p/2}P_1(|Y_k|)}{D+|Y_k|^2} \quad \text{and} \quad \frac{(D+|Y_k|^2)^{p/2}P_3(|Y_k|)}{(D+|Y_k|^2)^2}. \tag{3.10}$$

For $0 < p < 1$ the highest degrees of $|Y_k|$ in the numerators are $p+1$ and $p+3$ respectively, which are smaller than the corresponding highest degrees of $|Y_k|$ in the denominators. Thus for any $|Y_k| \in \mathbb{R}$ there exists an upper bound for both of the fractions. Also it is obvious that $C_2/(D+|Y_{k+1}|^2)^{i-p/2}$, $i = 1, 2, 3$ are bounded by some constant that depends on α and D. Substituting (3.7)–(3.9) into (3.6), then using (3.1) and (3.2) and the argument for (3.10) we have that

$$\begin{aligned}
\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}|\mathcal{F}_{k\Delta t}) &\le (D+|Y_k|^2)^{p/2}\Bigg(1+\frac{p}{2(D+|Y_{k+1}|^2)}\left(2\langle Y_k, f(Y_k)\rangle + \sum_{i=1}^m |g_i(Y_k)|^2\right)\Delta t \\
&\quad + \frac{p(p-2)}{2(D+|Y_k|^2)^2}\sum_{i=1}^m \langle Y_k, g_i(Y_k)\rangle^2\Delta t + C_1'\Delta t^2\Bigg) + C_2'\Delta t \\
&= (D+|Y_k|^2)^{p/2}\Bigg[1+p\Delta t\Bigg(\frac{\langle Y_k, f(Y_k)\rangle + \frac{1}{2}\sum_{i=1}^m |g_i(Y_k)|^2}{D+|Y_k|^2} \\
&\quad - \frac{\sum_{i=1}^m \langle Y_k, g_i(Y_k)\rangle^2}{(D+|Y_k|^2)^2}\Bigg) + \frac{p^2\Delta t\sum_{i=1}^m \langle Y_k, g_i(Y_k)\rangle^2}{2(D+|Y_k|^2)^2} + C_1'\Delta t^2\Bigg] + C_2'\Delta t \\
&\le (D+|Y_k|^2)^{p/2}\left(1 - p\lambda\Delta t + \frac{mp^2\Delta t K}{2} + C_1'\Delta t^2\right) + C_2'\Delta t,
\end{aligned}$$

where C_1' is a positive constant dependent on K and p, C_2' is a positive constant dependent on K, α, p and D, and both of them may change from line to line. For any given $\varepsilon \in (0, \lambda)$, choose $p^* \in (0, 1)$ sufficiently small for $mp^*K < \varepsilon$, then choose $\Delta t^* \in (0, 1)$ sufficiently small for $p^*\lambda\Delta t^* \le 1$ and $C_1'\Delta t^* \le \frac{1}{2}p^*\varepsilon$. For any $p \in (0, p^*)$ and any $\Delta t \in (0, \Delta t^*)$ we have

$$\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}|\mathcal{F}_{k\Delta t}) \le (D+|Y_k|^2)^{p/2}(1-p(\lambda-\varepsilon)\Delta t) + C_2'\Delta t.$$

Taking expectations on both sides yields

$$\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}) \le \mathbb{E}((D+|Y_k|^2)^{p/2})(1-p(\lambda-\varepsilon)\Delta t) + C_2'\Delta t. \tag{3.11}$$

By iteration we have

$$\mathbb{E}((D+|Y_k|^2)^{p/2}) \le \mathbb{E}((D+|Y_0|^2)^{p/2})(1-p(\lambda-\varepsilon)\Delta t)^k + \frac{1-(1-p(\lambda-\varepsilon)\Delta t)^{k-1}}{p(\lambda-\varepsilon)}C_2'. \tag{3.12}$$

