J. Math. Anal. Appl. 348 (2008) 34–37

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Automatic search for multiple limit cycles in three-dimensional Lotka–Volterra competitive systems with classes 30 and 31 in Zeeman's classification ☆

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ARTICLE INFO

*Article history:*
Received 8 April 2008
Available online 28 June 2008
Submitted by M. Fila

*Keywords:*
Lotka–Volterra systems
Center manifold
Hopf bifurcation
Limit cycles

ABSTRACT

Among the six classes of Zeeman's classification for three-dimensional Lotka–Volterra competitive systems with limit cycles, besides the classes 26, 27, 28 and 29, multiple limit cycles are found in classes 30 and 31 by an algorithmic method proposed by Hofbauer and So [J. Hofbauer, J.W. So, Multiple limit cycles for three-dimensional Lotka–Volterra equations, Appl. Math. Lett. 7 (1994) 65–70]. This also gives an answer to a problem proposed in [J. Hofbauer, J.W. So, Multiple limit cycles for three-dimensional Lotka–Volterra equations, Appl. Math. Lett. 7 (1994) 65–70].



## 1. Introduction

It is well known that a two-dimensional Lotka–Volterra system cannot admit isolated periodic orbit [7]; that is, if the system has periodic orbits, then these orbits are nonisolated.

In May and Leonard [12], a three-dimensional competitive system was studied and a general class of solutions with non-periodic oscillation was displayed. It was treated in subsequent papers by Chenciner [2] for Liapunov function construction and asymptotic behavior, Coste et al. [3] for the existence of limit cycles, Schuster et al. [13] for the structure of $\omega$-limit sets, Hofbauer [6] for the existence of limit cycles for higher-dimensional systems.

Hirsch's theorem [5] ensures that there is an invariant manifold (called the carrying simplex) for a three-dimensional Lotka–Volterra competitive system which is homeomorphic to the two-dimensional simplex and which attracts all orbits except the origin. By Hirsch's theorem, Zeeman [15] used geometric analysis of the surfaces $\dot{x}_i = 0$ of a system to define a combinatorial equivalence relation by inequalities on the parameters. A classification of 33 stable equivalence classes for three-dimensional Lotka–Volterra competitive systems was given in [15]. For 27 of these classes, the dynamical behaviors for the systems have been fully described [14,15]. The Hopf bifurcation theorem is applied to show that the remaining classes 26–31 can possess isolated periodic orbits or limit cycles. The question of how many limit cycles can appear in Zeeman's six classes 26–31 is open.

In 1994, two limit cycles for class 27 which has a heteroclinic cycle as in [12] were constructed by Hofbauer and So [8] based on Hirsch's monotone flow theorem, the center manifold theorem, and the Hopf bifurcation theorem. In their cases, the local stable positive equilibrium is surrounded by two limit cycles, in which one is from the Hopf bifurcation theorem

☆ Partially supported by a National Key Basic Research Project of China (Grant No. 2004CB318000) and the National Natural Science Foundation of China (Grant No. 10726001).

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2008.06.040

and the other is guaranteed by the Poincaré–Bendixson theorem. In [8], Hofbauer and So proposed a question to determine which other classes (26, 28, 29, 30, 31) in Zeeman's classification can have two or more limit cycles. In 2002, Lu and Luo [9] constructed two limit cycles for classes 26, 28, 29. In 2003, Lu and Luo [10] and in 2006, Gyllenberg et al. [4] constructed three cycles for classes 27 and 29, respectively. In fact, as a referee of the present paper points out that the example 4 of class 29 in [9] is permanent and the outer bifurcating limit cycle is unstable, by Poincaré–Bendixson theorem, this implies the existence of a third limit cycles in class 29.

For the classes 30 and 31, the existence of multiple limit cycles seems unsolved. Based on the algorithmic method proposed by Hofbauer and So [8] and modified by Lu and Luo [9], in this note, multiple limit cycles for classes 26, 27, 28, 29, 30, 31 in Zeeman's classification are found automatically.

