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A Note on Clean Rings*

Zhou Wang **Jianlong Chen**
Department of Mathematics, Southeast University, Nanjing 210096, China
E-mail: fylwangz@126.com *jlchen@seu.edu.cn*

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Abstract. Let R be a ring and $g(x)$ a polynomial in $C[x]$, where $C = C(R)$ denotes the center of R. Camillo and Simón called the ring $g(x)$-clean if every element of R can be written as the sum of a unit and a root of $g(x)$. In this paper, we prove that for $a, b \in C$, the ring R is clean and $b - a$ is invertible in R if and only if R is $g_1(x)$-clean, where $g_1(x) = (x - a)(x - b)$. This implies that in some sense the notion of $g(x)$-clean rings in the Nicholson–Zhou Theorem and in the Camillo–Simón Theorem is indeed equivalent to the notion of clean rings.

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1 Introduction

A ring R is called clean if every element of R is the sum of a unit and an idempotent. This notion was introduced by Nicholson in [4] as a sufficient condition for a ring to have the exchange property. Later, Camillo and Yu proved in [2] that all semiperfect rings and unit-regular rings are clean.

Generally, if $C = C(R)$ denotes the center of a ring R, and if $g(x)$ is a polynomial in $C[x]$, Camillo and Simón [1] called the ring $g(x)$-clean if every element of R can be written as the sum of a unit and a root of $g(x)$. If V is a vector space of countable dimension over a division ring D, they proved that $\mathrm{End}(_DV)$ is $g(x)$-clean provided that $g(x)$ has two toots in $C(D)$. Nicholson and Zhou [6] took a fixed polynomial $g(x) \in (x-a)(x-b)C[x]$, where $a, b \in C(R)$ such that b and $b-a$ are both units in R, and proved that $\mathrm{End}(_RM)$ is $g(x)$-clean if $_RM$ is a semisimple module over R.

Let $g_1(x) = (x - a)(x - b)$ with $a, b \in C$. In this paper, we prove that R is $g_1(x)$-clean if and only if R is clean and $b - a$ is invertible. Furthermore, for

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$g_2(x) \in (x-a)(x-b)C[x]$, where $a, b \in C$ and $b-a$ is a unit in R, if R is clean, then it is $g_2(x)$-clean; and if R is $g_2(x)$-clean for any $g_2(x) \in (x-a)(x-b)C[x]$, then it is clean. Finally, we construct an example which is $g(x)$-clean but not clean.

Throughout this paper, rings are associative with identity and modules are unitary. Let $J(R)$ and $U(R)$ denote the Jacobson radical and the group of units of R, respectively. We write C_n and RC_n for the cyclic group of order n and its group ring over R, respectively.

2 $g(x)$-Clean Rings and Clean Rings

In this section, we will investigate the relations between $g(x)$-clean rings and clean rings. The following result is simple but useful.

Theorem 2.1. *Let $g_1(x) = (x-a)(x-b)$ with $a, b \in C$. Then R is $g_1(x)$-clean if and only if R is clean and $b-a$ is invertible.*

Proof. Suppose $r \in R$. Since R is $g_1(x)$-clean, there exist a unit u_1 and a root s_1 of $g_1(x)$ such that $b = s_1 + u_1$. Since $g_1(s_1) = (s_1-a)(s_1-b) = 0$, we have $s_1 = a$. This implies that $b-a$ is invertible. Again by hypothesis, there exist a unit u_2 and a root s_2 of $g_1(x)$ such that $(b-a)r + a = s_2 + u_2$. Set $e = (b-a)^{-1}(s_2-a)$, i.e., $s_2 = (b-a)e + a$. Then we get $r = e + (b-a)^{-1}u_2$. Note that $g_1(s_2) = (s_2-a)(s_2-b) = (b-a)e[(b-a)e + a - b] = (b-a)^2(e^2-e) = 0$ by $b-a \in C(R)$. Since $b-a \in U(R)$, we have $e^2 = e$, as required.