Since $\mathbb{E}(|Y_k|^p) \le \mathbb{E}((D+|Y_k|^2)^{p/2})$, we have

$$\mathbb{E}(|Y_k|^p) \le \mathbb{E}((D+|Y_0|^2)^{p/2})(1-p(\lambda-\varepsilon)\Delta t)^k + \frac{1-(1-p(\lambda-\varepsilon)\Delta t)^{k-1}}{p(\lambda-\varepsilon)}C_2'. \tag{3.13}$$

Let $k \to \infty$, then (3.4) follows. □

3.2. A linear scalar SDE example

Let us consider a linear scalar SDE,

$$dx(t) = (\alpha_1 + \alpha_2 x(t))dt + (\sigma_1 + \sigma_2 x(t))dB(t), \qquad x(0) \in \mathbb{R}, \tag{3.14}$$

where $\alpha_1, \alpha_2, \sigma_1, \sigma_2$ are real numbers. We impose the condition, $\alpha_2 - \sigma_2^2/2 < 0$. By using this example, we will illustrate

- the existence of the constant, D, in condition (3.2) and how to choose it.

Obviously both drift and diffusion coefficients of (3.14) satisfy the linear growth condition (3.1). Now we consider the condition (3.2),

$$\frac{\langle Y_k, f(Y_k)\rangle + \frac{1}{2}|g(Y_k)|^2}{D + |Y_k|^2} = \frac{\left(\alpha_2 + \frac{1}{2}\sigma_2^2\right) Y_k^2}{D + |Y_k|^2} + \frac{(\alpha_1 + \sigma_1\sigma_2)Y_k + \sigma_1^2}{D + |Y_k|^2}, \tag{3.15}$$

and

$$\begin{aligned}\langle Y_k, g(Y_k)\rangle^2 &= (\sigma_1 Y_k + \sigma_2 Y_k^2)^2 \\ &= \sigma_2^2 Y_k^4 + 2\sigma_1\sigma_2 Y_k^3 + \sigma_1^2 Y_k^2 \\ &= \sigma_2^2\left(Y_k^2 + \frac{\sigma_1^2}{2\sigma_2^2}\right)^2 - \frac{\sigma_1^4}{4\sigma_2^2} + 2\sigma_1\sigma_2 Y_k^3. \end{aligned} \tag{3.16}$$

Choose $D = (\sigma_1^2)/(2\sigma_2^2)$ then we have

$$\begin{aligned}\frac{\langle Y_k, f(Y_k)\rangle + \frac{1}{2}|g(Y_k)|^2}{D + |Y_k|^2} - \frac{\langle Y_k, g(Y_k)\rangle^2}{(D + |Y_k|^2)^2} &\le \left(\alpha_2 - \frac{1}{2}\sigma_2^2\right) + \frac{(\alpha_1 + \sigma_1\sigma_2)Y_k + \sigma_1^2}{D + |Y_k|^2} \\ &\quad + \frac{1}{(D + |Y_k|^2)^2}\left(\frac{\sigma_1^4}{4\sigma_2^2} - 2\sigma_1\sigma_2 Y_k^3\right). \end{aligned} \tag{3.17}$$

Thus $-\lambda = \alpha_2 - \sigma_2^2/2$, $P_1(Y_k) = (\alpha_1 + \sigma_1\sigma_2)Y_k + \sigma_1^2$ and $P_3(Y_k) = \sigma_1^4/(4\sigma_2^2) - 2\sigma_1\sigma_2 Y_k^3$. Then the similar process to the proof of Theorem 3.2 leads to the property (3.4) for the linear scalar SDE (3.14).