By using the program `CLVproc`, 15128 examples are searched with randomly chosen interaction matrices. And among these examples, 1483, 4533, 10, 38 belong to classes 26, 27, 28, 29, respectively, but only 1 belongs to class 30 and 2 belong to class 31.

In Section 2, the algorithmic method of Hofbauer and So [8] and Lu and Luo [9] is stated with a program implemented. Examples with three limit cycles for classes 30 and 31 are found based on the method of Section 2 in Section 3. In Section 4, some concluding remarks are given.

## 2. Algorithmic construction for limit cycles

The algorithmic construction method stated below is from Hofbauer and So [8] and Lu and Luo [9].

Consider a three-dimensional Lotka–Volterra competitive system

$$\dot{x}_i = x_i \sum_{j=1}^{3} a_{ij}(x_j - 1), \quad i = 1, 2, 3, \tag{1}$$

where $a_{ij} < 0$ and $\mathbf{1} = (1, 1, 1)$ is the unique positive equilibrium of system (1).

Suppose that matrix $A = (a_{ij})_{3\times 3}$ has one real eigenvalue $\lambda$ and a pair of purely imaginary eigenvalues $\pm\omega i (\omega \neq 0)$. Then there exists a transformation matrix $T$ such that

$$TAT^{-1} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{2}$$

Here, the submatrix $\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{smallmatrix}\right)$ has a pair of purely imaginary eigenvalues $\pm\omega i$ $(\omega \neq 0)$. That is, $c_{11} + c_{22} = 0$ and $c_{11}c_{22} - c_{12}c_{21} > 0$.

In our construction, an interaction matrix $A$ will be chosen randomly and the block diagonal matrix $TAT^{-1}$ is obtained by the subprograms `randA` and `procT`($A =$ `randA`(), $T =$ `procT`($A$)).

From the center manifold theorem [1], we can suppose that, under the transformation $y = T(x - 1)$, the transformed system with linear part $Cy$ has an approximation to the center manifold taking the form

$$y_3 = h(y_1, y_2) = f_2(y_1, y_2) + f_3(y_1, y_2) + f_4(y_1, y_2) + \text{h.o.t}, \tag{3}$$

where $y = (y_1, y_2, y_3)^T$, $f_i = \sum_{j=0}^{i} c_{ij} y_1^{i-j} y_2^j$ and h.o.t denotes the terms with orders greater than or equal to five.

Solving for $c_{ij}$, we can obtain the approximate center manifold $y_3 = h(y_1, y_2)$ which is fulfilled by subprogram `Cmanifold`.

Substituting $y_3 = h(y_1, y_2)$ into the transformed system, the three-dimensional system is reduced to a two-dimensional one with center-focus type. The focal values are calculated by subprogram `Fvalue`. After obtaining the focal values, we use real root isolation method (subprogram `mrealroot` [11]) to check the independence of the focal values to get multiple limit cycles. A subprogram `ZM_class` is used for checking to which class in Zeeman's classification the constructed system belongs.