Conversely, for any $r \in R$, by hypothesis we may write $(b-a)^{-1}(r-a) = e+u$, where $e^2 = e \in R$ and $u \in U(R)$. Thus, we have $r = [(b-a)e + a] + (b-a)u$. Note that $(b-a)u$ is a unit since $b-a \in U(R)$. Now we have $g_1((b-a)e+a) = (b-a)e[(b-a)e+a-b] = (b-a)^2e(e-1) = 0$. So $(b-a)e+a$ is a root of $g_1(x)$. This completes the proof. $\square$

In fact, the condition $a, b \in C(R)$ in Theorem 2.1 can be replaced by $b-a \in C(R)$.

In [2], Camillo and Yu showed that if $2 \in U(R)$, then R is clean if and only if every element of R is the sum of a unit and a square root of 1. In fact, the condition $2 \in U(R)$ is necessary.

Corollary 2.2. *A ring R is clean and $2 \in U(R)$ if and only if every element of R is the sum of a unit and a square root of* 1.

Proof. Let $g_1(x) = (x+1)(x-1) = x^2 - 1$. Note that the condition that every element of R is the sum of a unit and a square root of 1 is equivalent to R being $g_1(x)$-clean. Hence, by Theorem 2.1, the proof is immediate. $\square$

Remark 2.3. Let $g_1(x) = (x-a)(x-b)$ and $g_2(x) \in (x-a)(x-b)C[x]$, where $a, b \in C$ and $b-a \in U(R)$. Then R is clean if and only if R is $g_1(x)$-clean, and in this case, R is also $g_2(x)$-clean. On the other hand, if R is $g_2(x)$-clean for any $g_2(x) \in (x-a)(x-b)C[x]$, then R is clean.

Next we will construct an example to show that a $g(x)$-clean ring is not necessarily clean. First, we need a lemma which extends [7, Proposition 3.2].

Lemma 2.4. *Let R be a commutative ring and let C_n be a cyclic group of order n generated by g. Then an element $x = \sum_{i=0}^{n-1} k_i g^i \in RC_n$ is invertible if and only if* $\det A \in R$ *is invertible, where $k_i \in R$ and*

$$A = \begin{pmatrix} k_0 & k_{n-1} & \cdots & k_1 \\ k_1 & k_0 & \cdots & k_2 \\ & & \ddots & \\ k_{n-1} & k_{n-2} & \cdots & k_0 \end{pmatrix}.$$

Proof. The element x is invertible if and only if there exists $y = \sum_{i=0}^{n-1} l_i g^i$ such that $xy = 1 = yx$, i.e.,

$$\begin{pmatrix} k_0 & k_{n-1} & \cdots & k_1 \\ k_1 & k_0 & \cdots & k_2 \\ & & \ddots & \\ k_{n-1} & k_{n-2} & \cdots & k_0 \end{pmatrix} \begin{pmatrix} l_0 \\ l_1 \\ \vdots \\ l_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let B be the matrix just as A replaced k_i by l_i. Then $xy = 1 = yx$ if and only if $AB = I = BA$.

If $xy = 1 = yx$, then $\det A \cdot \det B = 1 = \det B \cdot \det A$ and so $\det A \in U(R)$.

Conversely, since R is a commutative ring, $\det A \in U(R)$ implies that A is an invertible matrix, i.e., the above matrix equation has a solution. Hence, $x \in U(RC_n)$. □

Theorem 2.5. *Let R be a commutative local ring with $2 \in U(R)$ and let C_3 be a cyclic group of order 3 generated by g. Then RC_3 is $(x^6 - 1)$-clean.*

Proof. Let $\alpha = k + mg + lg^2$, where $k, m, l \in R$. Write

$$\begin{aligned} \alpha &= 1 + [(k-1) + mg + lg^2] = -1 + [(k+1) + mg + lg^2] \\ &= g + [k + (m-1)g + lg^2] = -g + [k + (m+1)g + lg^2] \\ &= g^2 + [k + mg + (l-1)g^2] = -g^2 + [k + mg + (l+1)g^2]. \end{aligned}$$