4. Backward Euler–Maruyama in small moment

So far, we have established some positive results on the asymptotic boundedness in small moment of the EM method under the linear growth condition (3.1). Now we consider to relax the constraint of the drift coefficient by imposing the one-sided Lipschitz condition,

$$\langle x - y, f(x) - f(y)\rangle \le \bar{\mu}|x - y|^2 + \bar{\alpha} \quad \forall x, y \in \mathbb{R}^n,$$

where $\bar{\mu} \in \mathbb{R}$ and $\bar{\alpha} \in \mathbb{R}^+$. Without losing generality, we further assume for $\forall x \in \mathbb{R}^n$

$$\langle x, f(x)\rangle \le \mu|x|^2 + \alpha, \tag{4.1}$$

where $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$. The diffusion coefficient still obeys the linear growth condition,

$$|g_i(x)|^2 \le K|x|^2 + \alpha, \quad 1 \le i \le m. \tag{4.2}$$

In this section, we start with a counter example to show that the EM solution will blow up under (4.1) and (4.2). Then we will show that the backward EM method can still preserve the boundedness property of the SDE solution under these conditions.

4.1. A counter example

Consider the following scalar SDE,

$$dx(t) = (-0.5x(t) - x^3(t) + 1)dt + (x(t) + 1)dB(t), \tag{4.3}$$

to which the EM method is applied.

Lemma 4.1. *Suppose $\Delta t \in (0, 1)$ and $p \in (0, 1)$, then for any $Y_0 \in \mathbb{R}$,*

$$\lim_{k\to\infty} \mathbb{E}|Y_k|^p = \infty. \tag{4.4}$$

Proof. By the property of conditional expectations, we have

$$\mathbb{E}|Y_{k+1}|^p = \mathbb{E}[\mathbb{E}(|Y_{k+1}|^p \big| Y_1)] \ge \mathbb{E}[\mathbf{1}_{\{|Y_1|^p \ge 2^3/\Delta t^{p/2}\}}\mathbb{E}(|Y_{k+1}|^p \big| Y_1)]. \tag{4.5}$$

Since there is a positive probability that the first Brownian motion increment will make $|Y_1|^p \ge 2^3/\Delta t^{p/2}$, we only need to show that for $|Y_1|^p \ge 2^3/\Delta t^{p/2}$, $\mathbb{E}(|Y_{k+1}|^p|Y_1) \ge 2^{k+3}/\Delta t^{p/2}$ for all $k \ge 0$. We show this by induction. Clearly, $\mathbb{E}(|Y_1|^p|Y_1) = |Y_1|^p \ge 2^3/\Delta t^{p/2}$. Suppose $\mathbb{E}(|Y_k|^p|Y_1) \ge 2^{k+2}/\Delta t^{p/2}$ for some $k \ge 1$, we will show that for any $\Delta t \in (0, 1)$, $\mathbb{E}(|Y_{k+1}|^p|Y_1) \ge 2^{k+3}/\Delta t^{p/2}$. Applying the EM method to the SDE (4.3), we have

$$|Y_{k+1}| = |Y_k - 0.5\Delta t Y_k - \Delta t Y_k^3 + Y_k \Delta B_k + \Delta t + \Delta B_k|.$$

Then by the fundamental inequality, $|a+b|^p > |a|^p - |b|^p$, we have

$$\begin{aligned} |Y_{k+1}|^p &\geq |\Delta t Y_k^3 + (0.5\Delta t - 1)Y_k + Y_k \Delta B_k|^p - |\Delta t|^p - |\Delta B_k|^p \\ &\geq \Delta t^p |Y_k|^{3p} - (1-0.5\Delta t)^p |Y_k|^p - |Y_k \Delta B_k|^p - |\Delta t|^p - |\Delta B_k|^p. \end{aligned}$$