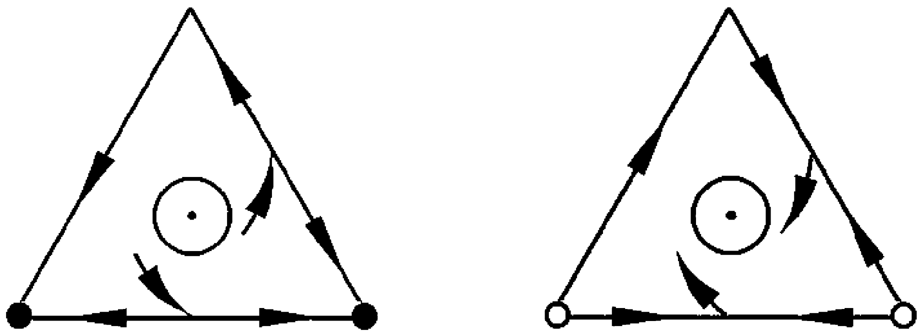
The main algorithm of the program `CLVproc` is as follows:

```
INPUT number
OUTPUT A,Znum
BEGIN
Znum := table([(26)=0,(27)=0,(28)=0,(29)=0,(30)=0,(31)=0]);
for j from 1 to number do
    A:=randA();
    T:=procT(A);
    Z:=Cmanifold(A,T,[x1,x2,x3],[y1,y2,y3],h);
    J:=Fvalues(Z[1],Z[2],[y1,y2],2);
    f1:=numer(J[1]);
    f2:=denom(J[1]);
    g1:=numer(J[2]);
    g2:=denom(J[2]);
```

```
    Mr:=Mrealroot([f1],[λ],1/10^20,[λ,f2,g1,g2,det(A),μ]);
    for i from 1 to nops(Mr) do
        if Mr[i,2,1]=- and Mr[i,2,2]<>0 and Mr[i,2,3]<>0 and
        Mr[i,2,4]<>0 and Mr[i,2,5]<>0 and Mr[i,2,6]=- then
            ds:=subs({x1=x1-1,x2=x2-1,x3=x3-1,λ=Mr[i,1,1]},Z[7]);
            ntemp:=ZM_class(ds,[x1,x2,x3]);
            if ntemp>=26 and ntemp<=31 then
                Znum[ntemp]:=Znum[ntemp]+ntemp;
                fd:=fopen("outA.txt",APPEND);
                fprintf(fd,"%a,%a",A,ntemp);
                close(fd);
                save j,Znum,"zm.txt";
            end if;
        end if;
   end do;
end do;
END:
```

## 3. Three limit cycles for classes 30 and 31

Based on Zeeman's classification, the dynamics of the classes 30 and 31 restricted on the carrying simplex are as follows:



class 30 class 31

Since the dynamics of classes 30 and 31 restricted to the carrying simplex are time-reversals of each other, we only construct limit cycles for class 31. Among the 15128 examples chosen randomly, there are only two belong to class 31 which have at least three limit cycles. The following is one of the examples.

The found system with an interaction matrix is as follows:

$$A=\begin{pmatrix} -\frac{1}{2} & -\frac{11}{21} & \lambda \\ -1 & -\frac{71}{59} & \mu \\ -1 & -100 & -\frac{36}{5} \end{pmatrix}.$$

To satisfy the necessary eigenvalue condition (see [8]) $\mathtt{det}(A)=(A_{11}+A_{22}+A_{33})\cdot\mathtt{trace}(A)$, we need $\mu=-\frac{799194693}{6146791100}-\frac{1334403\lambda}{10418290}$. By the transformation $y=T(x-1)$, with $T$ as follows:

$$\begin{pmatrix} -1 & \frac{77}{10} & -\frac{799194693}{6146791100}-\frac{1334403}{10418290\lambda} \\ -1 & -100 & \frac{201}{118} \\ \frac{2381247}{20836580}-\frac{13344030}{1041829\lambda} & -100\lambda+\frac{737}{826} & -\frac{7952186}{1041829\lambda}+\frac{2930380541}{43027537700} \end{pmatrix},$$

the three-dimensional system is transformed to be a new one whose linear part is in the block diagonal form

$$\begin{pmatrix} \frac{8812483}{1334403} & -\frac{1334403}{10418290}\lambda+\frac{64772793510359}{139021974308700} & 0 \\ -\frac{123022010}{1334403} & -\frac{8812483}{1334403} & 0 \\ 0 & 0 & -\frac{5253}{590} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The original three-dimensional system can be reduced to a two-dimensional system by the subprogram `Cmanifold`, and the first two focal values are obtained by the subprogram `Fvalues`,