The elements ± 1, $\pm g$ and $\pm g^2$ are the roots of the equation $x^6 = 1$. In order to show that α is $(x^6 - 1)$-clean, by Lemma 2.4, we only need to show that at least one of the following six elements is a unit in R:

(1) $(k-1)^3 + m^3 + l^3 - 3(k-1)ml$, (2) $(k+1)^3 + m^3 + l^3 - 3(k+1)ml$,
(3) $k^3 + (m-1)^3 + l^3 - 3k(m-1)l$, (4) $k^3 + (m+1)^3 + l^3 - 3k(m+1)l$,
(5) $k^3 + m^3 + (l-1)^3 - 3km(l-1)$, (6) $k^3 + m^3 + (l+1)^3 - 3km(l+1)$.

Suppose it is not true, i.e., all (1)–(6) belong to $J(R)$ since R is a commutative local ring.

By (1) and (2), we have $6k^2 + 2 - 6ml \in J(R)$. Since 2 is a unit in R, we have

$$3k^2 - 3ml + 1 \in J(R). \qquad (*)$$

If $3 \in J(R)$, then $1 \in J(R)$ by $(*)$, a contradiction. So $3 \in U(R)$. From $(*)$, we have $3k^3 - 3kml + k \in J(R)$. Similarly, $3m^3 - 3kml + m$, $3l^3 - 3kml + l \in J(R)$.

Thus, we obtain $3(k^3 + m^3 + l^3 - 3kml) + (k + m + l) \in J(R)$. Since $3 \in U(R)$,

$$k^3 + m^3 + l^3 - 3kml + 3^{-1}(k + m + l) \in J(R). \qquad (**)$$

By (1), (2) and $(**)$, we have $3k - 3^{-1}(k + m + l) \in J(R)$. Similarly, we have $3m - 3^{-1}(k + m + l)$, $3l - 3^{-1}(k + m + l) \in J(R)$. So $3(k + m + l) - (k + m + l) = 2(k + m + l) \in J(R)$. From $2 \in U(R)$, it is true that $k + m + l \in J(R)$. Hence, $3k \in J(R)$, this implies $k \in J(R)$. Similarly, $m, l \in J(R)$. By $(*)$, $1 \in J(R)$, a contradiction. Thus, α is $(x^6 - 1)$-clean, as required. □

Corollary 2.6. *Let R be a commutative semiperfect ring with $2 \in U(R)$ and let C_3 be a cyclic group of order 3. Then RC_3 is $(x^6 - 1)$-clean.*

Proof. Since R is semiperfect, there exist orthogonal local idempotents $e_1, \dots, e_m$ such that $1 = e_1 + \cdots + e_m$. So $R = e_1Re_1 \times \cdots \times e_mRe_m$ is a direct product of commutative local rings. Note that $RC_3 \cong (e_1Re_1)C_3 \times \cdots \times (e_mRe_m)C_3$, so RC_3 is $(x^6 - 1)$-clean by Theorem 2.5. □

Example 2.7. Let $\mathbb{Z}_{(7)} = \{m/n \mid m, n \in \mathbb{Z}, \gcd(7, n) = 1\}$ (which is a commutative local ring) and let C_3 be a cyclic group of order 3. Note that $2 \in U(\mathbb{Z}_{(7)})$. Let $g_2(x) = x^6 - 1 = (x - 1)(x + 1)(x^4 + x^2 + 1)$. Then by Theorem 2.5, $\mathbb{Z}_{(7)}C_3$ is $g_2(x)$-clean. However, Han and Nicholson [3] showed that $\mathbb{Z}_{(7)}C_3$ is not clean. Thus, we obtain an example which is $g(x)$-clean but not clean.

Remark 2.8. Let $g(x) = (x-a)h(x) \in C[x]$. If the equation $h(x) = 0$ has no solution in R, then R cannot be $g(x)$-clean. In fact, suppose R is $g(x)$-clean, then there exist a unit u and a root s of $g(x)$ such that $a = s + u$. Since $g(s) = (s - a)h(s) = 0$ and $s - a \in U(R)$, we get $h(s) = 0$. This is a contradiction.

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