By Hölder's inequality, we have $\mathbb{E}(|Y_k|^{3p}|Y_1) \geq (\mathbb{E}(|Y_k|^p|Y_1))^3$. Since ΔB_k is independent of Y_1 for all $k > 0$, $\mathbb{E}(|\Delta B_k|^p|Y_1) = \mathbb{E}(|\Delta B_k|^p) < 2$. Then taking conditional expectation on both sides we have

$$\begin{aligned} \mathbb{E}(|Y_{k+1}|^p|Y_1) &\geq \mathbb{E}(|Y_k|^p|Y_1)(\Delta t^p (\mathbb{E}(|Y_k|^p|Y_1))^2 - (1-0.5\Delta t)^p - \mathbb{E}|\Delta B_k|^p) - |\Delta t|^p - \mathbb{E}|\Delta B_k|^p \\ &\geq \mathbb{E}(|Y_k|^p|Y_1)(\Delta t^p (\mathbb{E}(|Y_k|^p|Y_1))^2 - 1 - 2) - 1 - 2 \\ &\geq \frac{2^{k+2}}{\Delta t^{p/2}}(2^{2k+4} - 3) - 3 \\ &\geq \frac{2^{k+3}}{\Delta t^{p/2}}. \end{aligned}$$

Then substituting it back to (4.5) we obtain

$$\mathbb{E}|Y_{k+1}|^p \geq \frac{2^{k+3}}{\Delta t^{p/2}} \mathbb{P}\left(|Y_1|^p \geq \frac{2^3}{\Delta t^{p/2}}\right).$$

Hence the assertion holds. □

This lemma states that for any initial value, the pth moment, $0 < p < 1$, of the EM solution will blow up. This contrasts to the initial-data-independent asymptotic boundedness of the underlying SDE solution, shown by Theorem 4.2. Hence the EM method is no longer a good candidate.

4.2. Asymptotic boundedness

Let us present another theorem on the asymptotic boundedness of the solution of the SDE (2.1). The condition used in Theorem 4.2 will be employed in Theorem 4.3 as well.

Theorem 4.2. *Let (4.1) and (4.2) hold. If there exists a constant D such that*

$$\frac{\sum_{i=1}^m |g_i(x)|^2}{D + |x|^2} - \frac{2\sum_{i=1}^m \langle x, g_i(x)\rangle^2}{(D+|x|^2)^2} \leq \rho + \frac{P_1(|x|)}{D+|x|^2} + \frac{P_3(|x|)}{(D+|x|^2)^2}, \tag{4.6}$$

where ρ is a constant with $\mu + \rho/2 < 0$, then there exists a $p^* \in (0, 1)$ such that for all $0 < p < p^*$ the solution of SDE (2.1) obeys

$$\limsup_{t\to\infty} \mathbb{E}(|x(t)|^p) \leq C, \quad \forall x(0) \in \mathbb{R}^n, \tag{4.7}$$

where C is a constant dependent on μ, α, K, p and D, but independent of $x(0)$.

It is straightforward to adapt the proof of Theorem 3.1 to show Theorem 4.2.

Let us now begin to discuss the asymptotic boundedness in small moment of the backward EM solution (2.3) under conditions (4.1), (4.2) and (4.6).

Theorem 4.3. *Let (4.1), (4.2) and (4.6) hold. Then there exists a pair of constants $p^* \in (0, 1)$ and $\Delta t^* \in (0, 1/(2|\mu|))$ such that for $\forall p \in (0, p^*)$ and $\forall \Delta t \in (0, \Delta t^*)$, the backward EM solution (2.3) satisfies*

$$\limsup_{k\to\infty} \mathbb{E}|Y_k|^p \leq \frac{C_2'}{p(\lambda - \varepsilon)}, \quad \forall Y_0 \in \mathbb{R}^n, \tag{4.8}$$

where $-\lambda = \mu + \rho/2 < 0$, $\varepsilon \in (0, |\mu + \rho/2|)$ and C_2' is a constant dependent on K, α, p and D, but independent of Y_0 and Δt.