$$LV_1=f(\lambda)=\frac{f_1(\lambda)}{f_2(\lambda)},$$
$$LV_2=g(\lambda)=\frac{g_1(\lambda)}{g_2(\lambda)},$$

where

$$f_1(\lambda) = 590321693949498\,(2583462210\,\lambda + 144294119)$$
$$\big(236660653229017446913055766390000\,\lambda^3 + 1104831703697239522133476646029 50\,\lambda^2$$
$$- 4221563749080913513732459518626 05\,\lambda - 302147201587821070643447518522 63\big),$$
$$f_2(\lambda) = 3783603736110025\,(359721278120400\,\lambda - 583621501722121)$$
$$(89930319530100\,\lambda - 598690136569291)^2,$$

and $g_1(\lambda)$ is a polynomial of 14 terms with degree 13 and $g_2(\lambda)$ is polynomial of 13 terms with degree 12.

Running the program mrealroot,

$$> \mathtt{mrealroot}\left(\left[f_1(\lambda)\right], [\lambda], \frac{1}{10^{20}}, \left[f_2(\lambda), g_1(\lambda), g_2(\lambda), \mathtt{det}(A), \lambda, \mu\right]\right),$$

we get

$$\left[\left[\frac{-10374223293533601281}{14757395258967641292 8}, \frac{-20262154870182815}{288230376151711744}\right], [-,-,-,-,-,-]\right].$$

This shows that it is a competitive system such that the real root of $LV_1 = 0$ in the interval form is $[\frac{-10374223293533601281}{147573952589676412928}, \frac{-20262154870182815}{288230376151711744}]$ around $-0.2180651959$ which makes $LV_2 > 0$. This ensures the existence of two limit cycles, the outer one being unstable.

The remaining work is to check to which class in Zeeman's classification the constructed system belongs.

Using Zeeman's notation, we have $R_{ij} = \mathrm{sgn}(\alpha_{ij})$ and $Q_{kk} = \mathrm{sgn}(\beta_{kk})$, with $\alpha_{ij} = \frac{b_i a_{ji}}{a_{ii}} - b_j = (AR_i)_j - b_j$ and $\beta_{kk} = (AQ_k)_k - b_k$, which are the algebraic invariants of $A$. Here $R_i$ is the equilibrium on the $x_i$-axis, and $Q_k$ is the positive equilibrium on the plane of $x_k = 0$ [9,15].

Since

$$R_{12} = -1, \quad Q_{33} = -1, \quad R_{21} = -1, \quad R_{23} = 1, \quad R_{32} = -1, \quad R_{31} = -1, \quad Q_{22} = -1, \quad R_{13} = -1,$$

the system belongs to class 31 in Zeeman's classification. It is checked that this system is permanent (as the referee points out), Poincaré–Bendixson theorem ensures the existence of a third limit cycle.

This shows that a system in class 31 can have at least three limit cycles.

## 4. Concluding remarks

In Section 3, based on the algorithm given by Hofbauer and So [8] and modified by Lu and Luo [9] described in Section 2, an example with three limit cycles belonging to 31 is constructed. The example together with ones in Hofbauer and So [8] and Lu and Luo [9] give an answer to a problem of existence of multiple limit cycles proposed by Hofbauer and So [8]. The procedure is well known, but the possibility to find the parameters of classes 30 and 31 is very low in the manipulation of constructing limit cycles, only around 0.01% as stated in Section 1.

In another aspect, as in Lu and Luo [9] and Gyllenberg et al. [4], for the example of class 31 in the present paper, the existence of a third limit cycle follows from the Poincaré–Bendixson theorem, since the boundary of the simplex is a repellor and the outer of the two limit cycles arising from the Hopf bifurcation is unstable.

## Acknowledgment

The authors are grateful to the referee for valuable comments, especially, for pointing out the existence of a third limit cycle in class 31.

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