Proof. From (2.3), we have

$$|Y_{k+1}|^2 = \left\langle Y_{k+1}, Y_k + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k} \right\rangle + \langle Y_{k+1}, f(Y_{k+1})\Delta t\rangle.$$

By (4.1), we obtain

$$|Y_{k+1}|^2 \leq \frac{1}{2}|Y_{k+1}|^2 + \frac{1}{2}\left|Y_k + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2 + \mu\Delta t|Y_{k+1}|^2 + \alpha\Delta t.$$

Hence

$$\frac{D}{1-2\mu\Delta t}+|Y_{k+1}|^2 \le \frac{D}{1-2\mu\Delta t}+\frac{1}{1-2\mu\Delta t}\left(|Y_k|^2+2\left\langle Y_k,\sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right\rangle+\left|\sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2+2\alpha\Delta t\right)$$
$$\le \frac{D+|Y_k|^2}{1-2\mu\Delta t}(1+\zeta_k),$$

where

$$\zeta_k=\frac{1}{D+|Y_k|^2}\left(2\left\langle Y_k,\sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right\rangle+\left|\sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2+2\alpha\Delta t\right).$$

It is clear that $\zeta_k > -1$ for all $k \ge 0$. For any $p \in (0, 1)$, by inequality (3.5) we have

$$\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}\big|\mathcal{F}_{k\Delta t}) \le \left(\frac{D+|Y_k|^2}{1-2\mu\Delta t}\right)^{p/2}\mathbb{E}\left(1+\frac{p}{2}\zeta_k+\frac{p(p-2)}{8}\zeta_k^2+\frac{p(p-2)(p-4)}{2^3\times 3!}\zeta_k^3|\mathcal{F}_{k\Delta t}\right). \tag{4.9}$$

Then following the same way as Theorem 3.2, by (4.2) we can show

$$\mathbb{E}(\zeta_k\big|\mathcal{F}_{k\Delta t})=\frac{1}{D+|Y_k|^2}\left(\sum_{i=1}^m|g_i(Y_k)|^2\Delta t+2\alpha\Delta t\right), \tag{4.10}$$

$$\mathbb{E}(\zeta_k^2\big|\mathcal{F}_{k\Delta t}) \ge \frac{4\sum_{i=1}^m\langle Y_k,g_i(Y_k)\rangle^2}{(D+|Y_k|^2)^2}\Delta t-\frac{P_2(|Y_k|)\Delta t^2}{(D+|Y_k|^2)^2}, \tag{4.11}$$

and

$$\mathbb{E}(\zeta_k^3\big|\mathcal{F}_{k\Delta t}) \le C_1\Delta t^2+\frac{P_4(|Y_k|)\Delta t^2}{(D+|Y_k|^2)^3}, \tag{4.12}$$

where C_1 is a constant dependent on K. Substituting (4.10)–(4.12) into (4.9), then using (4.2) and (4.6) and the similar argument in (3.10) we obtain

$$\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}\big|\mathcal{F}_{k\Delta t}) \le \left(\frac{D+|Y_k|^2}{1-2\mu\Delta t}\right)^{p/2}$$
$$\times\left(1+\frac{p}{2}\frac{\sum_{i=1}^m|g_i(Y_k)|^2}{D+|Y_k|^2}\Delta t+\frac{p(p-2)}{8}\frac{4\sum_{i=1}^m\langle Y_k,g_i(Y_k)\rangle^2}{(D+|Y_k|^2)^2}\Delta t+\frac{p(p-2)(p-4)}{2^3\times 3!}C_1\Delta t^2\right)+C_2'\Delta t$$
$$\le \frac{(D+|Y_k|^2)^{p/2}}{(1-2\mu\Delta t)^{p/2}}\left(1+\frac{1}{2}p\rho\Delta t+\frac{1}{2}p^2mK\Delta t+C_1'\Delta t^2\right)+C_2'\Delta t,$$

where C_1' is a positive constant dependent on K and p, C_2' is a positive constant dependent on K, α, μ, p and D, and both of them may change from line to line. Taking expectations on both sides, we obtain

$$\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}) \le \frac{1+\frac{1}{2}p\rho\Delta t+\frac{1}{2}p^2mK\Delta t+C_1'\Delta t^2}{(1-2\mu\Delta t)^{p/2}}\mathbb{E}((D+|Y_k|^2)^{p/2})+C_2'\Delta t. \tag{4.13}$$

For any $\varepsilon \in (0, |\mu+\rho/2|)$, by choosing p^* sufficiently small such that $p^*mK \le 1/(2\varepsilon)$ and sufficiently small Δt^*, for $p < p^*$ and $\Delta t < \Delta t^*$ we have

$$(1-2\mu\Delta t)^{p/2} \ge 1-p\mu\Delta t-C_3\Delta t^2 > 0, \tag{4.14}$$

where $C_3 > 0$ is a constant dependent on μ and p. Then further reducing Δt^* gives that for $\Delta t < \Delta t^*$

$$C_1'\Delta t < \frac{1}{8}p\varepsilon, \qquad C_3\Delta t < \frac{1}{4}\varepsilon, \qquad \left|p\left(\mu+\frac{1}{4}\right)\Delta t\right| < \frac{1}{2}.$$

Using these three inequalities together with (4.14), we have from (4.13) that

$$\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}) \leq \frac{1+\frac{1}{2}p\left(\rho+\frac{1}{2}\varepsilon\right)\Delta t}{1-p\left(\mu+\frac{1}{4}\varepsilon\right)\Delta t}\mathbb{E}((D+|Y_k|^2)^{p/2}) + C_2'\Delta t. \tag{4.15}$$

Since that for any $h \in [-0.5, 0.5]$

$$(1-h)^{-1} = 1+h+h^2\sum_{i=0}^{\infty}h^i \leq 1+h+h^2\sum_{i=0}^{\infty}0.5^i = 1+h+2h^2,$$

then by further reducing Δt^* such that for any $\Delta t < \Delta t^*$ we obtain

$$4p\left(\mu+\frac{1}{4}\varepsilon\right)^2\Delta t + \left(\rho+\frac{1}{2}\varepsilon\right)\left(p\left(\mu+\frac{1}{4}\varepsilon\right)\Delta t + 2\left(p\left(\mu+\frac{1}{4}\varepsilon\right)\Delta t\right)^2\right) < \varepsilon.$$

Together with (4.15), we arrive at

$$\begin{aligned}\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}) &\leq \left(1+\frac{1}{2}p\left(\rho+\frac{1}{2}\varepsilon\right)\Delta t\right)\left(1+p\left(\mu+\frac{1}{4}\varepsilon\right)\Delta t\right.\\ &\quad \left.+\,2\left(p\left(\mu+\frac{1}{4}\varepsilon\right)\Delta t\right)^2\right)\mathbb{E}((D+|Y_k|^2)^{p/2}) + C_2'\Delta t\\ &\leq \left[1+p\left(\mu+\frac{1}{2}\rho+\varepsilon\right)\Delta t\right]\mathbb{E}((D+|Y_k|^2)^{p/2}) + C_2'\Delta t.\end{aligned} \tag{4.16}$$

Then by iteration and letting $k \to \infty$, we have

$$\limsup_{k\to\infty}\mathbb{E}(|Y_{k+1}|^p) \leq \limsup_{k\to\infty}\mathbb{E}((D+|Y_{k+1}|^2)^{p/2}) \leq \frac{C_2'}{-p\left(\mu+\frac{1}{2}\rho+\varepsilon\right)}.$$

The proof is complete. □

5. The second moment

In this section, we discuss the asymptotic boundedness in the second moment for both the EM method and the backward EM method. Following the same structure as in the previous sections, we first give the results for the underlying SDEs, then the results for numerical solutions are proved under the same conditions.

5.1. The EM method

For the asymptotic second moment boundedness of the underlying solution, we still require condition (3.1) but replace condition (3.2) by the following condition that

$$\langle x, f(x)\rangle + \frac{1}{2}\sum_{i=1}^{m}|g_i(x)|^2 \leq -\beta|x|^2 + a_1, \quad \forall x \in \mathbb{R}^n, \tag{5.1}$$

where β and a_1 are positive constants.

Theorem 5.1. *Let (3.1) and (5.1) hold, then the underlying solution of SDE (2.1) is asymptotically bounded in the second moment*

$$\limsup_{t\to\infty}\mathbb{E}(|x(t)|^2) \leq \frac{a_1}{\beta}, \quad \forall x(0) \in \mathbb{R}^n. \tag{5.2}$$

We refer the reader to Chapter 5 of [19] for the proof.

Now we consider to reproduce this boundedness property by the EM method.

Theorem 5.2. *Let (3.1) and (5.1) hold, then for any $\Delta t < 2\beta/K$ the EM solution (2.2) satisfies*

$$\limsup_{k\to\infty}\mathbb{E}|Y_k|^2 \leq \frac{2a_1+\alpha\Delta t}{2\beta-K\Delta t}, \quad \forall Y_0 \in \mathbb{R}^n.$$

Moreover, let the stepsize tend to zero, then

$$\lim_{\Delta t\to 0}\limsup_{k\to\infty}\mathbb{E}|Y_k|^2 \le \frac{a_1}{\beta},\quad \forall Y_0\in\mathbb{R}^n. \tag{5.3}$$

Proof. Since $\Delta B_{i,k}$, $i = 1,\dots,m$, is independent from each other, we have $\mathbb{E}(\Delta B_{i,k}) = 0$, $\mathbb{E}((\Delta B_{i,k})^2) = \Delta t$ and $\mathbb{E}(\Delta B_{i,k}\Delta B_{j,k}) = \mathbb{E}(\Delta B_{i,k})\mathbb{E}(\Delta B_{j,k}) = 0$, for $i \ne j$. Taking square and expectation on both sides of the EM solution (2.2), we have

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 &\le \mathbb{E}|Y_k|^2 + \Delta t^2\mathbb{E}(|f(Y_k)|^2) + \Delta t\mathbb{E}\left(2\langle Y_k, f(Y_k)\rangle + \sum_{i=1}^m |g_i(Y_k)|^2\right)\\ &\le \mathbb{E}|Y_k|^2 + \Delta t^2(K\mathbb{E}|Y_k|^2+\alpha) + \Delta t(-2\beta\mathbb{E}|Y_k|^2 + 2a_1)\\ &\le (1-2\beta\Delta t + K\Delta t^2)\mathbb{E}|Y_k|^2 + (\alpha\Delta t^2 + 2a_1\Delta t).\end{aligned}$$

By iteration, we see

$$\mathbb{E}|Y_{k+1}|^2 \le (1-2\beta\Delta t+K\Delta t^2)^{k+1}\mathbb{E}|Y_0|^2 + (\alpha\Delta t^2+2a_1\Delta t)\frac{1-(1-2\beta\Delta t+K\Delta t^2)^{k+1}}{1-(1-2\beta\Delta t+K\Delta t^2)}.$$

Choosing $\Delta t < 2\beta/K$, we have $1-2\beta\Delta t+K\Delta t^2 < 1$. Let k tend to infinity and Δt tend to 0; then the assertion holds. □

It is interesting to see that for the case of second moment, the EM method can reproduce not only the boundedness property but also the upper bound accurately, that is the upper bounds in (5.2) and (5.3) are identical. From this point of view, the result for the second moment is better than that for the small moment. However, it should be noticed that the condition (5.1) is stronger than condition (3.2).

5.2. The backward EM method

To relax the constraint on the drift coefficient, we replace the linear growth condition by the following condition

$$\langle x, f(x)\rangle \le -\eta|x|^2 + a_2,\quad \forall x\in\mathbb{R}^n, \tag{5.4}$$

where η and a_2 are positive constants. We still need the linear growth condition (4.2) on the diffusion coefficient. For the asymptotic boundedness of the second moment of the underlying solution we state another theorem as follows and we refer the reader to Chapter 5 of [19] for the proof.

Theorem 5.3. *Let (4.2) and (5.4) hold. If $2\eta > mK$, the underlying solution of SDE (2.1) is asymptotically bounded in the second moment*

$$\limsup_{t\to\infty}\mathbb{E}(|x(t)|^2) \le \frac{2a_2+m\alpha}{2\eta - mK},\quad \forall x(0)\in\mathbb{R}^n. \tag{5.5}$$

However, in the same spirit of Lemma 4.1, we see the second moment of the EM solution may blow up under condition (5.4). So we turn to the backward EM method.

Theorem 5.4. *Let (4.2) and (5.4) hold. If $2\eta > mK$, then for any $\Delta t > 0$ the BE solution (2.3) satisfies*

$$\limsup_{k\to\infty}\mathbb{E}|Y_k|^2 \le \frac{2a_2+m\alpha}{2\eta-mK},\quad \forall Y_0\in\mathbb{R}^n. \tag{5.6}$$

Proof. Taking square on both sides of the backward EM solution, by (5.4) we obtain

$$\begin{aligned}|Y_{k+1}|^2 &= \left\langle Y_{k+1}, Y_k + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right\rangle + \langle Y_{k+1}, f(Y_{k+1})\Delta t\rangle\\ &\le \frac{1}{2}|Y_{k+1}|^2 + \frac{1}{2}\left|Y_k + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2 - \eta\Delta t|Y_{k+1}|^2 + a_2\Delta t\\ &\le \frac{1}{1+2\eta\Delta t}\left|Y_k + \sum_{i=1}^m g_i(Y_k)\Delta B_{i,k}\right|^2 + \frac{2a_2\Delta t}{1+2\eta\Delta t}.\end{aligned}$$

Then taking expectation on both sides, by (4.2) we see

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 &\le \frac{1}{1+2\eta\Delta t}(\mathbb{E}|Y_k|^2 + mK\Delta t\mathbb{E}|Y_k|^2 + m\alpha\Delta t) + \frac{2a_2\Delta t}{1+2\eta\Delta t}\\ &\le \frac{1+mK\Delta t}{1+2\eta\Delta t}\mathbb{E}|Y_k|^2 + \frac{(2a_2+m\alpha)\Delta t}{1+2\eta\Delta t}.\end{aligned}$$

By iteration, we have

$$\mathbb{E}|Y_{k+1}|^2 \le \left(\frac{1+mK\Delta t}{1+2\eta\Delta t}\right)^{k+1}\mathbb{E}|Y_0|^2 + \frac{(2a_2+m\alpha)\Delta t}{1+2\eta\Delta t}\times\frac{1-((1+mK\Delta t)/(1+2\eta\Delta t))^{k+1}}{1-(1+mK\Delta t)/(1+2\eta\Delta t)}.$$

Due to $2\eta > mK$, let $k \to \infty$; then the assertion holds. □

We have three comments on Theorem 5.4.

- Compare the upper bounds in (5.5) and (5.6), we observe the backward EM method can reproduce the asymptotic upper bound of the underlying solution accurately as well.
- There is no constraint on the stepsize for the backward EM method.
- The conditions we imposed in the case of second moment are stronger than those used in the small moment in Section 4.

6. Conclusions and further discussion

In this paper we have presented results on numerical asymptotic boundedness in both small moment and second moment. In both cases, the numerical methods are showed to be able to reproduce the asymptotic boundedness property of the underlying solution under certain conditions. It should be noted that the conditions for the small moment are weaker than those for the second moment, but better results are obtained for the second moment, that is the upper bound could be reproduced accurately and the requirement of the stepsize could be stated explicitly.

One obvious open question is in the case of small moment whether we could recover the upper bound of the true solution of the SDE accurately by using the numerical solution with carefully chosen D and Δt. Besides, although the asymptotic boundedness property for pth moment with $1 < p < 2$ could be implied by the second moment, it is still worth to investigate if there exists different (possibly weaker) conditions for $p \in (1, 2)$. Also, the existence of sufficient conditions for the case of $p > 2$ is interesting for further research.

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